Functions
A function $f$ from a set $A$ to a set $B$, written $f : A \rightarrow B$, is a relation $f \subseteq A \times B$ such that every element of $A$ is related to one element of $B$; in logical notation

1. $(a, b_1) \in f \land (a, b_2) \in f \Rightarrow b_1 = b_2$;

2. $\forall a \in A. \exists b \in B. (a, b) \in f$.

The set $A$ is called the domain and $B$ the co-domain of $f$.

If $a \in A$, then $f(a)$ denotes the unique $b \in B$ st. $(a, b) \in f$. 
If the domain $A$ is the $n$-ary product $A_1 \times \ldots \times A_n$, then we often write $f(a_1, \ldots, a_n)$ instead of $f((a_1, \ldots, a_n))$.

The intended meaning should be clear from the context.

Recall the difference between the following two Haskell functions:

\[
\begin{align*}
&\text{f :: A -> B -> C -> D} \\
&\text{f :: (A,B,C) -> D}
\end{align*}
\]

Our definition of function is not curried.
Let $f : A \to B$. For any $X \subseteq A$, define the image of $X$ under $f$ to be

$$f[X] \overset{\text{def}}{=} \{ f(a) : a \in X \}$$

The set $f[A]$ is called the image set of $f$. 
Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$.

Let $f \subseteq A \times B$ be defined by $f = \{(1, a), (2, b), (3, a)\}$.

The image set of $f$ is $\{a, b\}$.

The image of $\{1, 3\}$ under $f$ is $\{a\}$. 
Example

Let \( A = \{1, 2, 3\} \) and \( B = \{a, b\} \). Let \( f \subseteq A \times B \) be defined by \( f = \{(1, a), (1, b), (2, b), (3, a)\} \).

This \( f \) is not a well-defined function.
The following are examples of functions with infinite domains and co-domains:

1. the function \( f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \) defined by \( f(x, y) = x + y \);
2. the function \( f : \mathbb{N} \rightarrow \mathbb{N} \) defined by \( f(x) = x^2 \);
3. the function \( f : \mathbb{R} \rightarrow \mathbb{R} \) defined by \( f(x) = x + 3 \).

The binary relation \( R \) on the reals defined by \( x R y \) if and only if \( x = y^2 \) is not a function.
Let $A \rightarrow B$ denote the set of all functions from $A$ to $B$, where $A$ and $B$ are finite sets. If $|A| = m$ and $|B| = n$, then $|A \rightarrow B| = n^m$.

**Sketch proof**

For each element of $A$, there are $n$ independent ways of mapping it to $B$.

You do not need to remember this proof.
Partial Functions

A **partial** function $f$ from a set $A$ to a set $B$, written $f : A \rightarrow B$, is a relation $f \subseteq A \times B$ such that just some elements of $A$ are related to **unique** elements of $B$:

$$(a, b_1) \in f \land (a, b_2) \in f \Rightarrow b_1 = b_2.$$ 

The partial function $f$ is regarded as **undefined** on those elements which do not have an image under $f$.

We call this undefined value $\bot$ (pronounced *bottom*).

A partial function from $A$ to $B$ is a function from $A$ to $(B + \{\bot\})$. 
Examples of Partial Functions

1. Haskell functions which return run-time errors on some (or all) arguments.

2. The relation \( R = \{(1, a), (3, a)\} \subseteq \{1, 2, 3\} \times \{a, b\} \):

   ![Diagram of set A mapping to set B]

   Not every element in \( A \) maps to an element in \( B \).

3. The binary relation \( R \) on \( \mathcal{R} \) defined by \( x R y \text{ iff } \sqrt{x} = y \). It is not defined when \( x \) is negative.
Properties of Functions

Let $f : A \to B$ be a function.

