Mathematical Methods: Tutorial sheet 5

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Assessed question is number 3. Due Monday 19/11/2007.

Maclaurin's series

1. You all know (or can look up) the binomial theorem, which gives a finite expansion in powers of x for $(1+x)^n$ when n is a positive integer. Derive it by using Maclaurin's theorem. What is the result when n is *not* an integer?

   Solution: When n is a positive integer, the rth derivative of $(1+x)^n$ is

   $$n(n-1)\ldots(n-r+1)(1+x)^{n-r} = \frac{n!}{(n-r)!} \quad \text{when } x = 0,$$

   for $0 \le r \le n$, and is 0 for $r > n$. Hence the coefficient

   $$a_r = \frac{n!}{(n-r)!r!}$$

   for $0 \le r \le n$, noting that this gives 1 for a_0, and $a_r = 0$ for $r > n$. Thus

   $$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$$

   When n is not an integer – call it α instead – the same method applies but the number of non-zero derivatives is infinite (because you never get to a constant derivative). Thus the series is infinite *and only valid if it converges*:

   $$(1+x)^\alpha = \sum_{r=0}^{\infty} \frac{\alpha(\alpha-1)\ldots(\alpha-r+1)}{r!} x^r$$

2. Calculate to 4 decimal places $\sin\pi/2, \sin\pi/3, \sin\pi/4, \sin\pi/5, \sin\pi/6$ and $\sin\pi/7$ by using (in each case) any of:

   (a) a suitable geometric argument;

   (b) a protractor, ruler and very large piece of paper;

(c) Maclaurin's series.

N.B. I expect you to use method (a) for *at least* $\sin \pi/2$!

Calculate a bound on the error term (i.e. a number that the error is not greater than) in each case that you use Maclaurin's series, such that accuracy to 4 decimal places is assured.

Solution:

(a) $\sin \pi/2 = 1$ by considering a right-angled triangle where one acute angle approaches 0 and the other approaches $\pi/1$. Then the length of the side opposite the larger acute angle approaches the length of the hypotenuse.

(b) Drop a perpendicular (bisector) from a vertex of an equilateral triangle of side-length 2 to the opposite side. This creates two right-angled triangles with sides 2 (hypotenuse), 1 and $\sqrt{3}$ (by Pythagorus), and angles opposite these sides $\pi/2$, $\pi/6$ (because the angle at the vertex was bisected) and $\pi/3$. Thus $\sin \pi/3 = \sqrt{3}/2$.

(c) Consider a right-angled triangle with side-lengths $1, 1, \sqrt{2}$. Its angles are $\sin \pi/4, \sin \pi/4, \sin \pi/2$. Thus $\sin \pi/4 = 1/\sqrt{2}$.

(d) Use Maclaurin's series to up to the term $x^7/7!$, giving $\sin \pi/5 = 0.5878$. Since the 8^{th} derivative of $\sin x$ is $\sin x$ which has absolute value less than 1 (except when $x = \pi/2$), an error bound is $1/7! = 0.0000248016$ since $x < 1$. Actually, terms up to $x^5/5!$ suffice here.

(e) As in part (b), $\sin \pi/6 = 1/2$.

(f) Using Maclaurin's series as in part (d), $\sin \pi/7 = 0.4339$, using terms up to $x^7/7!$ Again, terms up to $x^5/5!$ suffice, and just two non-zero terms give a correct result to 3 d.p.

3. (a) Derive Maclaurin's series, i.e. power series expansions in x^i ($i = 1, 2, \ldots$), for the functions $e^x, \sin x, \cos x$.

(b) The differential equation:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \omega^2 y = 0$$

describes vibrations of various kinds, where y usually represents a distance and x is time. To solve it, suppose that a power series solution is postulated:

$$y = \sum_{i=0}^{\infty} a_i x^i$$

i. By substituting into the given differential equation and comparing coefficients of x^i for $i \geq 0$, show that if the power series solution is valid, then

$$a_{i+2} = -\frac{\omega^2 a_i}{(i+1)(i+2)}$$

ii. Deduce that, for $n \in \mathbb{N}$,

$$\begin{aligned} a_{2n} &= (-1)^n a_0 \frac{\omega^{2n}}{(2n)!} \\ a_{2n+1} &= (-1)^n a_1 \frac{\omega^{2n}}{(2n+1)!} \end{aligned}$$

iii. Hence show that, if $y = 1$ and $\frac{dy}{dx} = 1$ at $x = 0$, the solution of the differential equation is $y = \omega^{-1} \sin \omega x + \cos \omega x$.

