Mathematical Methods

for Computer Science

(Part 2)

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BASICS OF POWER SERIES

- Represent a function $f(x)$ by:

$$f(x) = \sum_{i=0}^{\infty} a_i x^i$$

for coefficients $a_i \in \mathbb{R}, i = 1, 2, \ldots$.
  - Called a *power series* because of the series of powers of the argument x
  - For example, $f(x) = (1 + x)^2 = 1 + 2x + x^2$ has $a_0 = 1, a_1 = 2, a_2 = 1, a_i = 0$ for $i > 2$
- But in general the series may be infinite *provided it converges*

What are the coefficients?

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

- Suppose the value of the function f is known at $x = 0$. Then we have straightaway, substituting $x = 0$

$$a_0 = f(0)$$

- Now differentiate $f(x)$ to get rid of the constant term:

$$f'(x) = a_1 + 2.a_2 x + 3.a_3 x^2 + 4.a_4 x^3 + \ldots$$

What are the coefficients? (2)

- Suppose the derivatives of the function f are known at $x = 0$ and set $x = 0$:

$$a_1 = f'(0)$$

- Differentiate again to get rid of the constant term:

$$f''(x) \equiv f^{(2)}(x) = 2.1.a_2 + 3.2.a_3 x + 4.3.a_4 x^2 + \ldots$$

- Set $x = 0$ and repeat the process:

$$a_2 = f^{(2)}(0)/2!, \ldots, a_n = f^{(n)}(0)/n!$$

for $n \geq 0$. More formally, we have .....

Maclaurin series

- Suppose $f(x)$ is differentiable infinitely many times and that it has a power series representation (*series expansion*) of the form $f(x) = \sum_{i=0}^{\infty} a_i x^i$, as above.
- Differentiating n times gives

$$f^{(n)}(x) = \sum_{i=n}^{\infty} a_i i(i-1)\ldots(i-n+1)x^{i-n}$$

- Setting $x = 0$, we have $f^{(n)}(0) = n!a_n$ because all terms but the first have x as a factor.

Maclaurin series (2)

- Hence we obtain Maclaurin's series:

$$f(x) = \sum_{i=0}^{\infty} f^{(i)}(0)\frac{x^i}{i!}$$

- *It is important to check the domain of convergence (set of valid values for x)*
- This rather sloppy argument will be tightened up later.

Example 1: $f(x) = (1+x)^3$

- $f(0) = 1$ so $a_0 = 1$
- $f'(x) = 3(1+x)^2$ so $f'(0) = 3$ and $a_1 = 3/1! = 3$
- $f''(x) = 3.2(1+x)$ so $f''(0) = 6$ and $a_2 = 6/2! = 3$
- $f'''(x) = 3.2.1$ so $f'''(0) = 6$ and $a_3 = 6/3! = 1$
- Higher derivatives are all 0 and so (as we know)

$$(1+x)^3 = 1 + 3x + 3x^2 + x^3$$

Example 2: $f(x) = (1-x)^{-1}$

- We probably know what the power series is for this function – namely the geometric series in x, in which all $a_i = 1$.
- $f(0) = 1$, so far so good!
- $f'(x) = -(1-x)^{-2}(-1) = (1-x)^{-2}$
- so $f'(0) = 1$

Example 2 (2)

- Differentiating repeatedly,

$$
\begin{aligned}
f^{(n)}(x) &= (-1)(-2)\dots(-n)(1-x)^{-(n+1)}(-1)^n \\
&= n!(1-x)^{-(n+1)}
\end{aligned}
$$

- so $a_n = f^{(n)}(0)/n! = n!(1)^{-(n+1)}/n! = 1$
- Thus

$$(1-x)^{-1} = \sum_{i=0}^{\infty} 1.x^i$$

provided this converges.

Example 3: $f(x) = \log_e(1+x)$

- $a_0 = f(0) = 0$ because $\log_e 1 = 0$ so no constant term
- $f'(x) = (1+x)^{-1}$ so $a_1 = 1$
- $f^{(n)}(x) = (-1)^{n-1}(n-1)!(1+x)^{-n}$ so

$$a_n = (-1)^{n-1}/n$$

- Therefore

$$\log_e(1+x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

provided this converges.

A look at convergence

- What about $\log_e 2$?
  - Is it true that

$$\log_e 2 = \sum_{n=1}^{\infty} (-1)^{n-1}/n \quad ?$$

  - i.e. is $\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$?
- It depends how you ‘add up the terms’, i.e. in what sequence
- *conditionally convergent series*
- Try it . . . how accurate is your result after 100,000 terms?

A look at convergence (2)

- What about when $x = -1$ giving $\log_e 0$?
  - Is

$$\log_e 0 = -\sum_{n=1}^{\infty} 1/n \quad ?$$

  - i.e. $-(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots)$?
- Well, we know that $\log_e 0 = -\infty$, so expect this series to diverge; *very slowly*, because $\log x$ diverges very slowly as $x \to \infty$ or 0.
  - What do think $\sum_{n=1}^{1000000} 1/n$ is?
- More about this later

Taylor series

- A more general result is:

$$f(a+h) = f(a)+$$

$$\frac{h}{1!}f^{(1)}(a)+\ldots+\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)+\frac{h^n}{n!}f^{(n)}(a+\theta h)$$

where $\theta \in (0,1)$

- Also called the n*th Mean Value Theorem*
- It is a nice result since it puts a bound on the error arising from using a truncated series

Power series solution of ODEs

- Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = ky$$

for constant k, given that $y = 1$ when $x = 0$.

- Try the *series solution*

$$y = \sum_{i=0}^{\infty} a_i x^i$$

- Find the coefficients a_i by differentiating term by term, to obtain the *identity*, for $i \geq 0$:

Matching coefficients

$$\sum_{i=1}^{\infty} a_i i x^{i-1} \equiv \sum_{i=0}^{\infty} k a_i x^i \equiv \sum_{i=1}^{\infty} k a_{i-1} x^{i-1}$$

- Comparing coefficients of x^{i-1} for $i \geq 1$

$$i a_i = k a_{i-1} \qquad \text{hence}$$

$$a_i = \frac{k}{i} a_{i-1} = \frac{k}{i} \cdot \frac{k}{i-1} a_{i-2} = \ldots = \frac{k^i}{i!} a_0$$

- When $x = 0, y = a_0$ so $a_0 = 1$ by the boundary condition. Thus

$$y = \sum_{i=0}^{\infty} \frac{(kx)^i}{i!} = e^{kx}$$

Answer ...

$$\sum_{n=1}^{1000000} 1/n = 14.3927$$

COMPLEX NUMBERS

A short history number systems:

- $\mathbb{N}$: for counting, not closed under subtraction;
- $\mathbb{Z}$: $\mathbb{N}$ with 0 and negative numbers, not closed under division;
- $\mathbb{Q}$: fractions, closed under arithmetic operations but can't represent the solution of non-linear equations, e.g. $\sqrt{2}$;
- $\mathbb{R}$: can do this for quadratic equations with real roots and some higher-order equations — *but not all*.
  - More on the reals when we consider limits

Missing numbers

- The first entity we cannot describe is the solution to the equation

$$x^2 + 1 = 0$$

  i.e. $\sqrt{-1}$ which we will call $i \equiv \sqrt{-1}$

- There is no way of squeezing this into $\mathbb{R}$ – it cannot be compared with a real number (contrast $\sqrt{2}$ or π which we can compare with rationals and get arbitrarily accurate approximations)
- So we treat i as an **imaginary** number, ‘orthogonal’ to the reals, and consider $\mathbb{R} \cup \{i\}$

Useful facts

From the definition of i we have

- $i^2 = -1; \quad i^3 = i^2 i = -i; \quad i^4 = (i^2)^2 = (-1)^2 = 1$
- more generally, for all $n \in \mathbb{N}$,

$$i^{2n} = (i^2)^n = (-1)^n; \quad i^{2n+1} = i^{2n} i = (-1)^n i$$

- $i^{-1} = \frac{1}{i} = \frac{i}{i^2} = -i$
- $i^{-2n} = \frac{1}{i^{2n}} = \frac{1}{(-1)^n} = (-1)^n$;
  $i^{-(2n+1)} = i^{-2n} i^{-1} = (-1)^{n+1} i$ for all $n \in \mathbb{N}$
- $i^0 = 1$

Closure under arithmetic operators

- Closing $\mathbb{R} \cup \{i\}$ under the 'arithmetic operators' gives the *complex numbers* $\mathbb{C}$.
- If $z_1, z_2 \in \mathbb{C}$, then $z_1 + z_2 \in \mathbb{C}$, $z_1 - z_2 \in \mathbb{C}$, $z_1 \times z_2 \in \mathbb{C}$ and $z_1/z_2 \in \mathbb{C}$.
- Any complex number can be written in the form $z = x + iy$ for $x, y \in \mathbb{R}$. We write:
  - $\Re(z) = x$, the **real part** of z
  - $\Im(z) = y$, the **imaginary part** of z

Arithmetic operators

- Arithmetic operations on $\mathbb{C}$ are defined *symbolically*
  - as if i were just a variable name
  - but replacing i^2 by -1
- Hence any operation results in a real constant (*real part*) added to a real constant (*imaginary part*) multiplied by i
- The precise definitions defined next *must* (and do) reduce to the well known operations on $\mathbb{R}$ when the imaginary parts of their operands are zero.

Addition

Definition: If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are complex numbers, then

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

and

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$$

- same as ‘adding brackets and collecting terms’
- addition is associative and commutative, because it is on real numbers (exercise)

Multiplication

Definition: If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are complex numbers, then

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$$

- same as ‘multiplying brackets and collecting terms’ *but also using the fact that* $i^2 = -1$
- multiplication is associative and commutative, because it is on real numbers (slightly harder exercise)

Complex conjugate

Definition: The **Complex conjugate** of a complex number $z = x + iy$ is $\overline{z} = x - iy$.

- $\Re\overline{z} = \Re z$
- $\Im\overline{z} = -\Im z$
- $z + \overline{z} = 2x = 2\Re z \in \mathbb{R}$
- $z - \overline{z} = 2iy = 2i\Im z$ which is purely imaginary
- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

Conjugate of a product

The conjugate of a product is the product of the conjugates:

$$\overline{z_1 z_2} = \overline{z_1}\;\overline{z_2}$$

- either by noting that the conjugate operation simply changes every occurrence of i to $-i$;
- or since

$$\begin{aligned}(x_1 + iy_1)(x_2 + iy_2) &= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)\\(x_1 - iy_1)(x_2 - iy_2) &= (x_1x_2 - y_1y_2) - i(x_1y_2 + y_1x_2)\end{aligned}$$

  which are conjugates

Modulus

Definition: The **modulus** or **absolute value** of z is $|z| = \sqrt{z\overline{z}}$.

- $z\overline{z} = (x+iy)(x-iy) = x^2 + y^2 \quad \in \mathbb{R}$
- Notice that the term 'absolute value' is the same as defined for real numbers when $\Im z = 0$, viz. $|x|$.
- $|z_1 z_2| = |z_1|\,|z_2|$ because

$$|z_1 z_2|^2 = z_1 z_2 \overline{z_1 z_2} = z_1 z_2 \overline{z_1}\,\overline{z_2} = z_1 \overline{z_1} z_2 \overline{z_2} = |z_1|^2 |z_2|^2$$

Reciprocal and division

- If $z = x + iy$, its reciprocal is

$$\frac{1}{z} = \frac{\overline{z}}{z\overline{z}} = \frac{\overline{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2}$$

- This can be written $z^{-1} = |z|^{-2}\overline{z}$, using only the *complex* operators multiply and add (but also real division which we already know).
- Complex division is now defined by $z_1/z_2 = z_1 \times z_2^{-1}$

Example

Calculate as a complex number

$$\frac{3+2i}{7-3i}$$

- Solution:

$$\begin{aligned}\frac{3+2i}{7-3i} &= \frac{(3+2i)(7+3i)}{(7-3i)(7+3i)} \\ &= \frac{15+23i}{49+9} \\ &= \frac{15}{58}+\frac{23}{58}i\end{aligned}$$

Uses

- This defines the complex numbers rigorously, consistent with the reals. *But why bother?*
- Lots of reasons!
  - The theory of complex numbers, complex variables and functions of a complex variable is very deep, with far-reaching results.
  - Often a 'real' problem can be solved by mapping it into complex space, deriving a solution, and mapping back again: a direct solution may not be possible.

Fundamental theorem of Algebra

- It can be shown that any polynomial equation of the form

$$1 + a_1 z + a_2 z^2 + \ldots + a_n z^n = 0$$

has n complex solutions (some of which might be coincident, e.g. for $z^2 = 0$).

- So we know that if we need a solution to such an equation, it *is* worth looking!
- Contrast in real space where we might try to locate a root of an equation with no real solutions.

Geometrical interpretation

- A complex number $z = x + iy$ is equivalent to the pair of real values (x, y), i.e. there is a 1-1 correspondence (bijective mapping) between $\mathbb{C}$ and $\mathbb{R} \times \mathbb{R}$
- *Thus each complex number is uniquely represented by a point in two dimensional space, i.e. has coordinates with respect to two axes.*
- The *distance* between two points z_1, z_2 is the *modulus* $|z_1 - z_2|$
- This two-dimensional space is called the **Argand diagram**.