1. $f$ is **onto** if and only if every element of $B$ is in the image of $f$:

   $$\forall b \in B. \exists a \in A. f(a) = b.$$ 

2. $f$ is **one-to-one** if and only if for each $b \in B$ there is at most one $a \in A$ with $f(a) = b$:

   $$\forall a, a' \in A. f(a) = f(a') \text{ implies } a = a'.$$

3. $f$ is a **bijection** iff $f$ is both one-to-one and onto.
Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. The function $f = \{(1, a), (2, b), (3, a)\}$ is onto, but not one-to-one:

We cannot define a one-to-one function from $A$ to $B$. There are too many elements in $A$ for them to map uniquely to $B$. 
Example

Let \( A = \{a, b\} \) and \( B = \{1, 2, 3\} \). The function \( f = \{(a, 3), (b, 1)\} \) is one-to-one, but not onto:

It is not possible to define an onto function from \( A \) to \( B \). There are not enough elements in \( A \) to map to all the elements of \( B \).
Let $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$. The function $f = \{(a, 1), (b, 3), (c, 2)\}$ is bijective:
The function $f$ on natural numbers defined by $f(x, y) = x + y$ is onto but not one-to-one.

To prove that $f$ is onto, take an arbitrary $n \in \mathbb{N}$. Then $f(n, 0) = n + 0 = n$.

To show that $f$ is not one-to-one, we need to produce a counter-example: that is, find $(m_1, m_2), (n_1, n_2)$ such that $(m_1, m_2) \neq (n_1, n_2)$, but $f(m_1, m_2) = f(n_1, n_2)$.

For example, $(1, 0)$ and $(0, 1)$. 
Examples

1. The function $f$ on natural numbers defined by $f(x) = x^2$ is one-to-one. The similar function $f$ on integers is not.

2. The function $f$ on integers defined by $f(x) = x + 1$ is onto. The similar function on natural numbers is not.

3. The function $f$ on the real numbers given by $f(x) = 4x + 3$ is a bijective function.

The proof is given in the notes.
The Pigeonhole Principle

Pigeonhole Principle

Let \( f : A \rightarrow B \) be a function, where \( A \) and \( B \) are finite. If \(|A| > |B|\), then \( f \) cannot be a one-to-one function.

Example

Let \( A = \{1, 2, 3\} \) and \( B = \{a, b\} \). A function \( f : A \rightarrow B \) cannot be one-to-one, since \( A \) is too big.

It is not possible to prove this property directly.

The pigeonhole principle states that we assume that the property is true.
Let $A$ and $B$ be finite sets, let $f : A \to B$ and let $X \subseteq A$. Then
\[ |f[X]| \leq |X|. \]

**Proof** Suppose for contradiction that $|f[X]| > |X|$. Define a function $p : f[X] \to X$ by
\[ p(b) = \text{some } a \in X \text{ such that } f(a) = b. \]
There is such an $a$ by definition of $f[X]$. We are placing the members of $f[X]$ in the pigeonholes $X$. By the pigeonhole principle, there is some $a \in X$ and $b, b' \in f[X]$ with $p(b) = p(b') = a$. But then $f(a) = b$ and $f(a) = b'$. Contradiction.
Proposition

Let $A$ and $B$ be finite sets, and let $f : A \to B$.

1. If $f$ is one-to-one, then $|A| \leq |B|$.

2. If $f$ is onto, then $|A| \geq |B|$.

3. If $f$ is a bijection, then $|A| = |B|$.

Proof

Part (a) is the contrapositive of the pigeonhole principle.

For (b), notice that if $f$ is onto then $f[A] = B$. Hence, $|f[A]| = |B|$. Also $|A| \geq |f[A]|$ by previous proposition. Therefore $|A| \geq |B|$ as required.

Part (c) follows from parts (a) and (b).
Let $A$, $B$ and $C$ be arbitrary sets, and let $f : A \rightarrow B$ and $g : B \rightarrow C$ be functions.

The **composition** of $f$ with $g$, written $g \circ f : A \rightarrow C$, is a function defined by

$$g \circ f (a) \overset{\text{def}}{=} g(f(a))$$

for every element $a \in A$. In Haskell notation, we would write

$$(g \cdot f) \; a = g \; (f \; a)$$

It is easy to check that $g \circ f$ is indeed a function.

The co-domain of $f$ **must** be the same as the domain of $g$. 
Let $A = \{1, 2, 3\}$, $B = \{a, b, c\}$, $f = \{(1, a), (2, b), (3, a)\}$ and $g = \{(a, 3), (b, 1)\}$.

