Functions

Introducing Functions

A **function** f from a set A to a set B, written $f : A \to B$, is a relation $f \subseteq A \times B$ such that **every** element of A is related to **one** element of B; in logical notation

1. $(a, b_1) \in f \wedge (a, b_2) \in f \Rightarrow b_1 = b_2$;
2. $\forall a \in A. \exists b \in B. (a, b) \in f$.

The set A is called the **domain** and B the **co-domain** of f.

If $a \in A$, then $f(a)$ denotes the unique $b \in B$ st. $(a, b) \in f$.

Comments

If the domain A is the n-ary product $A_1 \times \ldots \times A_n$, then we often write $f(a_1, \ldots, a_n)$ instead of $f((a_1, \ldots, a_n))$.

The intended meaning should be clear from the context.

Recall the difference between the following two Haskell functions:

```
f :: A -> B -> C -> D
f :: (A,B,C) -> D
```

Our definition of function is not curried.

Image Set

Let $f : A \to B$. For any $X \subseteq A$, define the **image** of X under f to be

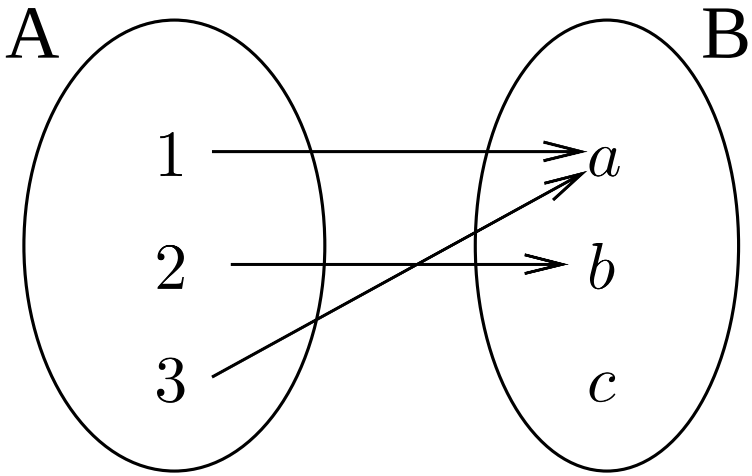
$$f[X] \stackrel{\text{def}}{=} \{f(a) : a \in X\}$$

The set $f[A]$ is called the **image set** of f.

Example

Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$.

Let $f \subseteq A \times B$ be defined by $f = \{(1, a), (2, b), (3, a)\}$.



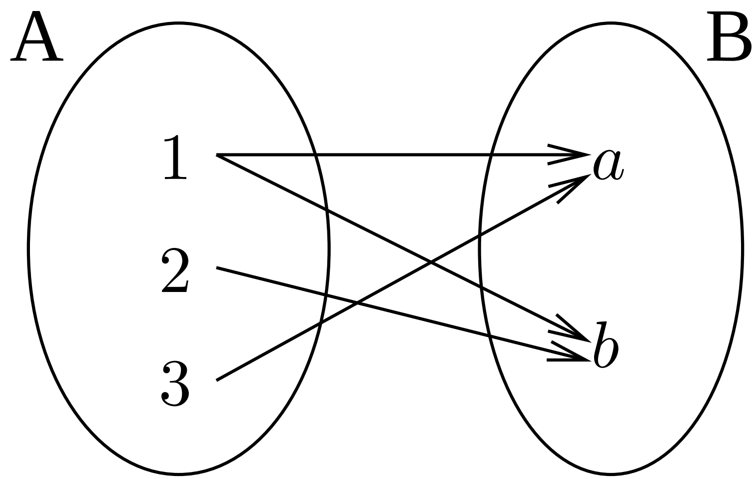
The image set of f is $\{a, b\}$.

The image of $\{1, 3\}$ under f is $\{a\}$.

Example

Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Let $f \subseteq A \times B$ be defined by $f = \{(1, a), (1, b), (2, b), (3, a)\}$.

This f is not a well-defined function.