Argand diagram

A point z can be represented

- in *Cartesian coordinates* by $z = x + iy$
- or in *polar coordinates* by $z = r(\cos\theta + i\sin\theta)$ where $|z|^2 = r^2(\sin^2\theta + \cos^2\theta) = r^2$, so $|z| = r$.
- Clearly $x = r\cos\theta$ and $y = r\sin\theta$
- We write $\text{Arg}\, z = \theta$ — the **argument** of z
- *Draw this for yourselves* and update the diagram as we go .....

Representation as vectors

- The addition rule is *exactly* the same as you had for vectors.
- Add the corresponding components:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

- Similarly with multiplication by a real / scalar – as complex numbers we get:

$$\lambda(x + iy) = \lambda x + i\lambda y \sim (\lambda x, \lambda y)$$

- Many two dimensional vector problems are solved using a complex number representation.

Products in the Argand diagram

- Geometrically, the definition of a product doesn't mean very much!
- But if we work in polar form we will see that if $z = z_1 z_2$, then
  - The modulus of z is the *product* of the moduli of z_1 and z_2 — as we would expect;
  - The argument of z is the *sum* of the arguments of z_1 and z_2.

DeMoivre's theorem

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

$$z_1 z_2 = r_1 r_2 (\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$$

- The proof is very easy. By definition of multiplication,

$$z_1 z_2 = r_1 r_2 \times$$

$$(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2))$$

- the result now follows by standard trigonometrical identities.

Back to the Argand diagram

So the product of the complex numbers z_1 and z_2 is identified graphically as that point z having:

- Arg z = Argz_1+Argz_2, i.e. the first point's polar angle rotates by an amount equal to the polar angle of he second point – this gives the *direction* of the result;
- $|z| = |z_1||z_2|$, i.e. the modulus of z, or distance along the now-known direction, is the product of the moduli of the two points.

Example

Multiply $3+3i$ by $(1+i)^3$

- Could expand $(1+i)^3$ and multiply by $3+3i$
- Alternatively, in polar form (using degrees),

$$\begin{aligned}(1+i)^3 &= [2^{1/2}(\cos 45 + i\sin 45)]^3 \\ &= 2^{3/2}(\cos 135 + i\sin 135)\end{aligned}$$

  by DeMoivre's theorem.
- $3+3i = 18^{1/2}(\cos 45 + i\sin 45)$ and so the result is

$$18^{1/2}2^{3/2}(\cos 180 + i\sin 180) = -12$$

Example(2)

- Geometrically, we just observe that the Arg of the second number is 3 times that of $1 + i$, i.e. $3 \times \pi/4$ (or 3×45 in degrees). The first number has the same Arg, so the Arg of the result is π or 180 degrees.
- The moduli of the numbers multiplied are $\sqrt{18}$ and $\sqrt{2^3}$, so the product has modulus 12.
- The result is therefore -12.

Triangle inequality

$$\forall z_1, z_2 \in \mathbb{C}, \quad |z_1 + z_2| \leq |z_1| + |z_2|$$

An alternative form, with $w_1 = z_1$ and $w_2 = z_1 + z_2$ is $|w_2| - |w_1| \leq |w_2 - w_1|$ and, switching w_1, w_2, $|w_1| - |w_2| \leq |w_2 - w_1|$. Thus, relabelling back to z_1, z_2:

$$\forall z_1, z_2 \in \mathbb{C}, \quad \big|\, |z_1| - |z_2| \,\big| \leq |z_2 - z_1|$$

- In the Argand diagram, this just says that: "In the triangle with vertices at O, Z_1, Z_2, the length of side $Z_1 Z_2$ is not less than the difference between the lengths of the other two sides"

Proof

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then

- The square of the left hand side is:

$$(x_1+x_2)^2+(y_1+y_2)^2 = |z_1|^2+|z_2|^2+2(x_1x_2+y_1y_2)$$

- The square of the right hand side is:

$$|z_1|^2 + |z_2|^2 + 2|z_1||z_2|$$

- So it is required to prove $x_1x_2 + y_1y_2 \leq |z_1||z_2|$.

Proof (2)

- You know this is true, since in vector notation $\vec{v}_1.\vec{v}_2 \leq |\vec{v}_1||\vec{v}_2|$.
- Otherwise, square and multiply out to require:

$$\begin{aligned} x_1^2x_2^2 + y_1^2y_2^2 + 2x_1x_2y_1y_2 &\leq x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + y_1^2x_2^2 \\ \text{i.e.}\quad 0 &\leq (x_1y_2 - y_1x_2)^2 \end{aligned}$$

  as required.
- The Argand diagram geometrical argument is usually considered an acceptable proof of the triangle inequality.

Complex power series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots$$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots$$

- *Same expansions hold in $\mathbb{C}$, e.g. because these functions are differentiable in $\mathbb{C}$ and Maclaurin's series applies.*

Euler's formula

Put $z = i\theta$ in the exponential series, for $\theta \in \mathbb{R}$:

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + i^2\frac{\theta^2}{2!} + i^3\frac{\theta^3}{3!} + i^4\frac{\theta^4}{4!} + \dots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \dots \\ &= \cos\theta + i\sin\theta \end{aligned}$$

- The polar form of a complex number may be written

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

and DeMoivre's theorem follows immediately.

More general form

- A more general form of Euler's formula is

$$z = re^{i(\theta+2n\pi)} \quad \text{for any } n \in \mathbb{Z}$$

since $e^{i2n\pi} = \cos 2n\pi + i \sin 2n\pi = 1$

- In terms of the Argand diagram, the points $e^{i(\theta+2n\pi)}$, $i = 1, 2, \ldots$ lie on top of each other, each corresponding to one more revolution (through 2π).
- The complex conjugate of $e^{i\theta}$ is $e^{-i\theta} = \cos\theta - i\sin\theta$ and so $\cos\theta = (e^{i\theta} + e^{-i\theta})/2, \ \ \sin\theta = (e^{i\theta} - e^{-i\theta})/2i$

nth roots of unity

Consider the equation $z^n = 1$ for $n \in \mathbb{N}$

- One root is $z = 1$, but by the Fundamental Theorem of Algebra, there are n altogether.
- Write this equation as

  $$z^n = e^{2k\pi i}$$

  for $k = 0, 1, \ldots$
- Then the solutions are $z = e^{2k\pi i/n}$ for $k = 0, 1, 2, \ldots, n-1$
- Note that the solutions repeat when $k = n, n+1, \ldots$

Example: cube roots of unity

- The 3rd roots of 1 are $z = e^{2k\pi i/3}$ for $k = 0, 1, 2,$ i.e. $1, e^{2\pi i/3}, e^{4\pi i/3}$.
- These simplify to

$$\begin{aligned} & 1 \\ \cos 2\pi/3 + i \sin 2\pi/3 &= (-1 + \sqrt{3}i)/2 \\ \cos 4\pi/3 + i \sin 4\pi/3 &= (-1 - \sqrt{3}i)/2 \end{aligned}$$

- Try cubing each solution directly ... and then do the 8th roots similarly!

Solution of $z^n = a + ib$

- These equations are solved (almost) the same way:
- Let $a + ib = re^{i\phi}$ in polar form. Then, for $k = 0, 1, \ldots, n-1$,

$$z^n = (a+ib)e^{2\pi k i} = re^{(\phi+2\pi k)i}$$

and so $$z = r^{\frac{1}{n}} e^{\frac{(\phi+2\pi k)}{n} i}$$

- E.g. cube roots of $1 - i$ ($r = \sqrt{2}, \phi = -\pi/4$) are: $2^{\frac{1}{6}}(\cos \pi/12 - i \sin \pi/12)$, $\quad 2^{\frac{1}{6}}(\cos 7\pi/12 + i \sin 7\pi/12)$ and $2^{\frac{1}{6}}(\cos 5\pi/4 + i \sin 5\pi/4) = -2^{-1/3}(1+i)$.

Multiple angle formulas

How can we calculate $\cos n\theta$ in terms of $\cos\theta$ and $\sin\theta$?

- Use DeMoivre's theorem to expand $e^{in\theta}$ and equate real and imaginary parts: e.g. for n=5, by the binomial theorem,

$$\begin{aligned}
&(\cos\theta + i\sin\theta)^5 \\
&= \cos^5\theta + i5\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta \\
&\quad -i10\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta
\end{aligned}$$

Multiple angle formulas (2)

- Comparing real and imaginary parts now gives:

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$

and

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$

Conversely ....

How can we calculate $\cos^n \theta$ in terms of $\cos m\theta$ and $\sin m\theta$ for $m \in \mathbb{N}$?

- Let $z = e^{i\theta}$ so that $z + z^{-1} = z + \overline{z} = 2\cos\theta$
- Similarly, $z^m + z^{-m} = 2\cos m\theta$ by DeMoivre's theorem.
- Hence by the binomial theorem, e.g. for $n = 5$,

$$\begin{aligned}(z + z^{-1})^5 &= (z^5 + z^{-5}) + 5(z^3 + z^{-3}) + 10(z + z^{-1}) \\ 2^5 \cos^5\theta &= 2(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)\end{aligned}$$

- Similarly, $z - z^{-1} = 2i\sin\theta$ gives $\sin^n\theta$

What happens when n is even?

- You get an extra term in the binomial expansion, which is *constant*.
- E.g. for $n = 6$:

$$
\begin{aligned}
(z + z^{-1})^6 &= (z^6 + z^{-6}) + 6(z^4 + z^{-4}) + 15(z^2 + z^{-2}) \\
2^6 \cos^6\theta &= 2(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)
\end{aligned}
$$

and so

$$
\cos^6\theta = \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)
$$

Summation of series

Some series with sines and cosines can be summed similarly, e.g.

$$C = \sum_{k=0}^{n} a^k \cos k\theta$$

- Let $S = \sum_{k=1}^{n} a^k \sin k\theta$. Then,

$$C + iS = \sum_{k=0}^{n} a^k e^{ik\theta} = \frac{1 - (ae^{i\theta})^{n+1}}{1 - ae^{i\theta}}$$

Summation of series (2)

- Hence

$$C + iS = \frac{(1-(ae^{i\theta})^{n+1})(1-ae^{-i\theta})}{(1-ae^{i\theta})(1-ae^{-i\theta})}$$

$$= \frac{1 - ae^{-i\theta} - a^{n+1}e^{i(n+1)\theta} + a^{n+2}e^{in\theta}}{1 - 2a\cos\theta + a^2}$$

Summation of series (3)

- Equating real and imaginary parts, the cosine series is:

$$C = \frac{1 - a\cos\theta - a^{n+1}\cos(n+1)\theta + a^{n+2}\cos n\theta}{1 - 2a\cos\theta + a^2}$$

- and the sine series is:

$$S = \frac{a\sin\theta - a^{n+1}\sin(n+1)\theta + a^{n+2}\sin n\theta}{1 - 2a\cos\theta + a^2}$$

Integrals

How about
$C = \int_0^x e^{a\theta} \cos b\theta d\theta, \qquad S = \int_0^x e^{a\theta} \sin b\theta d\theta$?

- Could do with reduction formulae if a or b is an integer, but .....

$$\begin{aligned} C + iS &= \int_0^x e^{(a+ib)\theta} d\theta \\ &= \frac{e^{(a+ib)x} - 1}{a+ib} = \frac{(e^{ax}e^{ibx} - 1)(a - ib)}{a^2 + b^2} \\ &= \frac{(e^{ax}\cos bx - 1 + ie^{ax}\sin bx)(a - ib)}{a^2 + b^2} \end{aligned}$$

Integrals (2)

- Result is therefore $C + iS =$

$$\frac{e^{ax}(a\cos bx + b\sin bx) - a + i(e^{ax}(a\sin bx - b\cos bx) + b)}{a^2+b^2}$$

- and so we get:

$$C = \frac{e^{ax}(a\cos bx + b\sin bx - a)}{a^2+b^2}$$

$$S = \frac{e^{ax}(a\sin bx - b\cos bx) + b}{a^2+b^2}$$

REAL NUMBERS

- Why do we need ‘real numbers’?
  - What’s wrong with just the rationals?
  - Aren’t fractions accurate enough – they have arbitrary precision?
- **Proposition**: $\sqrt{2}$ is not a rational number

Proof that $\sqrt{2}$ is not rational

Suppose $\exists p, q \in \mathbb{N}$ st. $\sqrt{2} = p/q$ and choose p, q st. they have no common factor.

Then $p^2 = 2q^2$ and so p^2 is even.

Therefore p is even (odd $\times$ odd is odd) and so p^2 is a multiple of 4.

Therefore $q^2 = p^2/2$ is even and hence so is q. But so is p, a contradiction.

Useful numbers

- So there are 'useful' numbers that are not rational.
- We call the 'useful' numbers the *real numbers* or just the *reals*, and denote them by $\mathbb{R}$.
- Clearly, $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$
- How many reals do you think there are, relative to the rationals?

How many real numbers?

- If r is irrational, then so is $r+q$ for any $q \in \mathbb{Q}$. (If $r+q=p \in \mathbb{Q}$, then $r=p-q \in \mathbb{Q}$, a contradiction.)
- so just $\sqrt{2}$ generates at least as many irrationals as there are rationals, and we haven't even considered the other arithmetic operations!
- in fact there are HUGELY many 'more' irrationals than rationals

Gaps in the real line

- Consider the real numbers in the closed interval [a] $[0,1] = \{x \mid 0 \leq x \leq 1\}$
- Number the rational numbers in [0,1] as

$$r_1, r_2, r_3, \ldots$$

- We can do this since the rationals are countable. Note that the ordering is not numerical, it can be anything.

[a] Similarly, an open interval has round brackets: $(0,1) = \{x \mid 0 < x < 1\}$ and there are 'mixed' intervals, open at one end, closed at the other, e.g. $(0,1]$.