Then $g \circ f = \{(1, 3), (2, 1), (3, 3)\}$. 
Let $f : A \rightarrow B$, $g : B \rightarrow C$ and $h : C \rightarrow D$ be arbitrary functions. Then $h \circ (g \circ f) = (h \circ g) \circ f$.

**Proof**  Let $a \in A$ be arbitrary. Then

\[
(h \circ (g \circ f))(a) = h((g \circ f)(a)) \\
= h(g(f(a))) \\
= (h \circ g)(f(a)) \\
= ((h \circ g) \circ f)(a)
\]
Proposition

Let $f : A \to B$ and $g : B \to C$ be arbitrary bijections. Then $g \circ f$ is a bijection.

Proof It is enough to show that

1. if $f, g$ are onto then so is $g \circ f$;
2. if $f, g$ are one-to-one then so is $g \circ f$.

Assume $f$ and $g$ are onto. Let $c \in C$. Since $g$ is onto, we can find $b \in B$ such that $g(b) = c$. Since $f$ is onto, we can find $a \in A$ such that $f(a) = b$. Hence $g \circ f(a) = g(f(a)) = g(b) = c$.

Assume $f$ and $g$ are one-to-one. Let $a_1, a_2 \in A$ such that $g \circ f(a_1) = g \circ f(a_2)$. Then $g(f(a_1)) = g(f(a_2))$. Since $g$ is one-to-one, $f(a_1) = f(a_2)$. Since $f$ is also one-to-one, $a_1 = a_2$. 
Identity

Let $A$ be a set. Define the identity function on $A$, denoted $\text{id}_A : A \rightarrow A$, by $\text{id}_A(a) = a$ for all $a \in A$.

In Haskell, we would declare the function

```
  id :: A -> A
  id x = x
```
Let $f : A \to B$ be an arbitrary function. The function $g : B \to A$ is an inverse of $f$ if and only if

for all $a \in A$, \hspace{1cm} g(f(a)) = a

for all $b \in B$, \hspace{1cm} f(g(b)) = b

Another way of stating the same property is that $g \circ f = \text{id}_A$

and $f \circ g = \text{id}_B$. 
Examples

1. The inverse relation of the function $f : \{1, 2\} \to \{1, 2\}$ defined by $f(1) = f(2) = 1$ is not a function. When an inverse function exists, it corresponds to the inverse relation.

2. Let $A = \{a, b, c\}$, $B = \{1, 2, 3\}$, $f = \{(a, 1), (b, 3), (c, 2)\}$ and $g = \{(1, a), (2, c), (3, b)\}$. Then $g$ is an inverse of $f$. 
Proposition

Let $f : A \to B$ be a bijection, and define $f^{-1} : B \to A$ by

$$f^{-1}(b) = a \text{ whenever } f(a) = b$$

The relation $f^{-1}$ is a well-defined function.

It is the inverse of $f$ (as shown in next proposition).

Proof

Let $b \in B$. Since $f$ is onto, there is an $a$ such that $f(a) = b$. Since $f$ is one-to-one, this $a$ is unique. Thus $f^{-1}$ is a function. It satisfies the conditions for being an inverse of $f$. 
Proposition

Let \( f : A \rightarrow B \). If \( f \) has an inverse \( g \), then \( f \) must be a bijection and the inverse is unique.

**Proof**  To show that \( f \) is onto, let \( b \in B \). Since \( f(g(b)) = b \), it follows that \( b \) must be in the image of \( f \).

To show that \( f \) is one-to-one, let \( a_1, a_2 \in A \). Suppose \( f(a_1) = f(a_2) \). Then \( g(f(a_1)) = g(f(a_2)) \). Since \( g \circ f = \text{id}_A \), it follows that \( a_1 = a_2 \).