Examples

The following are examples of functions with infinite domains and co-domains:

1. the function $f : \mathcal{N} \times \mathcal{N} \mapsto \mathcal{N}$ defined by $f(x, y) = x + y$;
2. the function $f : \mathcal{N} \mapsto \mathcal{N}$ defined by $f(x) = x^2$;
3. the function $f : \mathcal{R} \mapsto \mathcal{R}$ defined by $f(x) = x + 3$.

The binary relation R on the reals defined by $x\,R\,y$ if and only if $x = y^2$ is not a function

Cardinality

Let $A \longrightarrow B$ denote the set of all functions from A to B, where A and B are finite sets. If $|A| = m$ and $|B| = n$, then $|A \longrightarrow B| = n^m$.

Sketch proof

For each element of A, there are n independent ways of mapping it to B.

You do not need to remember this proof.

Partial Functions

A **partial** function f from a set A to a set B, written $f : A \rightharpoonup B$, is a relation $f \subseteq A \times B$ such that just **some** elements of A are related to **unique** elements of B:

$$(a, b_1) \in f \wedge (a, b_2) \in f \Rightarrow b_1 = b_2.$$

The partial function f is regarded as **undefined** on those elements which do not have an image under f.

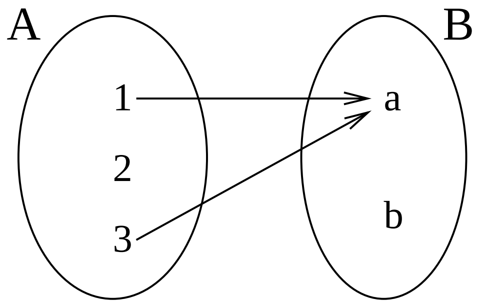
We call this undefined value $\perp$ (pronounced *bottom*).

A partial function from A to B is a function from A to $(B + \{\perp\})$.

Examples of Partial Functions

1. Haskell functions which return run-time errors on some (or all) arguments.

2. The relation $R = \{(1,a),(3,a)\} \subseteq \{1,2,3\} \times \{a,b\}$:



   Not every element in A maps to an element in B.

3. The binary relation R on $\mathcal{R}$ defined by $x\,R\,y$ iff $\sqrt{x} = y$. It is not defined when x is negative.

Properties of Functions

Let $f : A \to B$ be a function.

1. f is **onto** if and only if every element of B is in the image of f:

$$\forall b \in B. \exists a \in A. f(a) = b.$$

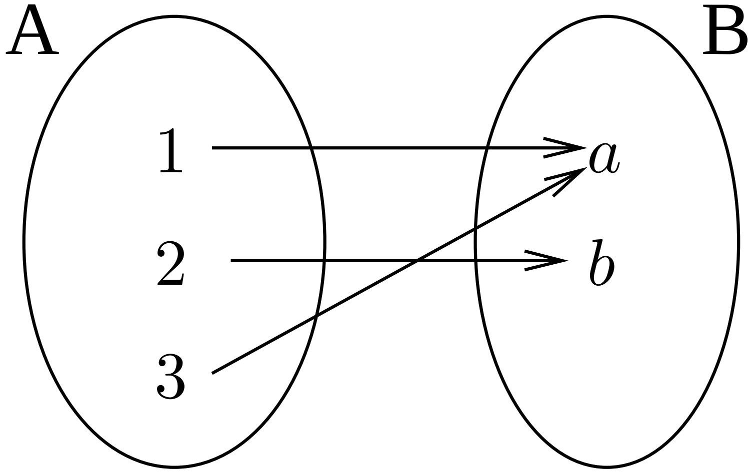
2. f is **one-to-one** if and only if for each $b \in B$ there is at most one $a \in A$ with $f(a) = b$:

$$\forall a, a' \in A. f(a) = f(a') \text{ implies } a = a'.$$

3. f is a **bijection** iff f is **both** one-to-one and onto.