The rationals' space

- Given any small value $\delta \in \mathbb{Q}$, put the closed interval

$$I_n = [r_n - \delta/2^n, r_n + \delta/2^n]$$

around the nth rational

- i.e. r_n is in the middle of an interval of length $\delta/2^{n-1}$

The rationals' space (2)

- The sum of the lengths of the intervals I_n is

$$\sum_{n=1}^{\infty} \delta/2^{n-1} = 2\delta$$

- This is because the sum is a geometric progression of the form

$$\sum_{i=0}^{\infty} x^i = 1/(1-x)$$

for $|x| < 1$; $x = 1/2$ in our case.

Continuum of numbers

- Some of the intervals overlap, but it doesn't matter, their combined length is less than 2δ *for any value of* δ, *however small*
- Their combined length is therefore 0 (why?) and so the rationals take up 'zero space'
- The rest of $[0, 1]$ is taken up with real, irrational numbers.
- We want the reals to form a 'continuum' so we can move smoothly along the real line without falling into gaps, e.g. to gradually approach the solution of an equation by iteration.

Digression on bounds

- The number $U \in \mathbb{R}$ is an **upper bound** of the set of real numbers X if $r \leq U$ for all $r \in X$. Similarly for a lower bound.
- A set of reals is **bounded above** if it has an upper bound, and **bounded below** if it has a lower bound.
- A set which is bounded above and below is just called **bounded**

Digression on bounds (2)

- The smallest element (if it exists) of a set of upper bounds is called the least upper bound or the **supremum** of a set X, abbreviated to $\sup(X)$
- The largest element (if it exists) of a set of lower bounds is called the greatest lower bound or the **infimum** of a set X, abbreviated to $\inf(X)$
- What are the $\sup$ and $\inf$ of $(0, 1)$?

Fundamental Axiom

- To get a continuum of reals, we make an assumption: the **Fundamental Axiom**:
  An increasing sequence $r_1, r_2, \ldots$ *of real numbers that is bounded above converges to a limit which is itself a real number*
  - Compare the definition of a **Complete Partial Ordering** (CPO) used in semantics of programming languages (maybe next year or in 'domain theory')
  - 'complete' means 'closed w.r.t. limits'.

Alternative definition

- An equivalent form of the Fundamental Axiom is:
  The set of upper bounds of any set of real numbers has a least member (assuming it is non-empty, of course)
- The proof of equivalence is non-trivial (but not too hard either): uses the ‘Chinese box theorem’
- Similarly for lower bounds

Decimal numbers

- What we know about is fractions and decimals!
  - Fractions are just rationals, so are also reals because $\mathbb{Q} \subset \mathbb{R}$
  - Decimals, finite and infinite, define all rationals also and all of the irrationals in every day use, like square roots, π, e etc.
  - Can decimals characterise *all* the reals?

Real Numbers as decimals

- We write a decimal in [0,1) in the form:

$$0.d_1 d_2 \ldots = \sum_{i=1}^{\infty} d_i 10^{-i}$$

  where $d_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \;\; \forall i \in \mathbb{N}$
- For a *finite* decimal of length n, $d_i = 0 \;\; \forall i > n$

Real Numbers as decimals (2)

- It can be shown that the decimals provide a *complete characterisation of the reals*
  - every decimal denotes a real number
  - every real number can be written as a decimal, e.g. .....

Real Numbers as decimals (3)

- the natural number n is written $n.0$
- 3/4 = 0.75
- 1/3 = $0.\dot{3} = 0.333333\ldots$ (recurring infinite decimal)
- $\pi = 3.14159265358979323846 2\ldots$ (non-recurring infinite decimal)
- The fundamental axiom is crucial in the proof.
- This is a nice result as it means our intuitive view of real numbers (as decimals) is sufficient ..... but no coincidence, of course!

SEQUENCES AND CONVERGENCE

- A sequence is a countable, ordered set of real numbers $\{a_i \in \mathbb{R} \mid i \in \mathbb{N}\}$, usually written

$$a_1, a_2, \ldots, a_n, \ldots$$

  or simply

$$a_1, a_2, \ldots$$

- Alternatively it is *function*, $a : \mathbb{N} \longrightarrow \mathbb{R}$ with the obvious definition
- examples
  - $1, 4, 9, \ldots, n^2, \ldots$
  - $1, -0.25, 0.\dot{1}, \ldots, (-1)^{n+1}/n^2, \ldots$

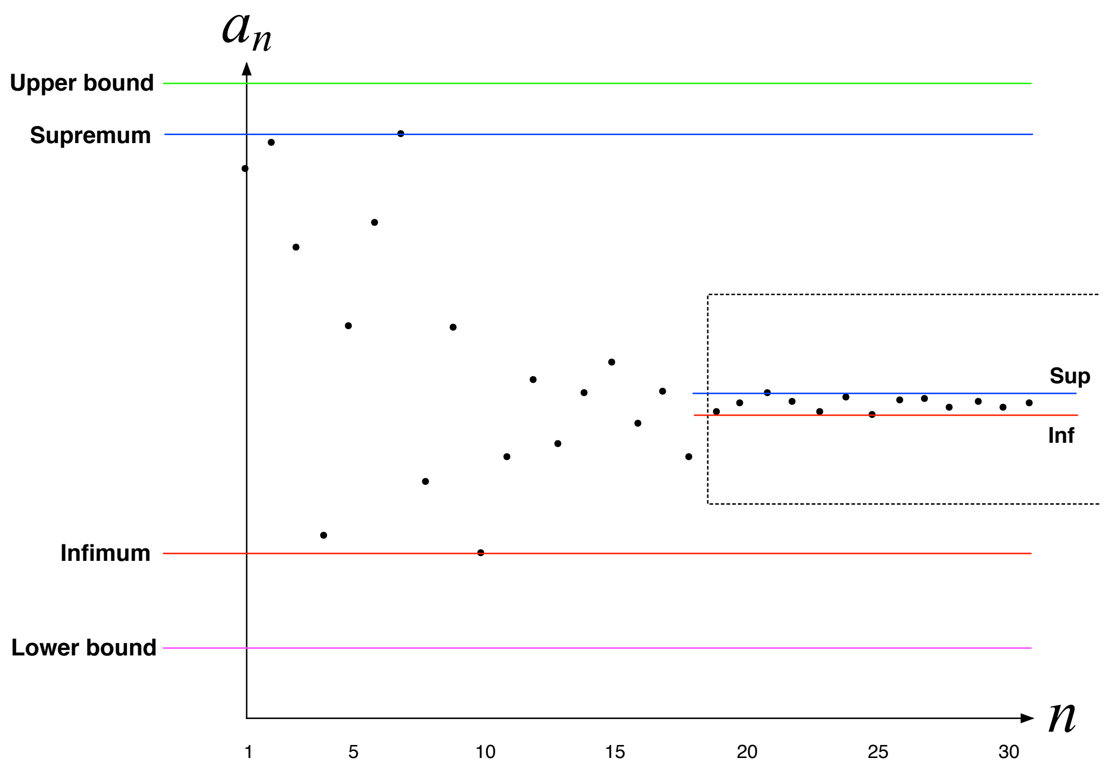
Convergence

Definition: A sequence $a_1, a_2, \ldots$ converges to a limit $l \in \mathbb{R}$, written $a_n \to l$ as $n \to \infty$ or $\lim_{n\to\infty} a_n = l$, iff

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n > N, |a_n - l| < \epsilon$$

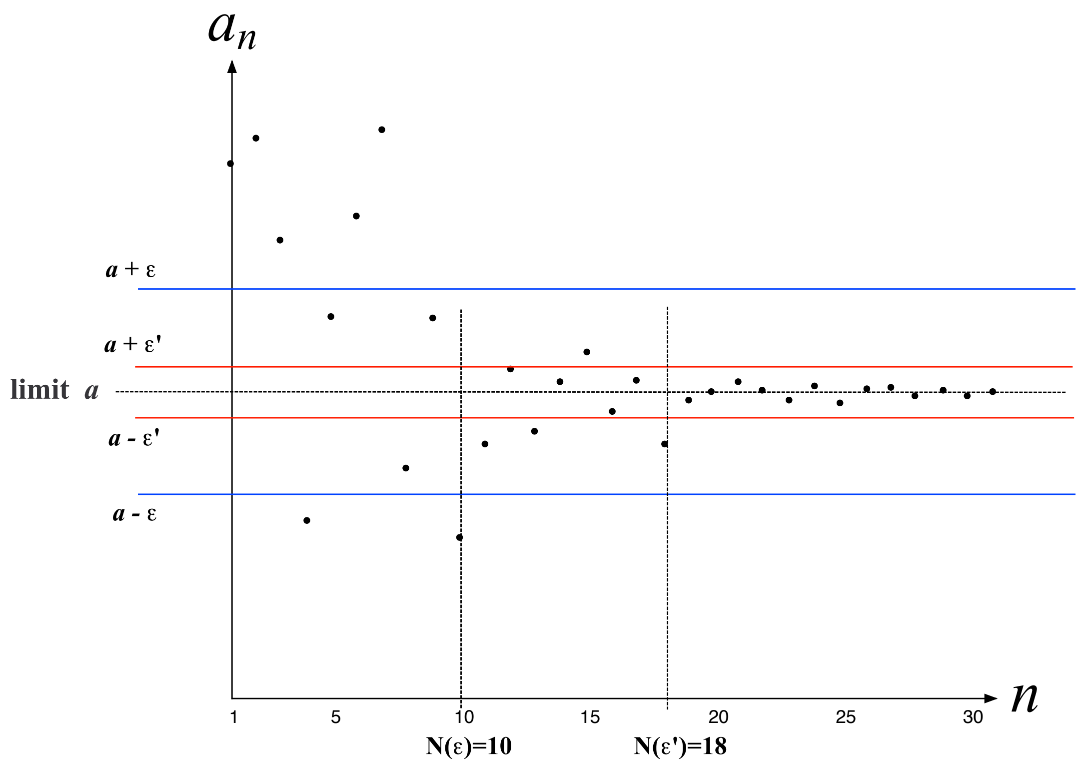
- equivalently, $l - \epsilon < a_n < l + \epsilon$
- 'tramlines' ϵ away from the limit value l

Illustration of bounds, Sup and Inf



Notice how the supremum *decreases* and the infimum *increases* for the subsets $\{a_n, a_{n+1}, \ldots\}$ as n increases.

Illustration of convergence



Need a bigger N as ϵ decreases

Convergence (2)

- Important in *any numerical algorithms & programs that use iteration*
  - i.e. quite a lot! – graphics, performance analysis, engineering applications like CFD and FEM .....
  - iteration no use unless it *converges*
  - if it does, how fast? Can we calculate the result directly?

Convergence and boundedness

- *For a bounded increasing sequence of positive values $p_1, p_2, \ldots$ the limit p is equal to the supremum $s = \sup p_n$*
  - Limit p exists by Fundamental Axiom
  - $\forall \epsilon > 0$ the ‘upper tramline’ is an upper bound
  - similarly, every upper bound is above the lower tramline
  - therefore $p - \epsilon < s < p + \epsilon$ and so $s = p$

Convergence and boundedness (2)

- *A convergent sequence is bounded*
  - Let $a_1, a_2, \ldots$ have limit l.
  - Then $\exists N$ s.t. $l - 1 < a_n < l + 1 \; \forall n > N$
  - So, for all $i \in \mathbb{N}$,

$$\min(l-1, a_1, \ldots, a_N) \leq a_i \leq \max(l+1, a_1, \ldots, a_N)$$

Proof that $s = p$ by the ϵ-N method

1. Suppose $p_m > p$ for some m.
   Pick $\epsilon = (p_m - p)/2$ so that $\forall n > m$,
   $p_n - p \geq p_m - p = 2\epsilon > \epsilon$. Hence $p_1, p_2, \ldots$ does not converge, a contradiction. Thus p is an upper bound, so $p \geq s$.

2. Now suppose that u is an upper bound.
   Since $p_1, p_2, \ldots$ converges,
   $\forall \epsilon > 0, \exists N$ s.t. $p_N > p - \epsilon$. Hence $p - \epsilon < u$ and so $p \leq u$ since ϵ can be arbitrarily small.
   In particular, $p \leq s$.

$p \geq s$ and $p \leq s \Rightarrow p = s$.

Example: $a_n = 1/n$

- Intuitively, $1/n$ decreases, getting closer and closer to zero, as n increases.
- This (correct) intuition is made rigorous as follows:
  Given any $\epsilon > 0$, $a_N \leq \epsilon$ if $N \geq 1/\epsilon$. Choose $N = \lceil 1/\epsilon \rceil$. Then

  $$\forall n > N, \ |a_n| < \epsilon$$

  since a_n is decreasing. Thus, $a_n \to 0$ as $n \to \infty$.
- Similarly for $a_n = 1/n^\alpha$ for any $\alpha > 0$ (exercise).

Trapping

Theorem: Given convergent sequences $b_1, b_2, \ldots$ and $c_1, c_2, \ldots$, each with limit l, suppose the sequence $a_1, a_2, \ldots$ satisfies

$$b_n \leq a_n \leq c_n$$

$\forall n \geq N$ for some $N \in \mathbb{N}$. Then $a_n \to l$ as $n \to \infty$.

- Intuitively, the sequence a_n becomes 'trapped' between b_n and c_n.
- Commonly called the *sandwich theorem*.