To show that the inverse is unique, suppose that \( g, g' \) are both inverses of \( f \). Let \( b \in B \). Then \( f(g(b)) = f(g'(b)) \) since \( g, g' \) are inverses. Hence \( g(b) = g'(b) \) since \( f \) is one-to-one.
Consider the function $f : \mathbb{N} \rightarrow \mathbb{N}$ defined by

$$f(x) = \begin{cases} x + 1, & x \text{ odd} \\ x - 1, & x \text{ even} \end{cases}$$

It is easy to check that $(f \circ f)(x) = x$, considering the cases when $x$ is odd and even separately. Therefore $f$ is its own inverse, and we can deduce that it is a bijection.
**Cardinality of Sets**

**Definition**

For any sets $A$, $B$, define $A \sim B$ if and only if there is bijection from $A$ to $B$.

**Proposition** Relation $\sim$ is reflexive, symmetric and transitive.

**Proof** Relation $\sim$ is reflexive, as $\text{id}_A : A \rightarrow A$ is a bijection.

To show that it is symmetric, $A \sim B$ implies that there is a bijection $f : A \rightarrow B$. By previous proposition, it follows that $f$ has an inverse $f^{-1}$ which is also a bijection. Hence $B \sim A$.

The fact that the relation $\sim$ is transitive follows from previous proposition.
Let $A, B, C$ be arbitrary sets, and consider the products $(A \times B) \times C$ and $A \times (B \times C)$.

There is a natural bijection $f : (A \times B) \times C \rightarrow A \times (B \times C)$:

$$f : ((a, b), c) \mapsto (a, (b, c))$$

Also define the function $g : A \times (B \times C) \rightarrow (A \times B) \times C$:

$$g : (a, (b, c)) \mapsto ((a, b), c)$$

Function $g$ is the inverse of $f$. 

Example
Example

Consider the set $\text{Even}$ of even natural numbers.

There is a bijection between $\text{Even}$ and $\mathcal{N}$ given by $f(n) = 2n$.

Not all functions from $\text{Even}$ to $\mathcal{N}$ are bijections.

The function $g : \text{Even} \to \mathcal{N}$ given by $g(n) = n$ is one-to-one but not onto.

To show that $\text{Even} \sim \mathcal{N}$, it is enough to show the existence of such a bijection.
Recall that the cardinality of a finite set is the number of elements in that set. Let $|A| = n$. There is a bijection

$$c_A : \{1, 2, \ldots, n\} \rightarrow A.$$ 

Let $A$ and $B$ be two finite sets. If $A$ and $B$ have the same number of elements, we can define a bijection $f : A \rightarrow B$ by

$$f(a) = (c_B \circ c_A^{-1})(a).$$

Two finite sets have the same number of elements if and only if there is a bijection between them.
Cardinality

Given two *arbitrary* sets $A$ and $B$, then $A$ has the same *cardinality* as $B$, written $|A| = |B|$, if and only if $A \sim B$.

Notice that this definition is for *all* sets.
Exploring Infinite Sets

The set of natural numbers is one of the simplest infinite sets. We can build it up by stages:

0
0, 1
0, 1, 2
...

Infinite sets which can be built up in finite portions by stages are particularly nice for computing.
A set $A$ is **countable** if and only if $A$ is finite or $A \sim \mathbb{N}$.

The elements of a countable set $A$ can be listed as a finite or infinite sequence of distinct terms: $A = \{a_1, a_2, a_3, \ldots\}$. 
Example

The integers \( \mathbb{Z} \) are countable, since they can be listed as:

\[
0, -1, 1, -2, 2, -3, 3, \ldots
\]

This ‘counting’ bijection \( g : \mathbb{Z} \to \mathbb{N} \) is defined formally by

\[
g(x) = \begin{cases} 
2x, & x \geq 0 \\
-1 - 2x, & x < 0
\end{cases}
\]

The set of integers \( \mathbb{Z} \) is like two copies of the natural numbers.
Example

The set $\mathbb{N}^2$ is countable:

Comment

The rational numbers are also countable.
Uncountable Sets

Cantor showed that there are uncountable sets.

An important example is the set of reals $\mathbb{R}$.

Another example is the power set $\mathcal{P}(\mathbb{N})$.

We cannot manipulate reals in the way we can natural numbers.

Instead, we use approximations: for example, the floating point decimals of type Float in Haskell.

For more information, see Truss, section 2.4.