Example

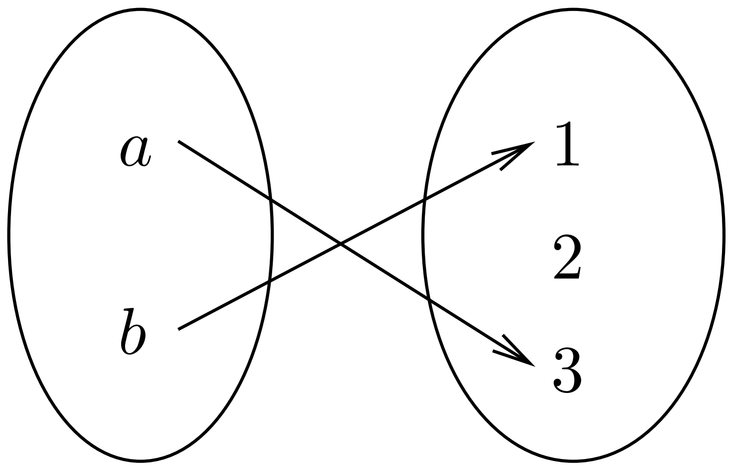
Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. The function $f = \{(1, a), (2, b), (3, a)\}$ is onto, but **not** one-to-one:



We cannot define a one-to-one function from A to B. There are too many elements in A for them to map uniquely to B.

Example

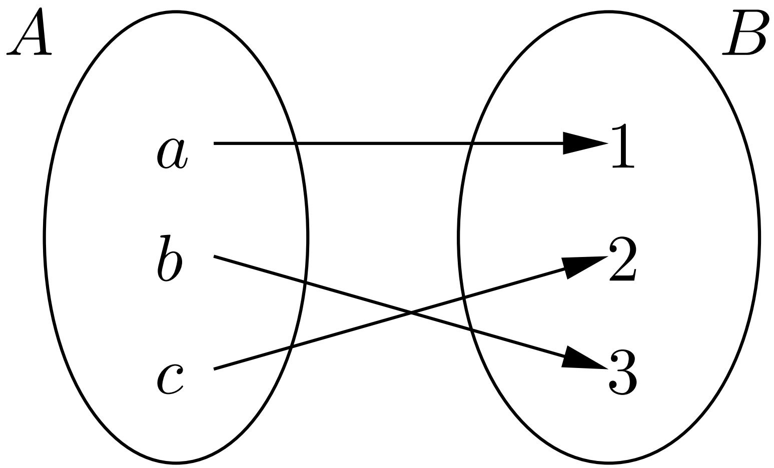
Let $A = \{a, b\}$ and $B = \{1, 2, 3\}$. The function $f = \{(a, 3), (b, 1)\}$ is one-to-one, but not onto:



It is not possible to define an onto function from A to B. There are not enough elements in A to map to all the elements of B.

Example

Let $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$. The function $f = \{(a, 1), (b, 3), c, 2)\}$ is bijective:



Example

The function f on natural numbers defined by $f(x, y) = x + y$ is onto but not one-to-one.

To prove that f is onto, take an arbitrary $n \in \mathcal{N}$. Then $f(n, 0) = n + 0 = n$.

To show that f is not one-to-one, we need to produce a counter-example: that is, find $(m_1, m_2), (n_1, n_2)$ such that $(m_1, m_2) \neq (n_1, n_2)$, but $f(m_1, m_2) = f(n_1, n_2)$.

For example, $(1, 0)$ and $(0, 1)$.

Examples

1. The function f on natural numbers defined by $f(x) = x^2$ is one-to-one. The similar function f on integers is not.

2. The function f on integers defined by $f(x) = x + 1$ is onto. The similar function on natural numbers is not.

3. The function f on the real numbers given by $f(x) = 4x + 3$ is a bijective function.

   The proof is given in the notes.

The Pigeonhole Principle

Pigeonhole Principle

Let $f : A \rightarrow B$ be a function, where A and B are **finite**. If $|A| > |B|$, then f cannot be a one-to-one function.

Example

Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. A function $f : A \rightarrow B$ cannot be one-to-one, since A is too big.

It is not possible to prove this property directly.

The pigeonhole principle states that we assume that the property is true.

Proposition

Let A and B be finite sets, let $f : A \to B$ and let $X \subseteq A$. Then

$$|f[X]| \leq |X|.$$

Proof Suppose for contradiction that $|f[X]| > |X|$. Define a function $p : f[X] \to X$ by

$$p(b) = \text{some } a \in X \text{ such that } f(a) = b.$$

There is such an a by definition of $f[X]$. We are placing the members of $f[X]$ in the pigeonholes X. By the pigeonhole principle, there is some $a \in X$ and $b, b' \in f[X]$ with $p(b) = p(b') = a$. But then $f(a) = b$ and $f(a) = b'$. Contradiction.