Proof of sandwich theorem

- Pick $\epsilon > 0$
- Since the sequences b_n and c_n converge, $\exists N_1, N_2$ s.t. $\forall n > \max(N_1, N_2),\ l - \epsilon < b_n < l + \epsilon$ and $l - \epsilon < c_n < l + \epsilon$, i.e.

$$l - \epsilon < b_n < a_n < c_n < l + \epsilon$$

- Hence, $\exists N(= \max(N_1, N_2))$ s.t. $\forall n > N, |a_n - l| < \epsilon$
- So $a_n \to l$ as $n \to \infty$

Special cases

- If $b_n = l$ for all $n > 0$, the greatest lower bound (infimum) on a_n is the constant l
- An upper bound is c_n and the supremum is l
- E.g. the sequence $1/n^2$ is trapped between 0 and $1/n$, which we just showed has limit 0
- Similarly, if $c_n = l$ for all $n > 0$, the supremum on a_n is the constant l and a lower bound is b_n with infimum l

Example

- Suppose $a_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \ldots + \frac{1}{\sqrt{n^2+n}}$
- $a_n > \frac{n}{\sqrt{n^2+n}} = \frac{1}{\sqrt{1+1/n}}$
- $a_n < \frac{n}{\sqrt{n^2+1}} = \frac{1}{\sqrt{1+1/n^2}}$
- Hence a_n is trapped between two sequences that tend to 1 as $n \to \infty$, so $a_n \to 1$

Ratio convergence test

Theorem: If $|a_{n+1}/a_n| < c < 1$ for some $c \in \mathbb{R}$ and for all sufficiently large n (i.e. $\forall n \geq N$ for some integer N), then $a_n \to 0$ as $n \to \infty$.

- A convergent sequence with limit 0 is called a *null sequence*.
- The proof is that, for $n \geq N$,

$$|a_n| < c|a_{n-1}| < \ldots < c^{n-N}|a_N| = kc^n$$

  where k is the constant $|a_N|/c^N$
- But $c^n \to 0$ as $n \to \infty$ and so the theorem is proved by the sandwich theorem

Ratio divergence test

Theorem: If $|a_{n+1}/a_n| > c > 1$ for some $c \in \mathbb{R}$ and for all sufficiently large n, then the sequence a_n diverges.

- The analogous proof is that, for $n \geq N$,

$$|a_n| > c|a_{n-1}| > \ldots > c^{n-N}|a_N| = kc^n$$

- But c^n has no upper bound, and hence neither does $|a_n|$

Alternative form of ratio tests

Simpler forms of the ratio tests use the *limit* of the ratio $|a_{n+1}/a_n|$, when this exists – call it r:

- Then if $r < 1$ the sequence converges and if $r > 1$, it diverges.
- The proof is simple: e.g. if $r < 1$, then $\exists N$ s.t. $\forall n > N, |a_{n+1}/a_n| < (r+1)/2 < 1$ and we can pick $c = (r+1)/2$

Combinations of sequences

Theorem: Given convergent sequences a_n and b_n with limits a and b respectively, then

- $\lim_{n \to \infty} (a_n + b_n) = a + b$
- $\lim_{n \to \infty} (a_n - b_n) = a - b$
- $\lim_{n \to \infty} (a_n b_n) = ab$
- $\lim_{n \to \infty} (a_n / b_n) = a/b$ provided that $b \neq 0$

Sample proof: product

$$\begin{aligned} |a_n b_n - ab| &= |a_n(b_n - b) + b(a_n - a)| \\ &\leq |a_n||b_n - b| + |b||a_n - a| \end{aligned}$$

- Let A be any upper bound of $\{|a_n|\}$
- Given $\epsilon > 0$, $\exists N_1$ s.t. $|a_n - a| < \epsilon/(A + |b|)$ for all $n > N_1$ and $\exists N_2$ s.t. $|b_n - b| < \epsilon/(A + |b|)$ for all $n > N_2$
- Hence $|a_n b_n - ab| < \epsilon$ for all $n > \max(N_1, N_2)$

Example

$$a_n = \frac{3n^2 + n}{n^2 + 3n + 1}$$

- Divide numerator and denominator by n^2:

$$a_n = \frac{3 + 1/n}{1 + 3/n + 1/n^2}$$

- $1/n \to 0$, so $1/n^2 \to 0$ (product of sequences or trapping)

Example (2)

- numerator and denominator converge to 3 and 1 respectively (sum of sequences, 3 times)
- so $a_n \to 3$ by the division rule (denominator non-zero)
- *rigorous justification of 'domination of largest term' rule*

General convergence theorem

NB: *This is not examinable*

Theorem (Cauchy): The sequence $a_1, a_2, \ldots$ is convergent if and only if $\forall \epsilon > 0, \exists$N s.t. $|a_n - a_m| < \epsilon$ for all $n, m > N$.

- This theorem is useful because you don't need to know what the limit is (when it exists), e.g.
  - when a_n is defined by a recurrence relation;
  - when a_n is defined by a recursive Haskell function
- It is also a test for divergence

Example

$$a_n = \sum_{i=1}^{n} \frac{1}{i(i+1)}$$

$$\begin{aligned}
a_n - a_m &= \frac{1}{(m+1)(m+2)} + \ldots + \frac{1}{n(n+1)} \\
&= \left(\frac{1}{m+1} - \frac{1}{m+2}\right) + \left(\frac{1}{m+2} - \frac{1}{m+3}\right) + \\
&\quad \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= \frac{1}{m+1} - \frac{1}{n+1} \to 0 \quad \text{as } n > m \to \infty
\end{aligned}$$

Iteration and fixpoints

Consider the simple iteration:

$$a_{n+1} = \frac{2 + a_n}{3 + a_n}$$

with initial value $a_1 = 1$.

- If this converges, its limit is l given by

$$l^2 + 2l - 2 = 0$$

  so that $l = -1 \pm \sqrt{3}$.

- So will it converge, and to which root, $l = l^+$ or l^-?

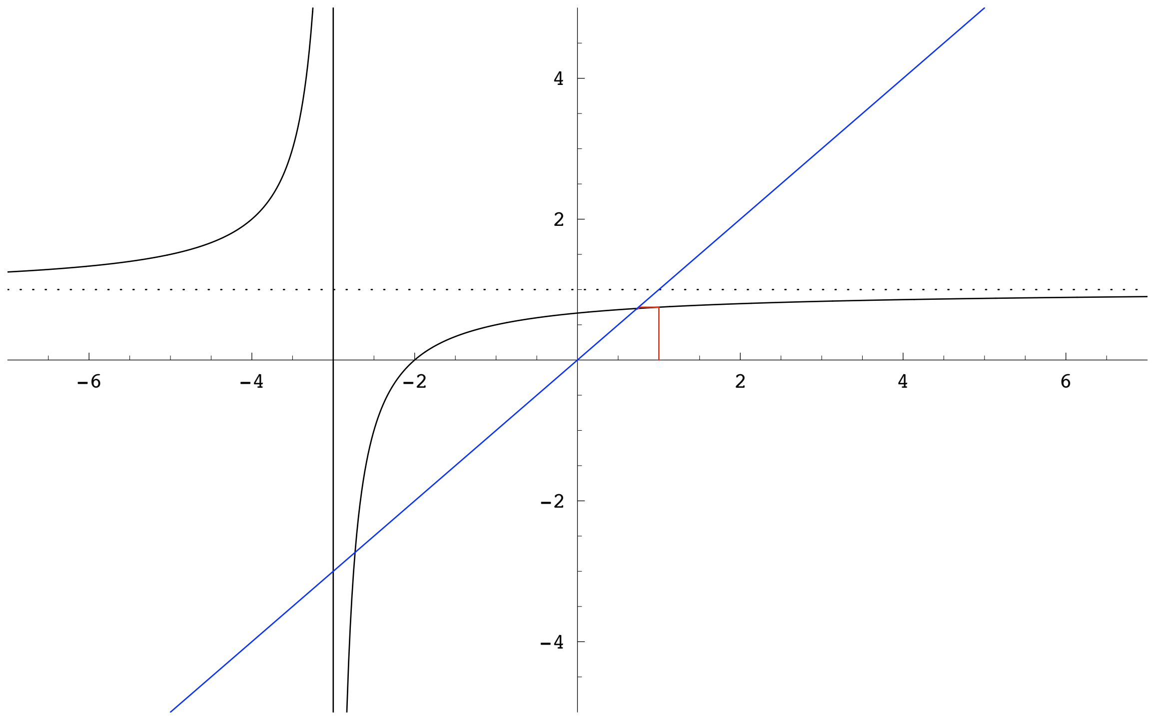
Convergence

- Clearly, every $a_n > 0$ (rigorous proof by induction), so can't converge to l^-.
- Let $x_n = a_n - l^+$ for $n \geq 1$ and try to prove $x_n \to 0$
- Aiming to use the ratio test for sequences:

$$x_{n+1} = \frac{2 + a_n}{3 + a_n} - \frac{2 + l^+}{3 + l^+} = \frac{x_n}{(3 + a_n)(3 + l^+)}$$

- Thus $|x_{n+1}| < |x_n|/9$ since a_n and $l^+ > 0$
- So the iteration does converge to $l^+ = \sqrt{3} - 1$

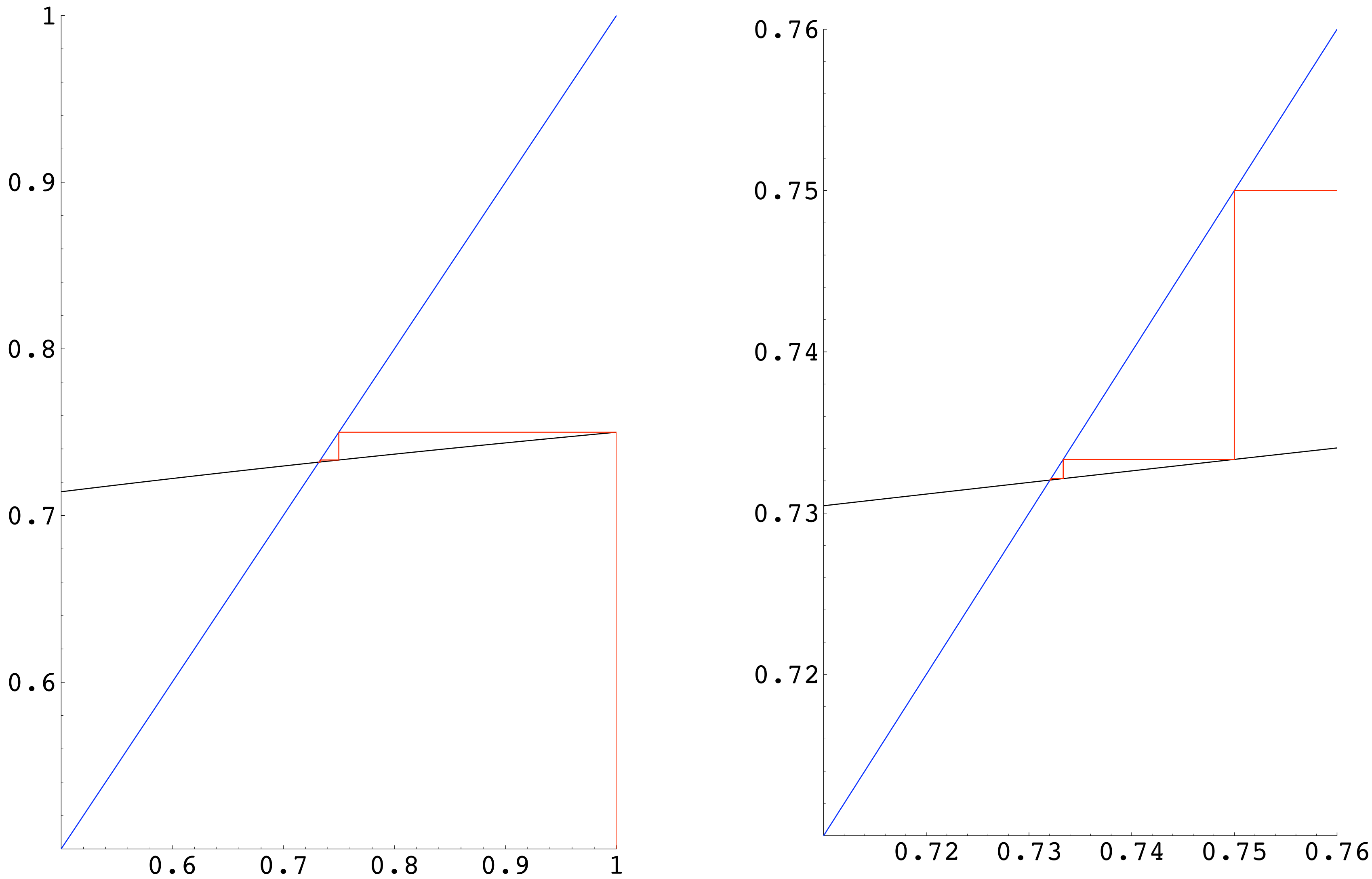
Graphically ....



