Bitopological duality for distributive lattices and Heyting algebras

Nick Bezhanishvili
Department of Computing
Imperial College London
nbezhani@doc.ic.ac.uk
http://www.doc.ic.ac.uk/~nbezhani/

(Joint work with Guram Bezhanishvili, David Gabelaia and Alexander Kurz)

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Stone, Priestley and Esakia dualities

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- Esakia (1974) developed an order-topological duality for Heyting algebras by means of special Priestley spaces we call Esakia spaces.

Stone duality for distributive lattices

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A purely topological characterization of (X_D, τ_D) is as follows:

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In fact, each bounded distributive lattice is represented as the lattice of compact open subsets of the corresponding spectral space.

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Theorem (Stone 1937). $\mathbf{DLat} \simeq \mathbf{Spec}^{op}$.

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A purely order-topological characterization of $(X_D, \tau_P, \subseteq)$ is as follows:

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An ordered-topological space $(X, \tau, \leq)$ is called a Priestley space if (X, τ) is a Stone space and for all $x, y \in X$

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From every Priestley space $(X, \tau, \leq)$ we first construct (X, τ_1, τ_2), where τ_1 are open upsets of $(X, \tau, \leq)$ and τ_2 are open downsets of $(X, \tau, \leq)$.

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From (X, τ_1, τ_2) we obtain the spectral space (X, τ_1) by simply forgetting the topology τ_2.

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How do we characterize $\mathcal{C}$ purely in bitopological terms?

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We define the two topologies τ_D^+ and τ_D^- on X_D by declaring $\{\phi_+(a) : a \in D\}$ and $\{\phi_-(a) : a \in D\}$ as bases for τ_D^+ and τ_D^-, respectively.

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An internal bitopological characterization of $(X_D, \tau_D^+, \tau_D^-)$ is as follows:

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Definition (Salbany, 1974) (X, τ_1, τ_2) is called pairwise compact if for each cover $\{U_i : i \in I\}$ of X with $U_i \in \tau_1 \cup \tau_2$, there exists a finite subcover.

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Definition (Salbany, 1974). (X, τ_1, τ_2) is called pairwise Hausdorff if for any two distinct points $x, y \in X$ there exist disjoint $U \in \tau_1$ and $V \in \tau_2$ such that $x \in U$ and $y \in V$ or there are disjoint $U \in \tau_2$ and $V \in \tau_1$ such that $x \in U$ and $y \in V$.

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Definition (Reilly, 1973) (X, τ_1, τ_2) is pairwise zero-dimensional if $\tau_1 \cap \delta_2$ forms a basis for τ_1 and $\tau_2 \cap \delta_1$ forms a basis for τ_2.

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Theorem

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- $\mathbf{Bool} \simeq^{op} \mathbf{Stone}$.
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- $\mathbf{DLat} \simeq^{op} \mathbf{Spec} \simeq \mathbf{PStone} \simeq \mathbf{Pries}$

Duality dictionary

DLat	**Pries**	**PStone**	**Spec**
filter	closed upset	τ_2-closed set	compact saturated set
ideal	open upset	τ_1-open set	open set
prime filter	$\uparrow x$	$\mathrm{Cl}_2(x)$	$\mathrm{Sat}(x)$
prime ideal	$(\downarrow x)^c$	$[\mathrm{Cl}_1(x)]^c$	$[\mathrm{Cl}(x)]^c$
maximal filter	$\uparrow x = \{x\}$	$\mathrm{Cl}_2(x) = \{x\}$	$\mathrm{Sat}(x) = \{x\}$
maximal ideal	$(\downarrow x)^c = \{x\}^c$	$[\mathrm{Cl}_1(x)]^c = \{x\}^c$	$[\mathrm{Cl}(x)]^c = \{x\}^c$
homomorphic image	closed subset	pairwise compact subset	spectral subset
subalgebra	$Q \in \mathsf{Q}_X$	$(\tau_1', \tau_2') \in \mathsf{Z}_X$	$\tau' \in \mathsf{SC}_X$
canonical completion	$\mathsf{Up}(X)$	$\mathsf{S}_1(X) = \mathsf{CS}_2(X)$	$\mathsf{S}(X)$
MacNeille completion	$\mathsf{RgOpUp}(X)$	$\mathsf{RgOp}_{12}(X)$	$\mathsf{SatOp}(X)$
complete lattice	$\mathsf{RgOpUp}(X) = \mathsf{CpUp}(X)$	$\beta_1 = \mathsf{RgOp}_{12}(X)$	$\mathcal{E}(X) = \mathsf{SatOp}(X)$

Table 1: Dictionary for **DLat**, **Pries**, **PStone**, and **Spec**.

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A map $f : X \to Y$ is a p-morphism if f is order-preserving and $f(x) \leq y$ implies there exists $z \in \uparrow x$ such that $f(z) = y$.

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We call a pairwise Stone space an Esakia bitopological space if for each $A \in \tau_1 \cap \delta_2$ and $B \in \tau_2 \cap \delta_1$ we have $\mathsf{Cl}_1(A \cap B) \in \tau_2 \cap \delta_1$.

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Let (X, τ_1, τ_2) and (X', τ_1', τ_2') be Esakia bitopological spaces. A map $f : X \to X'$ is called an Esakia bitopological morphism if f is bi-continuous and $f(\mathsf{Cl}_2(x)) = \mathsf{Cl}_2'(f(x))$ for each $x \in X$.

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This provides an alternative representation and duality of Heyting algebras by means of special spectral spaces we call spectral Esakia spaces. Thus we obtain that the category **ESpec** of spectral spectral spaces is dually equivalent to **Heyt**, and so we obtain:

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- The bitopological duality for distributive lattices is the most natural analogue of the Stone duality for Boolean algebras.
- This provides one more argument in support of Jung and Moshier's claim that the bitopological duality is the most natural one for distributive lattices.

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- The same can be done also for co-Heyting algebras and bi-Heyting algebras.
- In fact, the asymmetry present in the definition of an Esakia bitopological space goes away in the bi-Heyting case, and the whole picture becomes symmetric again.
- This provides a rather convenient bitopological explanation of the lack of symmetry for Heyting algebras, first discussed by McKinsey and Tarski back in 1946.