Proposition

Let A and B be **finite** sets, and let $f : A \to B$.

1. If f is one-to-one, then $|A| \leq |B|$.
2. If f is onto, then $|A| \geq |B|$.
3. If f is a bijection, then $|A| = |B|$.

Proof

Part (a) is the contrapositive of the pigeonhole principle.

For (b), notice that if f is onto then $f[A] = B$. Hence, $|f[A]| = |B|$. Also $|A| \geq |f[A]|$ by previous proposition. Therefore $|A| \geq |B|$ as required.

Part (c) follows from parts (a) and (b).

Composition

Let A, B and C be arbitrary sets, and let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be functions.

The **composition** of f with g, written $g \circ f : A \longrightarrow C$, is a function defined by

$$g \circ f(a) \stackrel{\text{def}}{=} g(f(a))$$

for every element $a \in A$. In Haskell notation, we would write

```
(g.f) a = g (f a)
```

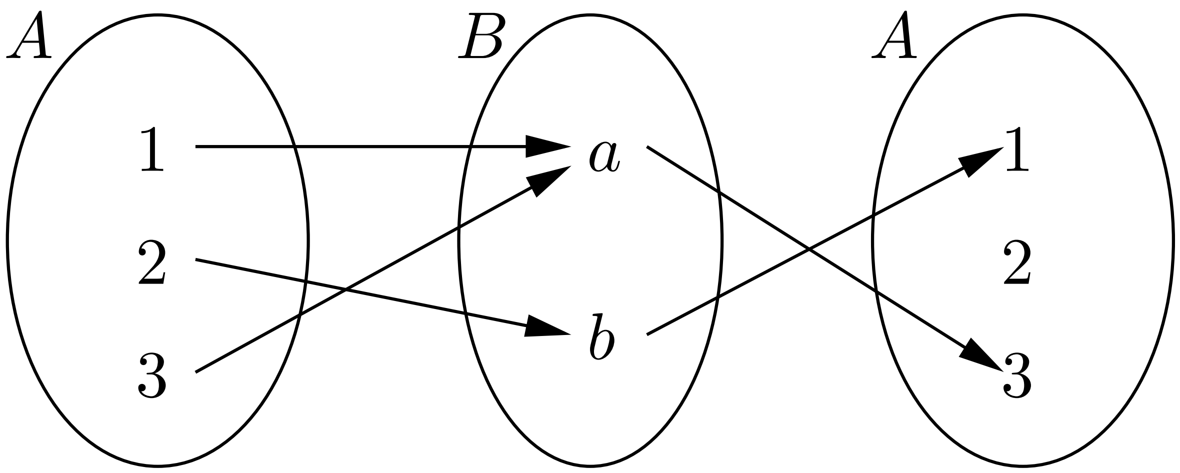
It is easy to check that $g \circ f$ is indeed a function.

The co-domain of f **must** be the same as the domain of g.

Example

Let $A = \{1, 2, 3\}$, $B = \{a, b, c\}$, $f = \{(1, a), (2, b), (3, a)\}$ and $g = \{(a, 3), (b, 1)\}$.

Then $g \circ f = \{(1, 3), (2, 1), (3, 3)\}$.



Associativity

Let $f : A \to B$, $g : B \to C$ and $h : C \to D$ be arbitrary functions. Then $h \circ (g \circ f) = (h \circ g) \circ f$.

Proof Let $a \in A$ be arbitrary. Then

$$\begin{aligned}
(h \circ (g \circ f))(a) &= h((g \circ f)(a)) \\
&= h(g(f(a))) \\
&= (h \circ g)(f(a)) \\
&= ((h \circ g) \circ f)(a)
\end{aligned}$$

Proposition

Let $f : A \to B$ and $g : B \to C$ be arbitrary bijections. Then $g \circ f$ is a bijection.