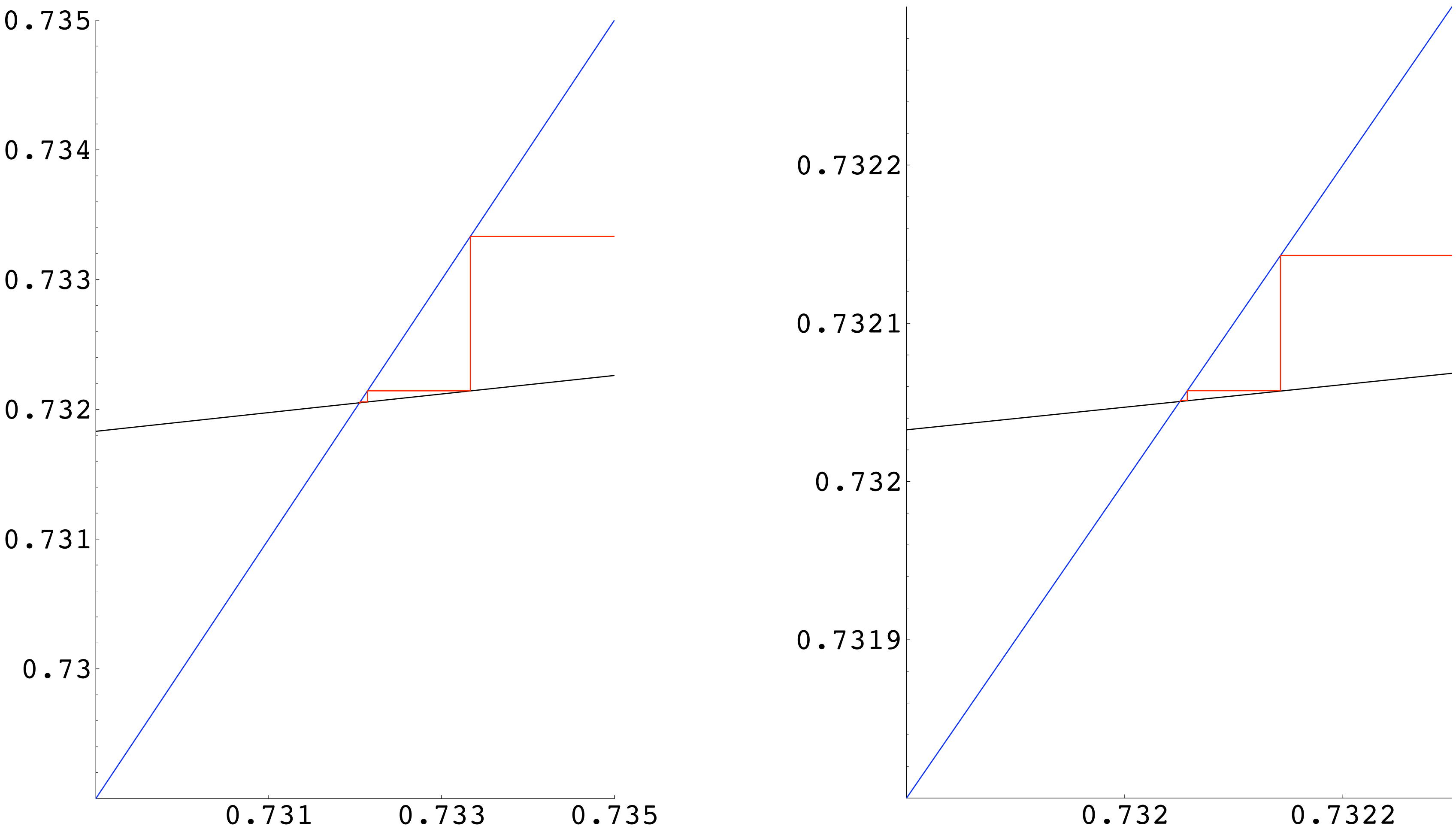
The iteration follows the red path, starting at the initial point $(1, 0)$ and repeating:

- vertical segment up to the blue line $y = x$
- horizontal to the curve $y = \frac{2+x}{3+x}$

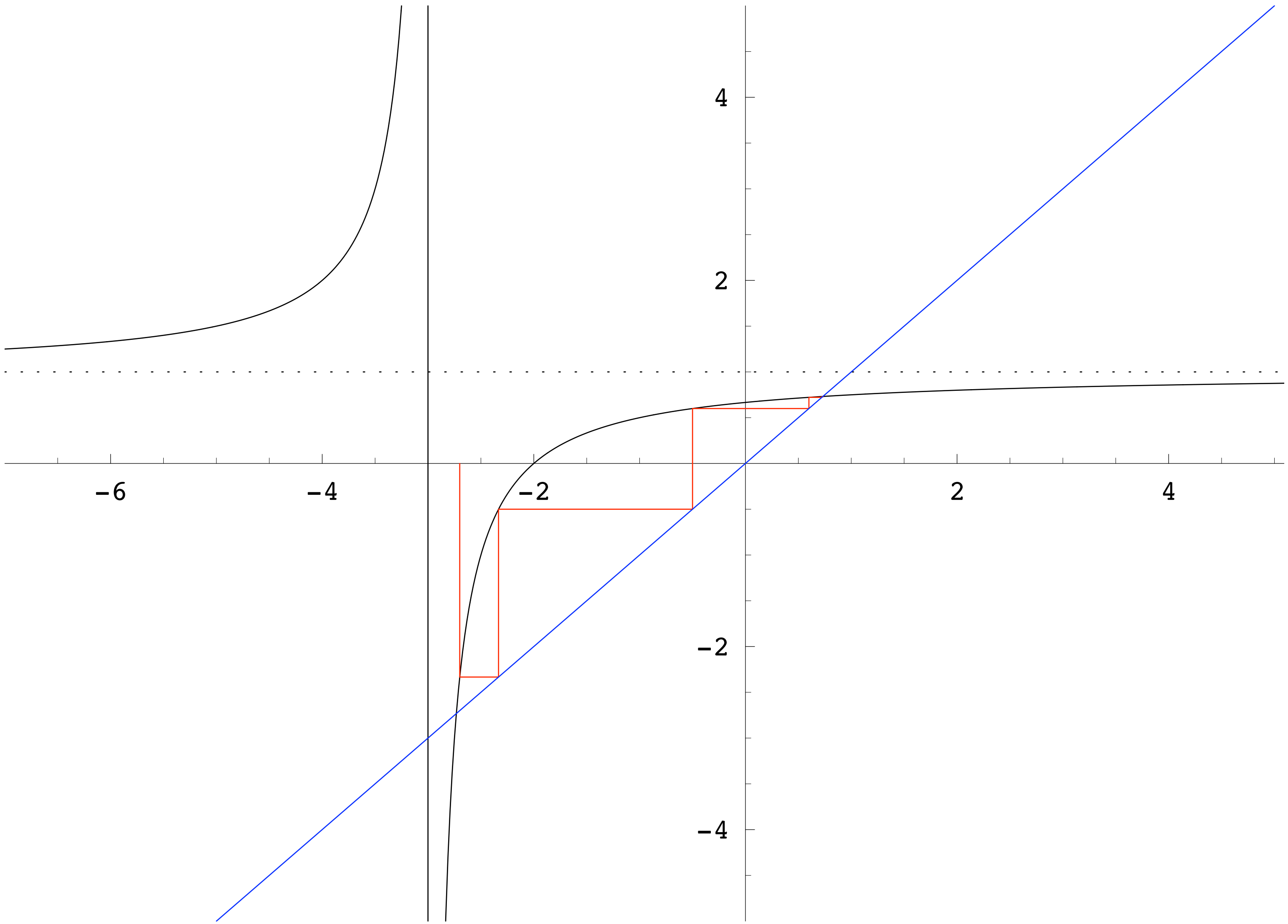
Smaller plot range and zoom 10×



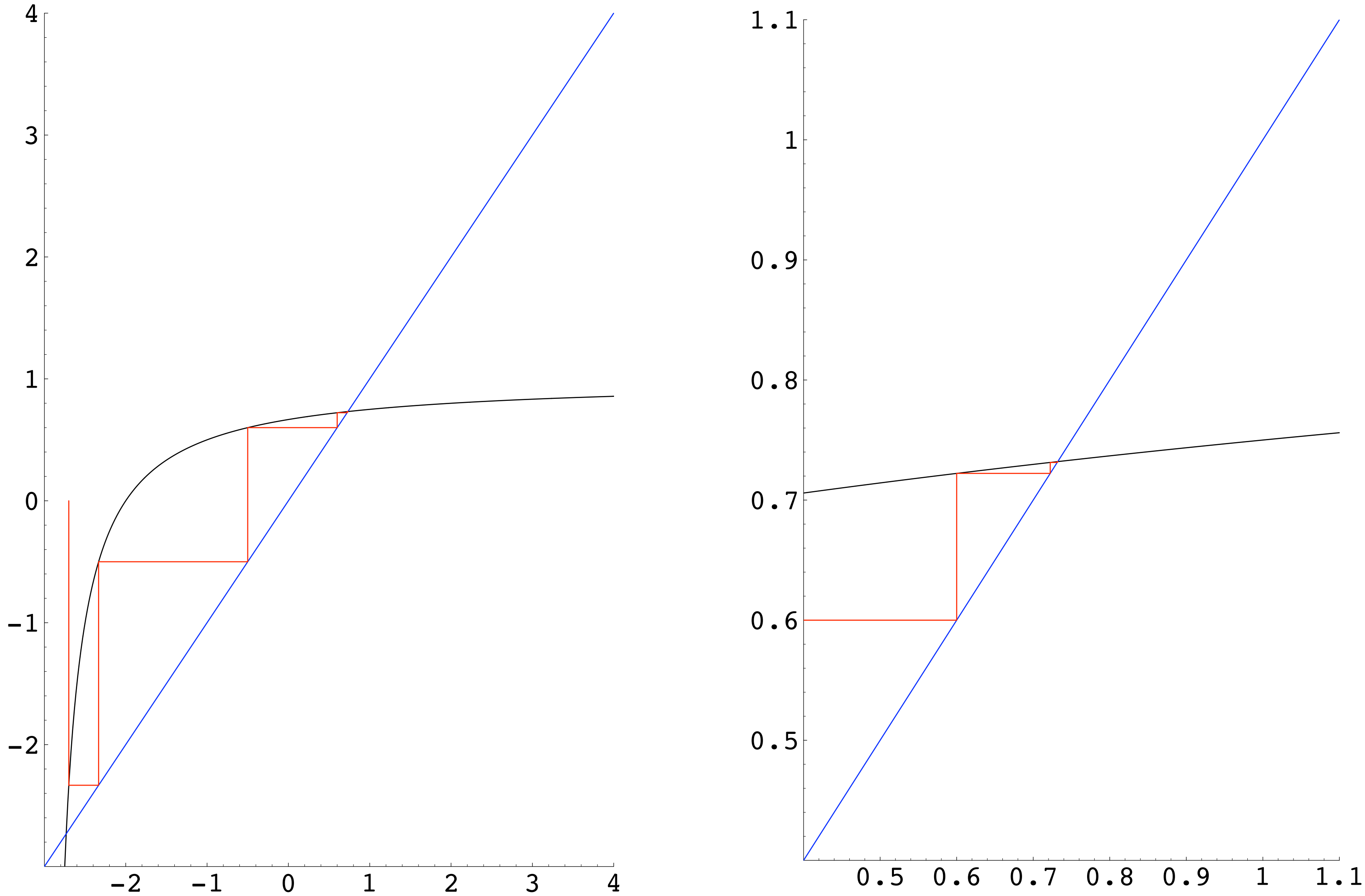
Zoom 100× and 1000×



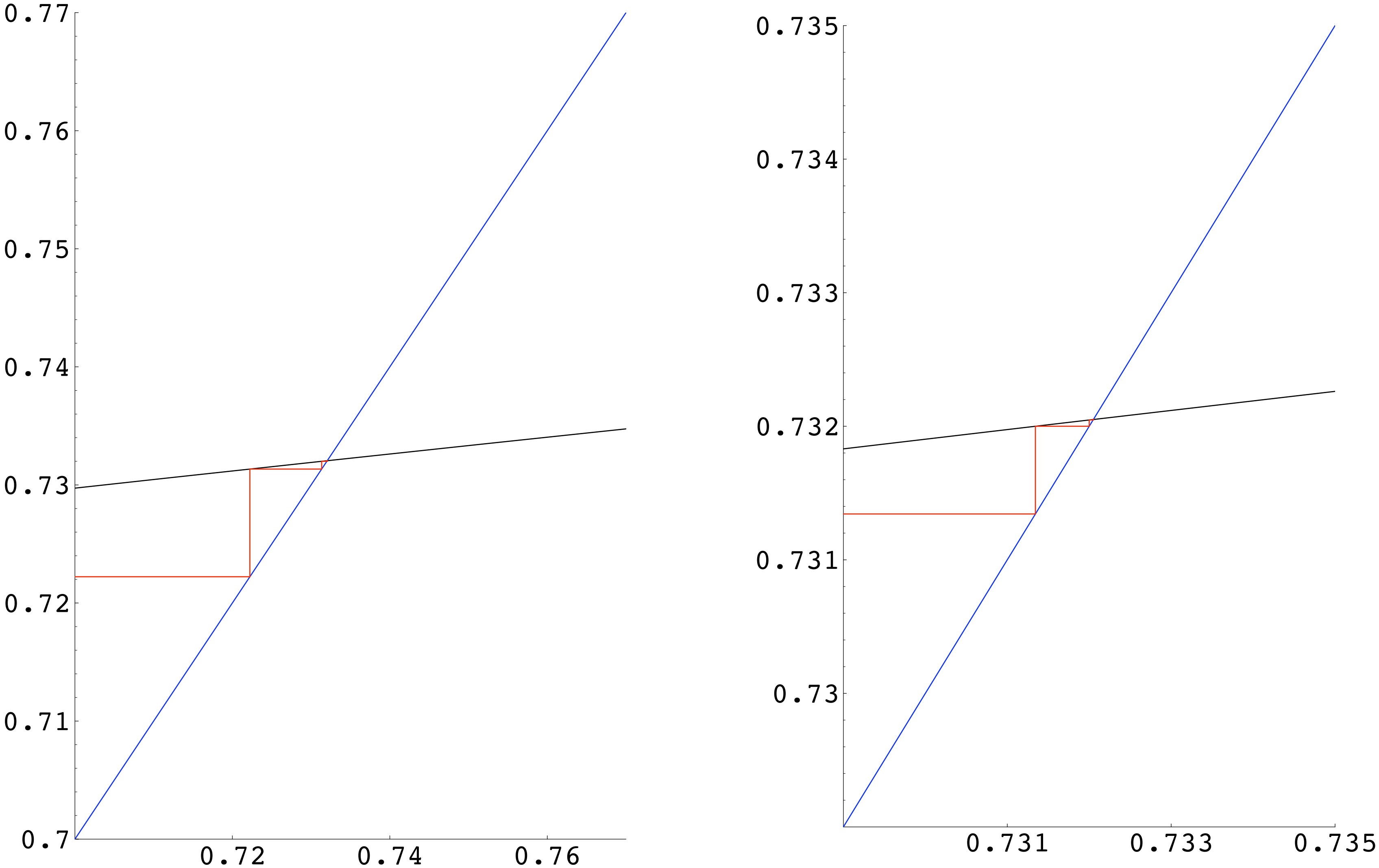
Starting at $x = -2.7$ near negative root