Solution:

(a) Let D denote differentiation wrt x. $D^n e^x = e^x = 1$ at $x = 0$. Thus, Maclaurin's series gives

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

$D^{2n} \sin x = (-1)^n \sin x = 0$ at $x = 0$, $D^{2n+1} \sin x = (-1)^n \cos x = (-1)^n$ at $x = 0$. Thus, Maclaurin's series gives

$$\sin x = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)!}$$

$D^{2n} \cos x = (-1)^n \cos x = (-1)^n$ at $x = 0$, $D^{2n+1} \cos x = -(-1)^n \sin x = 0$ at $x = 0$. Thus, Maclaurin's series gives

$$\cos x = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!}$$

i. Let $y = \sum_{i=0}^{\infty} a_i x^i$. Substituting in the differential equation, we get:

$$\sum_{i=2}^{\infty} a_i i(i-1) x^{i-2} + \sum_{i=0}^{\infty} a_i \omega^2 x^i = 0$$

Changing the summation variable in the left hand sum to $i + 2$,

$$\sum_{i=0}^{\infty} [a_{i+2}(i+2)(i+1) + a_i \omega^2] x^i = 0$$

Comparing coefficients the result follows.

ii. The recurrence 'goes up in 2s' and even and odd terms depend respectively on a_0 and a_1. Thus,

$$a_{2n} = -\frac{\omega^2 a_{2n-2}}{2n(2n-1)} = \ldots = (-1)^n a_0 \frac{\omega^{2n}}{(2n)!}$$

a_{2n+1} follows similarly.

iii. Substituting into the power series, the solution is

$$\begin{aligned} y &= a_0 \sum_{i=0}^{\infty} (-1)^i \frac{\omega^{2i} x^{2i}}{(2i)!} + a_1 \sum_{i=0}^{\infty} (-1)^i \frac{\omega^{2i} x^{2i+1}}{(2i+1)!} \\ &= a_0 \sum_{i=0}^{\infty} (-1)^i \frac{(\omega x)^{2i}}{(2i)!} + a_1/\omega \sum_{i=0}^{\infty} (-1)^i \frac{(\omega x)^{2i+1}}{(2i+1)!} \\ &= a_0 \cos \omega x + (a_1/\omega) \sin \omega x \end{aligned}$$

Since $y = 1$ at $x = 0$, $a_0 = 1$. Since $Dy = -a_0 \omega \sin \omega x + a_1 \cos \omega x = a_1$ at $x = 0$, the result follows.

4. **(Exam question Q6C1452005)**

   Calculate the first three non-zero terms of the Maclaurin series for the function $\tan x$.

   Solution: The first six derivatives (starting at 0th) of $\tan x$ are:

$$\begin{aligned} \tan x \\ \sec^2 x \\ 2\sec^2 x \tan x \\ 6\sec^4 x - 4\sec^2 x \\ (24\sec^4 x - 8\sec^2 x)\tan x \\ (\ldots)' \tan x + (24\sec^4 x - 8\sec^2 x)\sec^2 x \end{aligned}$$

   which evaluate at $x = 0$ to: $0, 1, 0, 2, 0, 16$

   Maclaurin's series therefore starts:

$$\tan x = x + 2x^3/3! + 16x^5/5! + \ldots = x + x^3/3 + 2x^5/15 + \ldots$$

5. Verify numerically using Maclaurin's series that $\sin \pi/4 = \sin 9\pi/4$ to four decimal places. How many terms did you need, and why is this more than you needed in question 2?

   Solution: This time you need 13 terms, up to $x^{25}/25!$, when $x = 9\pi/4$. The error term is bounded above by $x^{26}/26! = 0.0000299932$, so guaranteed correct to 4 d.p. Leaving out the 13th term, we are not accurate to 4 d.p. (we get 0.7070), and the error is only bounded by $x^{24}/24! = 0.000390186$ which is insufficient.