Proof It is enough to show that

1. if f, g are onto then so is $g \circ f$;
2. if f, g are one-to-one then so is $g \circ f$.

Assume f and g are onto. Let $c \in C$. Since g is onto, we can find $b \in B$ such that $g(b) = c$. Since f is onto, we can find $a \in A$ such that $f(a) = b$. Hence $g \circ f(a) = g(f(a)) = g(b) = c$.

Assume f and g are one-to-one. Let $a_1, a_2 \in A$ such that $g \circ f(a_1) = g \circ f(a_2)$. Then $g(f(a_1)) = g(f(a_2))$. Since g is one-to-one, $f(a_1) = f(a_2)$. Since f is also one-to-one, $a_1 = a_2$.

Identity

Let A be a set. Define the **identity** function on A, denoted $\mathsf{id}_A : A \longrightarrow A$, by $\mathsf{id}_A(a) = a$ for all $a \in A$.

In Haskell, we would declare the function

```
id :: A -> A
id x = x
```

Inverse

Let $f : A \longrightarrow B$ be an arbitrary function. The function $g : B \longrightarrow A$ is an **inverse** of f if and only if

$$\begin{aligned} \text{for all } a \in A, \quad g(f(a)) &= a \\ \text{for all } b \in B, \quad f(g(b)) &= b \end{aligned}$$

Another way of stating the same property is that $g \circ f = \mathsf{id}_A$ and $f \circ g = \mathsf{id}_B$.

Examples

1. The inverse relation of the function $f : \{1, 2\} \to \{1, 2\}$ defined by $f(1) = f(2) = 1$ is not a function.

   When an inverse function exists, it corresponds to the inverse relation.

2. Let $A = \{a, b, c\}$, $B = \{1, 2, 3\}$,
   $f = \{(a, 1), (b, 3), (c, 2)\}$ and $g = \{(1, a), (2, c), (3, b)\}$.
   Then g is an inverse of f.

Proposition

Let $f : A \longrightarrow B$ be a bijection, and define $f^{-1} : B \longrightarrow A$ by

$$f^{-1}(b) = a \text{ whenever } f(a) = b$$

The relation f^{-1} is a well-defined function.

It is **the** inverse of f (as shown in next proposition).

Proof

Let $b \in B$. Since f is onto, there is an a such that $f(a) = b$. Since f is one-to-one, this a is unique. Thus f^{-1} is a function. It satisfies the conditions for being **an** inverse of f.

Proposition

Let $f : A \to B$. If f has an inverse g, then f must be a bijection and the inverse is unique.

Proof To show that f is onto, let $b \in B$. Since $f(g(b)) = b$, it follows that b must be in the image of f.

To show that f is one-to-one, let $a_1, a_2 \in A$. Suppose $f(a_1) = f(a_2)$. Then $g(f(a_1)) = g(f(a_2))$. Since $g \circ f = \mathsf{id}_A$, it follows that $a_1 = a_2$.

To show that the inverse is unique, suppose that g, g' are both inverses of f. Let $b \in B$. Then $f(g(b)) = f(g'(b))$ since g, g' are inverses. Hence $g(b) = g'(b)$ since f is one-to-one.

Example

Consider the function $f : \mathcal{N} \to \mathcal{N}$ defined by

$$\begin{aligned} f(x) &= x + 1, \qquad x \text{ odd} \\ &= x - 1, \qquad x \text{ even} \end{aligned}$$

It is easy to check that $(f \circ f)(x) = x$, considering the cases when x is odd and even separately. Therefore f is its own inverse, and we can deduce that it is a bijection.

Cardinality of Sets

Definition

For any sets A, B, define $A \sim B$ if and only if there is bijection from A to B.

Proposition Relation $\sim$ is reflexive, symmetric and transitive.

Proof Relation $\sim$ is reflexive , as $\mathsf{id}_A : A \longrightarrow A$ is a bijection.

To show that it is symmetric, $A \sim B$ implies that there is a bijection $f : A \longrightarrow B$. By previous proposition, it follows that f has an inverse f^{-1} which is also a bijection. Hence $B \sim A$.

The fact that the relation $\sim$ is transitive follows from previous proposition.