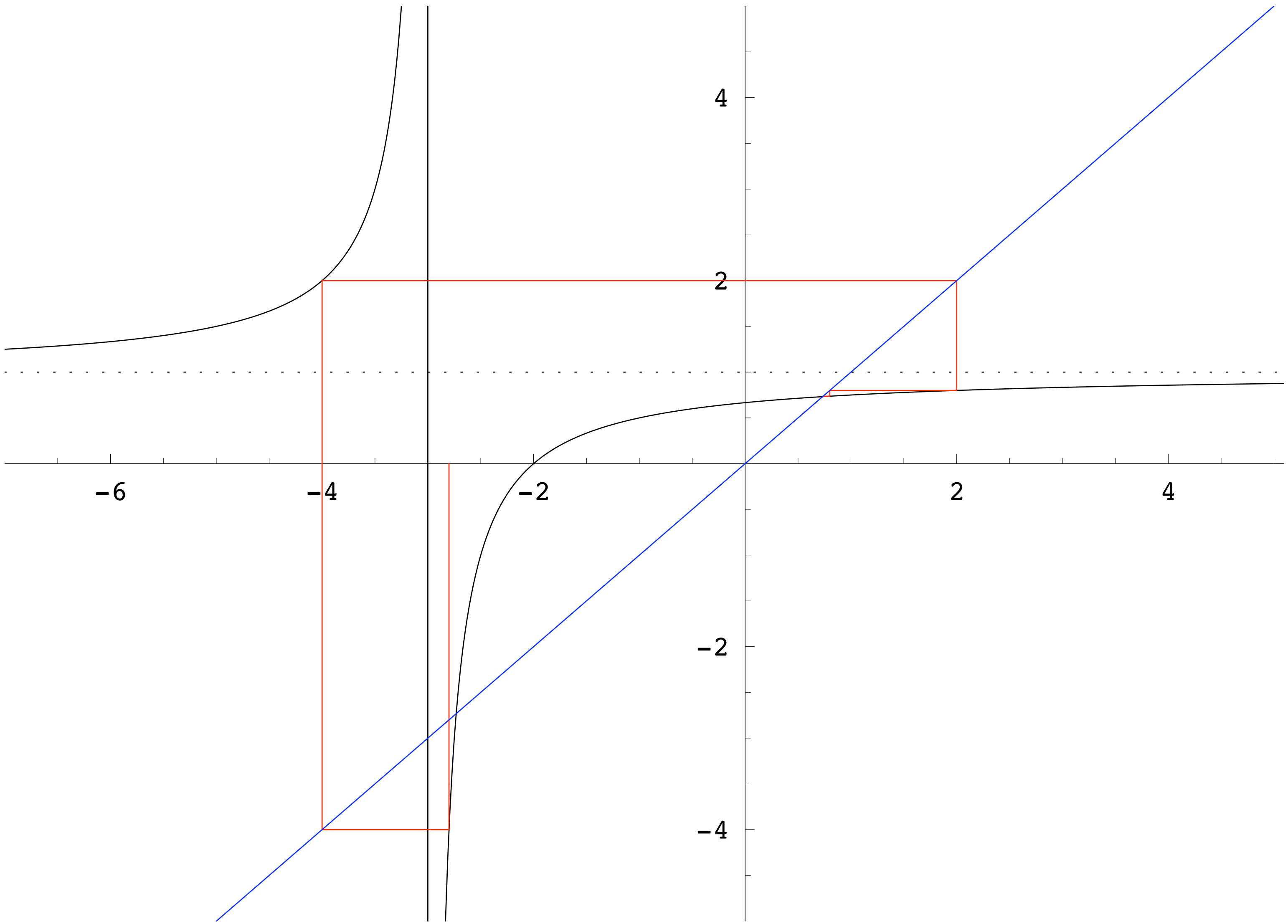
Smaller plot range and zoom 10×



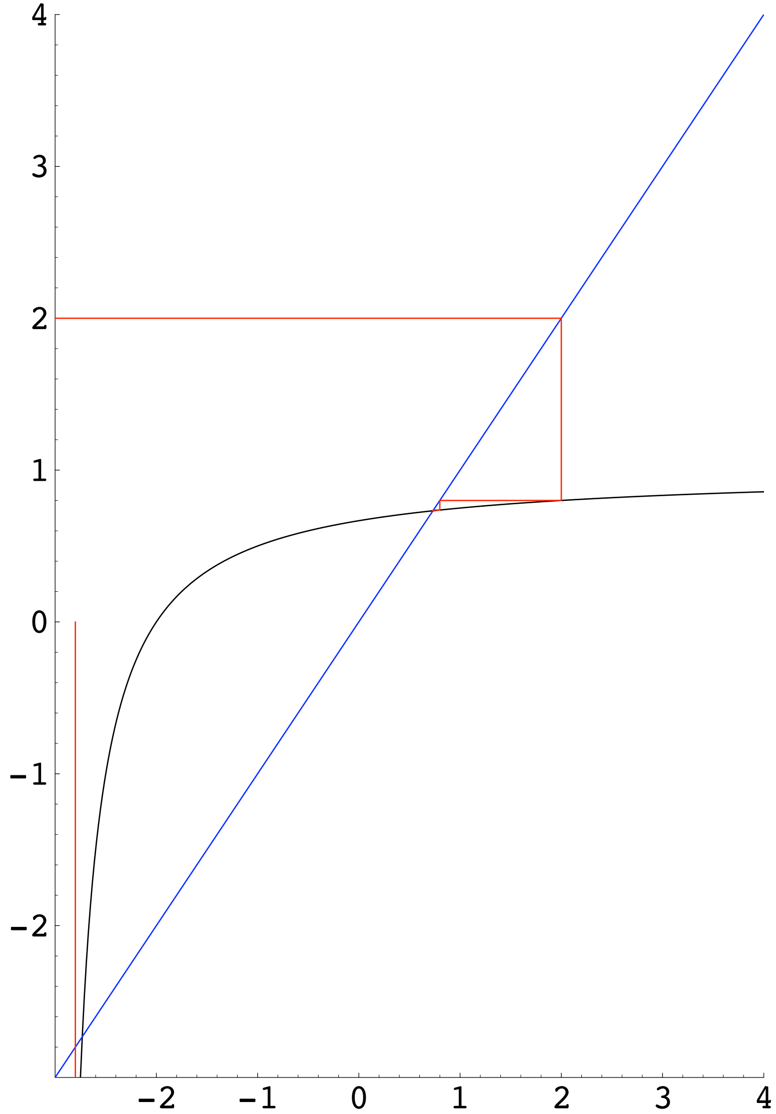
Zoom 100× and 1000×

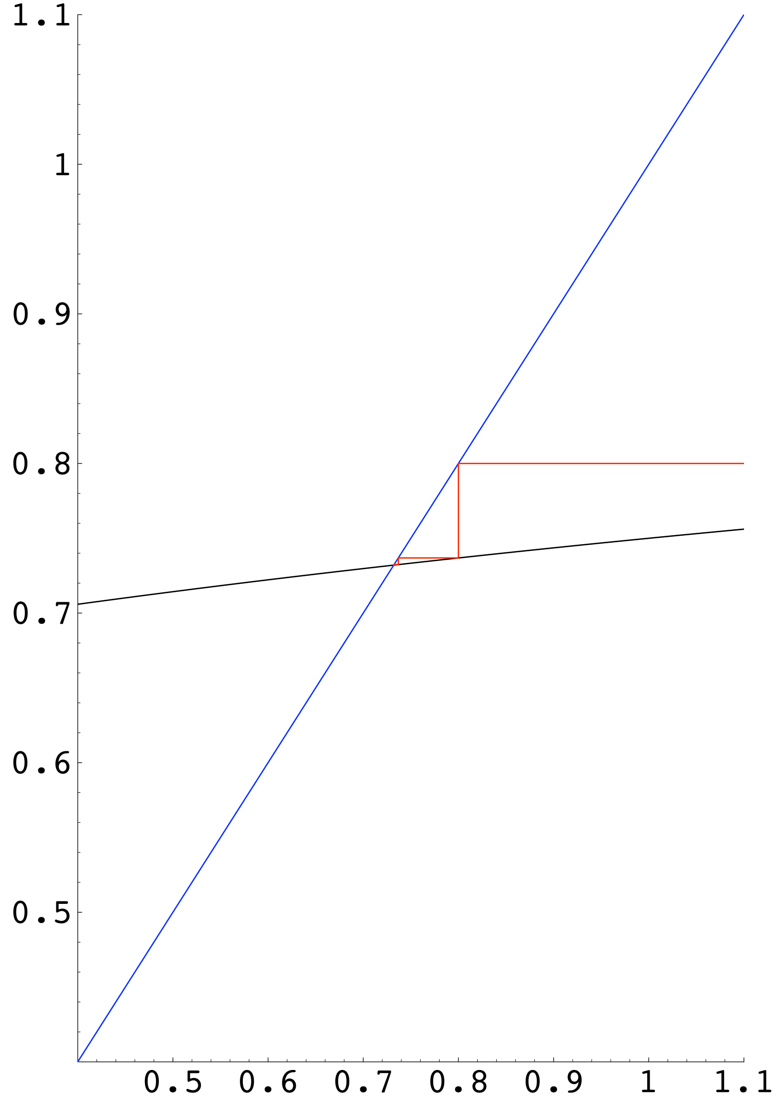


Starting at $x = -2.8$ (other side)

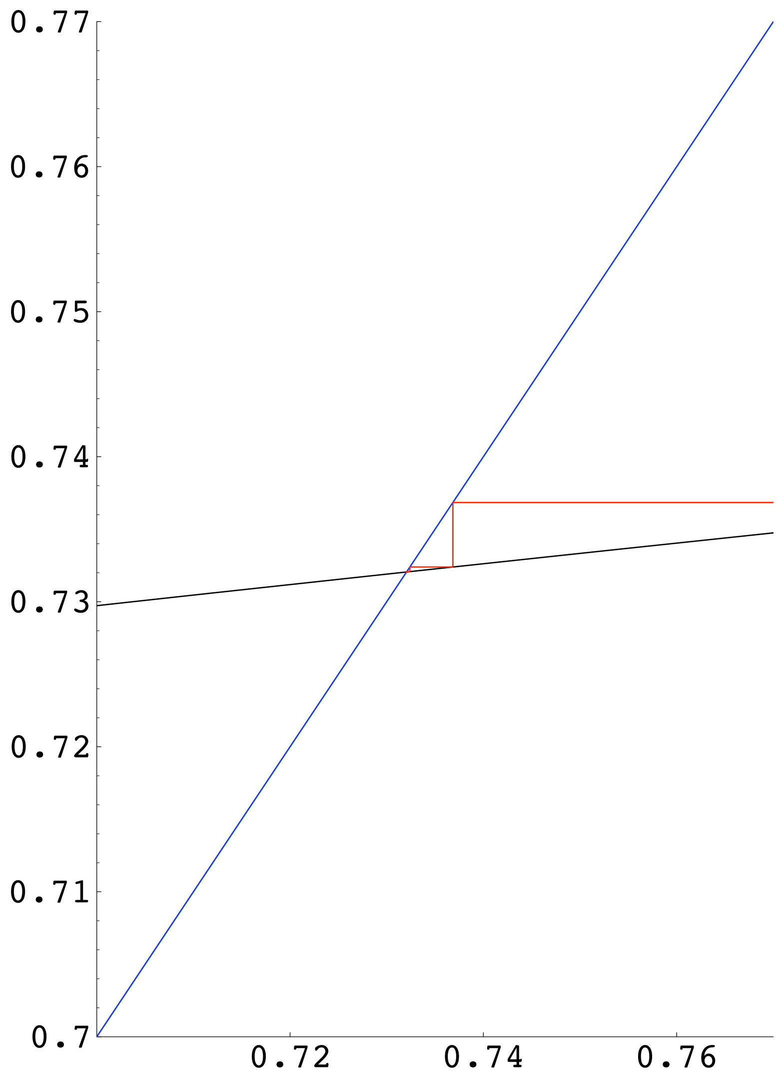


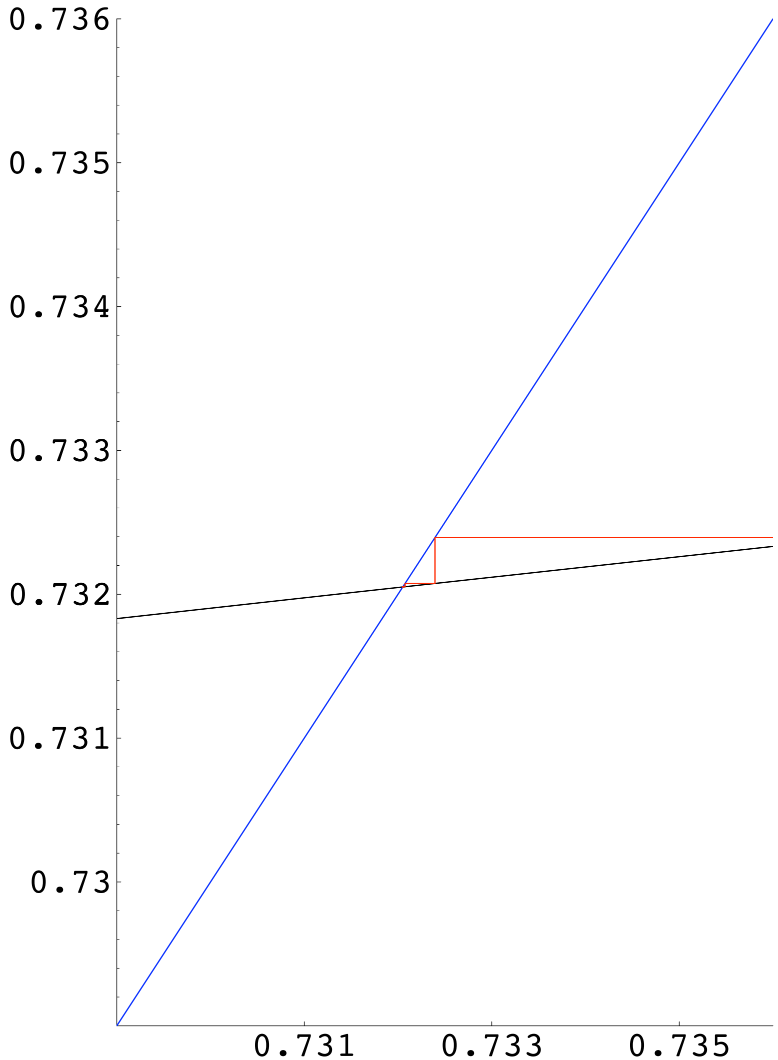
Smaller plot range and zoom 10×





Zoom 100× and 1000×





INFINITE SERIES

An infinite series is a summation of the form $S = \sum_{i=1}^{\infty} a_i$ for a real *sequence* $a_1, a_2, \ldots$

- E.g. the decimal numbers
- Finite if $\exists N \in \mathbb{N}$ s.t. $a_n = 0 \;\; \forall n > N$
- nth partial sum $S_n = \sum_{i=1}^{n} a_i$
- Partial sums $S_1, S_2, \ldots$ form a *sequence*:
  - A series converges or diverges iff its sequence of partial sums does
  - Often the best means of analysis

Geometric series

- A ubiquitous example is $G = \sum_{i=1}^{\infty} x^i$ – the *geometric progression*
- *Provided G exists*,

  $G = x + \sum_{i=2}^{\infty} x^i = x + x\sum_{i=1}^{\infty} x^i = x + xG$ so:

$$G = \frac{x}{1 - x}$$

- When does G exist? When the series (or sequence of partial sums) is convergent!

Convergence of the geometric series

- Similarly, nth partial sum

  $$G_n = x + \sum_{i=2}^{n} x^i = x + x \sum_{i=1}^{n-1} x^i = x + x(G_n - x^n),$$

  so:

  $$G_n = \frac{x - x^{n+1}}{1 - x}$$

  - For $|x| < 1,\ G_n \to x/(1-x)$ as $n \to \infty$ by rules for sequences.
  - Similarly, for $|x| > 1,\ G_n$ diverges as $n \to \infty$.
  - For $x = 1$, $G_n = n$ which also diverges.

Result

- If $|x| < 1$, i.e. $-1 < x < 1$,

$$G = \sum_{i=1}^{\infty} x^i = \frac{x}{1-x}$$

- If $|x| \geq 1,\ G = \sum_{i=1}^{\infty} x^i = \infty,$ i.e. the series *diverges*.

Another example

- Consider the convergence properties of the series
$$S = \sum_{i=1}^{\infty} \frac{1}{i(i+1)}$$
- Using partial fractions, we can write the nth partial sum
$$S_n = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1} \right) = 1 - \frac{1}{n+1}$$
- So S_n converges, therefore so does the series and $S = 1$

Sum of inverse squares

- What about the series $S = \sum_{i=1}^{\infty} \frac{1}{i^2}$?
- $\frac{1}{i(i+1)} < \frac{1}{i^2} < \frac{1}{(i-1)i}$ for $i \geq 2$. So, summing from $i = 2$ to n and adding 1:

$$1/2 + \sum_{i=1}^{n} \frac{1}{i(i+1)} < S_n < 1 + \sum_{i=1}^{n-1} \frac{1}{i(i+1)}$$

Sum of inverse squares (2)

- Thus, from the previous slide,

$$3/2 - 1/(n+1) < S_n < 2 - 1/n$$

- Since S_n is increasing, the series converges (by the fundamental axiom, 2 is an upper bound) to a value in $(1.5, 2)$.

Dodgy series

Consider the series

$$S = \sum_{i=1}^{\infty} (-1)^{i+1}/i = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

- $S_{2n} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{2n-1} - \frac{1}{2n}\right) > 0.5$ and increasing
- $S_{2n} = 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \ldots - \left(\frac{1}{2n-2} - \frac{1}{2n-1}\right) - \frac{1}{2n} < 1$

Dodgy series (2)

- Thus S_{2n} is increasing and bounded, hence convergent.
- $S_{2n+1} = S_{2n} + \frac{1}{2n+1}$ and so all partial sums converge to the same limit, l say. Hence S converges to l.

Rearrangements

- Now consider the sub-series formed by taking two positive terms and a negative term: $B_{3n} = \sum_{i=1}^{n} b_i$ where $b_i = \frac{1}{4i-3} + \frac{1}{4i-1} - \frac{1}{2i}$
- Clearly, as $n \to \infty$, B_{3n} includes all the terms of S: it is a *rearrangement* of S
- Now, $B_{3n} = S_{4n} + \frac{1}{2n+2} + \frac{1}{2n+4} + \ldots + \frac{1}{4n} > S_{4n} + 0.25$
- Hence, B_{3n} converges to a different limit than S_{4n} (limit l)!

Sums of series

Theorem: Suppose $\sum a_i$ and $\sum b_i$ are convergent with sums a and b respectively. Then if $c_i = a_i + b_i$, $\sum c_i$ is convergent with sum $a + b$, and $\sum \lambda a_i$ is convergent with sum λa.

- Easy to prove by considering the partial sums
- Further expected properties hold for series without negative terms .....

Series of non-negative terms

- In a series of non-negative terms, the partial sums are increasing and hence either
  - converge, if the partial sums are bounded
  - diverge, if they are not
- Notation:
  - p_i is a non-negative term in the series $\sum p_i$
  - $\sum c_i$ is a convergent series with sum c
  - $\sum d_i$ is a divergent series

Comparison test

Theorem: Let $\lambda > 0$ and $N \in \mathbb{N}$. Then

1. if $p_i \leq \lambda c_i \; \forall i > N$, then $\sum p_i$ converges;
2. if $p_i \geq \lambda d_i \; \forall i > N$, then $\sum p_i$ diverges.

Sometimes the following form is easier:

- if $\lim \frac{p_i}{c_i}$ exists, then $\sum p_i$ converges;
- if $\lim \frac{d_i}{p_i}$ exists, then $\sum p_i$ diverges.

D'Alembert's ratio test

- This is a very useful – and even over-used – technique:

Theorem: For $N \in \mathbb{N}$,

1. if $p_{i+1}/p_i \geq 1 \; \forall i > N$, then $\sum p_i$ diverges;
2. if $\exists k \in \mathbb{R}$ s.t. $p_{i+1}/p_i < k < 1 \; \forall i > N$, then $\sum p_i$ converges.

Exercise: Consider the series with $p_i = 1/i$

Proof of part 2

- $p_{i+1} < kp_i$ for $i > N$. Thus, (formally by induction)

$$p_i < k^i (p_{N+1}/k^{N+1}) \text{ if } i > N+1$$

- Thus $\sum p_i$ converges by the comparison test with $c_i = k^i$ and $\lambda = p_{N+1}/k^{N+1}$ (Note $k > 0$.)
- Proof of part 1 is analogous.

Absolute convergence

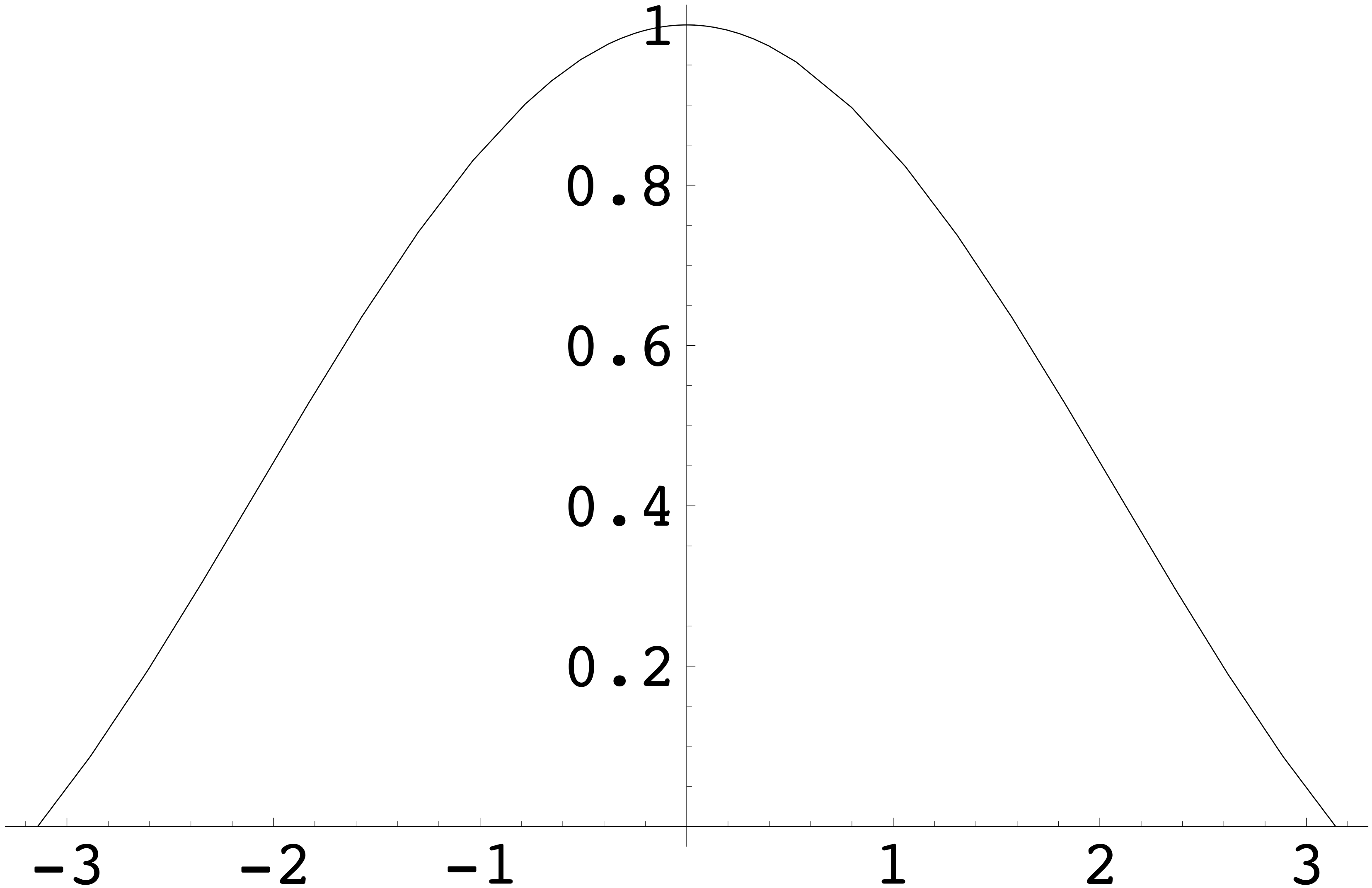
A series $\sum a_i$ is *Absolutely Convergent* if $\sum |a_i|$ converges, i.e. the sum of the absolute values of its terms is convergent.