Example

Let A, B, C be arbitrary sets, and consider the products $(A \times B) \times C$ and $A \times (B \times C)$.

There is a natural bijection $f : (A \times B) \times C \to A \times (B \times C)$:

$$f : ((a, b), c) \mapsto (a, (b, c))$$

Also define the function $g : A \times (B \times C) \to (A \times B) \times C$:

$$g : (a, (b, c)) \mapsto ((a, b), c)$$

Function g is the inverse of f.

Example

Consider the set Even of even natural numbers.

There is a bijection between Even and $\mathcal{N}$ given by $f(n) = 2n$.

Not **all** functions from Even to $\mathcal{N}$ are bijections.

The function $g : \mathsf{Even} \rightarrow \mathcal{N}$ given by $g(n) = n$ is one-to-one but not onto.

To show that $\mathsf{Even} \sim \mathcal{N}$, it is enough to show the **existence** of such a bijection.

Example

Recall that the cardinality of a **finite** set is the number of elements in that set. Let $|A| = n$. There is a bijection

$$c_A : \{1, 2, \ldots, n\} \longrightarrow A.$$

Let A and B be two finite sets. If A and B have the same number of elements, we can define a bijection $f : A \longrightarrow B$ by

$$f(a) = (c_B \circ c_A^{-1})(a).$$

Two finite sets have the same number of elements if and only if there is a bijection between then.

Cardinality

Given two **arbitrary** sets A and B, then A has the same **cardinality** as B, written $|A| = |B|$, if and only if $A \sim B$.

Notice that this definition is for **all** sets.

Exploring Infinite Sets

The set of natural numbers is one of the simplest infinite sets. We can build it up by stages:

$$0$$

$$0, 1$$

$$0, 1, 2$$

$$\ldots$$

Infinite sets which can be built up in finite portions by stages are particularly nice for computing.

Countable

A set A is **countable** if and only if A is finite or $A \sim \mathcal{N}$.

The elements of a countable set A can be listed as a finite or infinite sequence of distinct terms: $A = \{a_1, a_2, a_3, \ldots\}$.

Example

The integers $\mathcal{Z}$ are countable, since they can be listed as:

$$0, -1, 1, -2, 2, -3, 3, \ldots$$

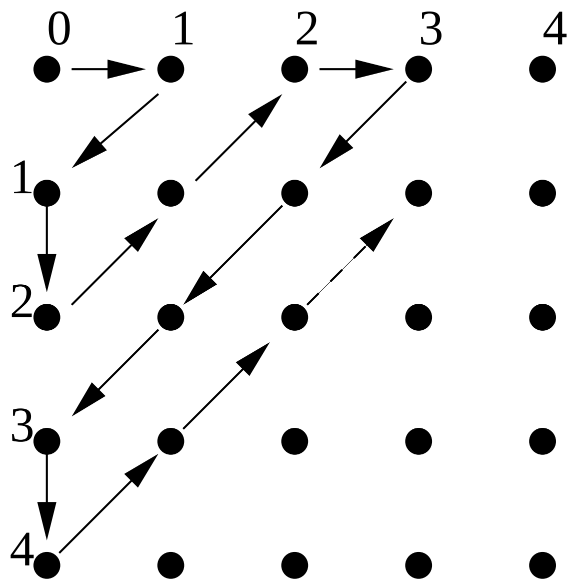
This 'counting' bijection $g : \mathcal{Z} \rightarrow \mathcal{N}$ is defined formally by

$$\begin{aligned} g(x) &= 2x, \quad x \geq 0 \\ &= -1 - 2x, \quad x < 0 \end{aligned}$$

The set of integers $\mathcal{Z}$ is like two copies of the natural numbers.

Example

The set $\mathcal{N}^2$ is countable:



Comment

The rational numbers are also countable.

Uncountable Sets

Cantor showed that there are **uncountable** sets.

An important example is the set of reals $\mathcal{R}$.

Another example is the power set $\mathcal{P}(\mathcal{N})$.

We cannot manipulate reals in the way we can natural numbers.

Instead, we use approximations: for example, the floating point decimals of type `Float` in Haskell.

For more information, see Truss, section 2.4.