- The sum of absolute values is a sum of positive terms
- An absolutely convergent series is convergent (proof by Cauchy's test)
- A series which is convergent but not absolutely convergent is called *conditionally convergent*
  - E.g. the 'dodgy series'

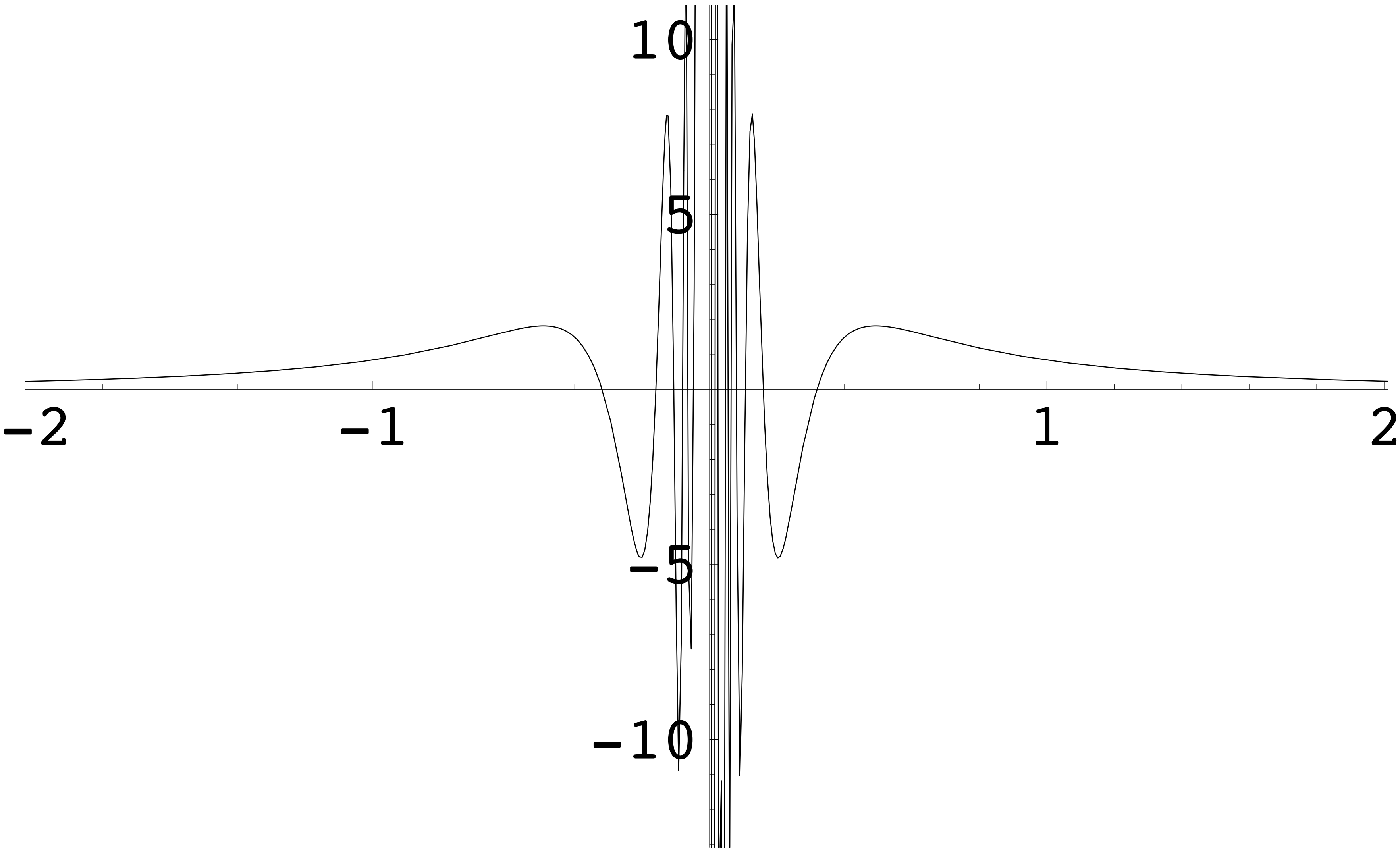
CONTINUITY

- A function $f(x)$ is *continuous* at $x = a$ if $f(x) \to f(a)$ as $x \to a$
- I.e. there is no 'jump' in the graph of $f(x)$ at $x = a$ or *'you can draw the graph without taking your pen off the paper'*
  - E.g. the *step-function* $f(x) = \lfloor x \rfloor$ is *not* continuous.
  - $f(x) = (1/x) \sin x$ *is* continuous at all x, including $x = 0$ if we define $f(0) = 1$.
  - $f(x) = (1/x) \sin(1/x)$ *is not* continuous at $x = 0$
- What does it mean to say 'as $x \to a$'?

Graph of $f(x) = (1/x)\sin x$



Graph of $f(x) = (1/x)\sin(1/x)$



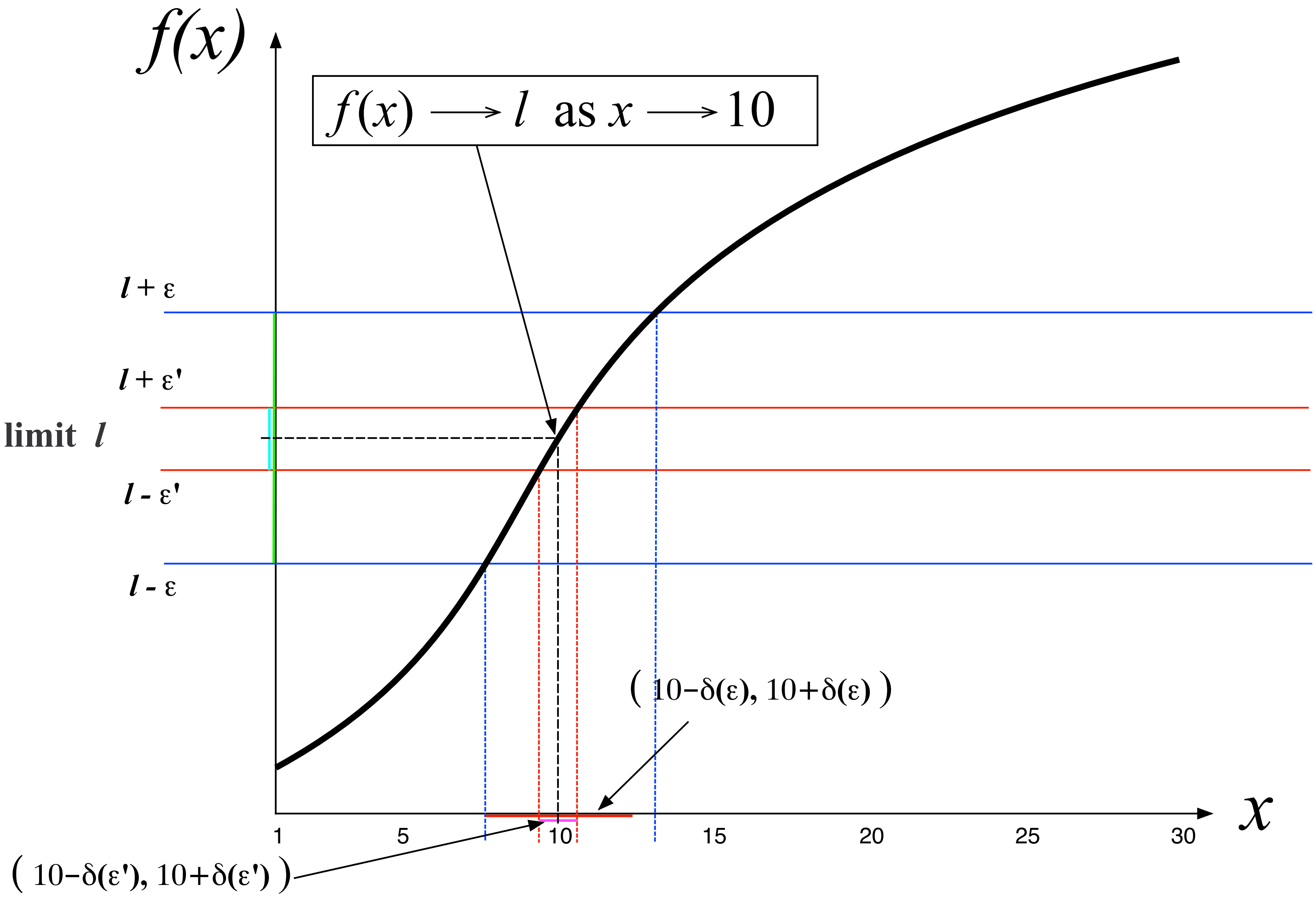
Limit of a function

Definition: $f(x) \to l$ as $x \to a$ if

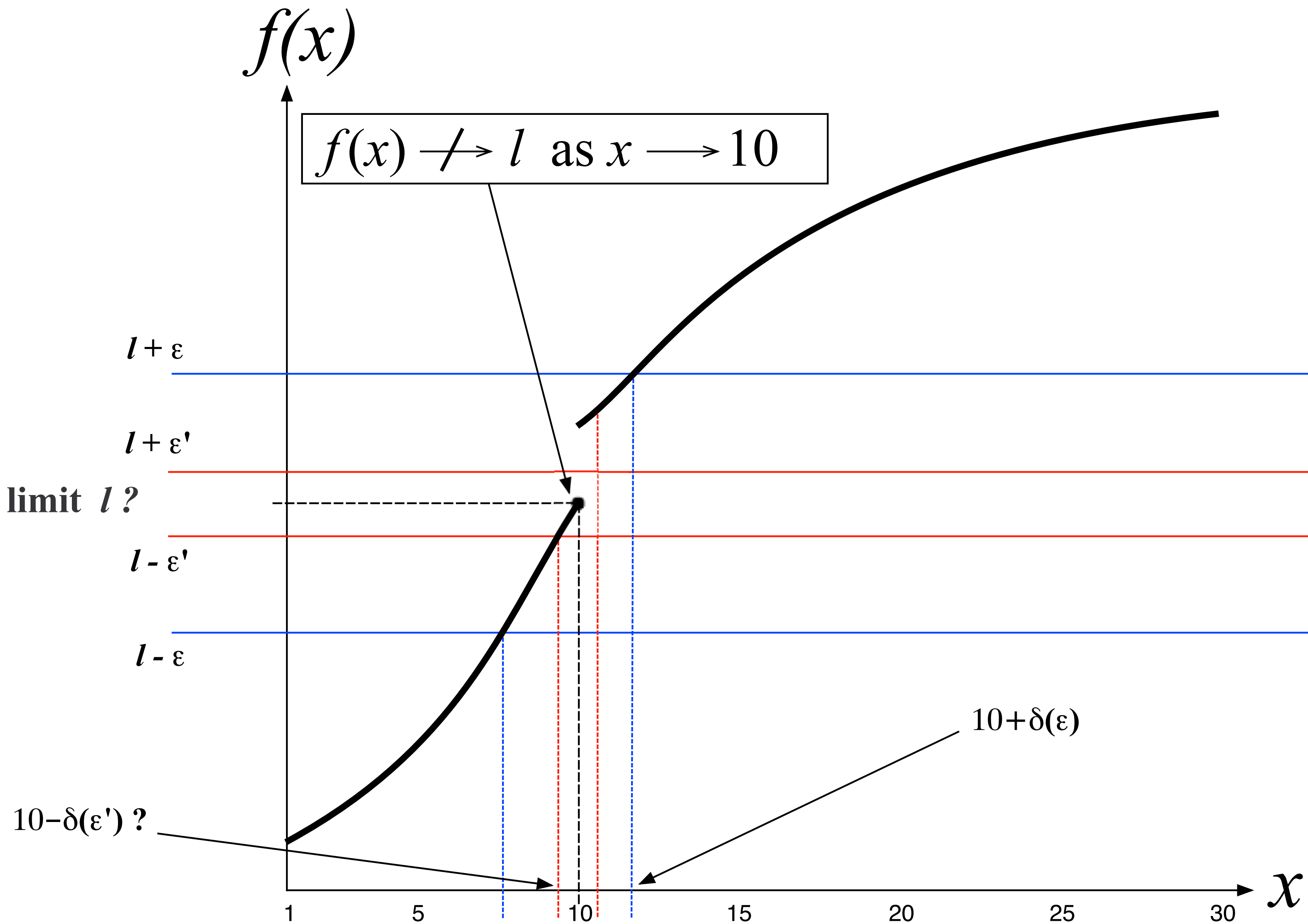
$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } |x - a| < \delta \Rightarrow |f(x) - l| < \epsilon$$

- The rigorous definition of continuity is therefore $\forall \epsilon > 0, \exists \delta > 0$ s.t. $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$
  - In words, as x gets closer and closer to a, $f(x)$ gets closer and closer to $f(a)$.
  - I.e. $f(x)$ can't suddenly 'jump' to $f(a)$, skipping over intermediate values, leaving a gap, 'taking the pen off the page'.

A continuous function



A discontinuous function



Comments

- In the continuous function, as x gets closer and closer to 10, $f(x)$ gets closer and closer to l.
  - If $f(10)$ is defined to be l, f is continuous at $x = 10$
  - Points in the arbitrary ‘green’ intervals on the y-axis must be the images of ‘red’ intervals on the x-axis
- Note the discontinuity at $x = 10$ in the discontinuous function:
  - Cannot find any ‘red’ interval when the ‘green’ interval gets too small.

Simple properties

- Sums and products of (a finite number of) continuous functions are continuous – $f(x) + \lambda g(x), f(x)g(x)$ are continuous if f and g are ($\lambda \in \mathbb{R}$).
- Same for quotients $f(x)/g(x)$ where $g(x) \neq 0$.
- A continuous function of a continuous function is continuous – i.e. the composition $f(g(x))$ is continuous.

Differentiability and continuity

- If $f(x)$ is differentiable at $x = a$, it is continuous there. Why?
- Recalling the definition of a derivative, $\lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} < \infty$ and so $f(x + \delta x) \to f(x)$ as $x + \delta x \to x$
- But $f(x) = x \sin(1/x)$ is continuous at $x = 0$, where $f(x) = 0$, but *not* differentiable there [$f'(x) = \sin(1/x) - (1/x)\cos(1/x)$ for $x \neq 0$]

Graph of $f(x) = x \sin(1/x)$