Mathematical Methods: Tutorial sheet 6

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Assessed questions: 3, 4 and 5. Due: Monday 3 December 2007.

Complex numbers

1. Find the complex numbers z that satisfy the following equations:

   (a) $3z + (2 - 2i) = 1 + i$

   (b) $(1 + 3i)z - 3(1 - 2i) = 3 + 2i - 2(4 - i)z$

   Solution:

   (a) $3z + (2 - 2i) = 1 + i$, so $3z = -1 + 3i, z = -\frac{1}{3} + i$

   (b) $(1 + 3i + 8 - 2i)z = 3(1 - 2i) + 3 + 2i = 6 - 4i$, so

   $$z = \frac{6 - 4i}{9 + i} = \frac{(6 - 4i)(9 - i)}{82} = \frac{50 - 42i}{82} = \frac{25}{41} - \frac{21}{41}i$$

2. Write the complex number $i^{4000021}$ in the form $a + ib$.

   Solution: $i^{4000021} = i^{4(1000005)+1} = (i^4)^{1000005} i = i$

3. (a) Let X be the set of complex numbers z of the form $3e^{i\theta}$, where $-\pi/2 \leq \theta \leq \pi/2$. Plot X on the Argand diagram.

   (b) Let Y be the set of complex numbers z such that $|z - (1 + i)| = 1$. Plot Y on the Argand diagram.

   Solution:

   (a) Every $z \in X$ has modulus 3. The Arg ranges from $-\pi/2$ (negative imaginary axis) to $\pi/2$ (positive imaginary axis). So X is the right half circle, radius 3, centre O.

   (b) $|z - (1 + i)|$ is the distance from z to the point representing $1 + i$. Hence Y is the circle with centre $1 + i$ and radius 1.

4. Let $z = e^{i\theta}$. By expressing $\sin\theta$ in terms of z and $1/z$, expand $(\sin\theta)^4$ to show that

   $$\cos 4\theta = 8(\sin\theta)^4 + 4\cos 2\theta - 3$$

Solution: $\sin\theta = (z - 1/z)/2i$ so

$$\begin{aligned}(\sin\theta)^4 &= ((z-1/z)/2i)^4 \\ &= \frac{1}{16}[(z^4+z^{-4}) - 4(z^2+z^{-2}) + 6z^2z^{-2}] \\ &= \frac{1}{8}[\cos 4\theta - 4\cos 2\theta + 3]\end{aligned}$$

5. (a) Find the square roots of $-2\sqrt{2} - 2\sqrt{2}i$ and plot them on the Argand diagram.

   (b) Find all solutions $z \in \mathcal{C}$ to the equation $z^5 = -16\sqrt{2}(1+i)$ and plot them on the Argand diagram.

   Solution:

   (a) $-2\sqrt{2} - 2\sqrt{2}i = e^{i\pi}2^{3/2}(1+i) = e^{i\pi}2^{3/2}2^{1/2}e^{i(2k\pi+\pi/4)}$ for $k = 0, 1, \ldots$. So the square root is

   $$\begin{aligned}2[e^{i(2k\pi+5\pi/4)}]^{1/2} &= 2e^{i(k\pi+5\pi/8)} \\ &= 2e^{5\pi i/8},\ \ 2e^{13\pi i/8}\end{aligned}$$

   Roots are symmetrical in the origin (Args differ by π).

   (b) $z^5 = 2^5e^{i(\pi+\pi/4+2k\pi)} = 2^5e^{i(5\pi/4+2k\pi)}$ and so

   $$z = 2e^{i(\pi/4+2k\pi/5)}$$

   for $k = 0, 1, 2, 3, 4$. That is:

   $$z = 2e^{\pi i/4} = \sqrt{2}(1+i), 2e^{13\pi i/20}, 2e^{21\pi i/20}, 2e^{29\pi i/20}, 2e^{37\pi i/20}$$

   Roots lie on the circle, centre O, radius 2, separated by an angle $2\pi/5$, starting at Arg $\pi/4$.

6. **(Exam question Q6C1452005)**

   (a) Express the complex number $\sqrt{3} + i$ in polar form, i.e. in the form $re^{i\theta}$ for certain positive real numbers r and θ.
   Solution: Modulus is $r = \sqrt{3+1} = 2$. Argument is $\theta = \arctan 1/\sqrt{3}[= \pi/6$

   (b) Using the fact that $\sin(A+B) = \sin A\cos B + \cos A\sin B$ and a corresponding result for $\cos(A+B)$, which you should state, prove that:

   $$r_1(\cos\theta_1 + i\sin\theta_1)r_2(\cos\theta_2 + i\sin\theta_2) = r_1r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$$

   Solution:

   $$r_1(\cos\theta_1 + i\sin\theta_1)r_2(\cos\theta_2 + i\sin\theta_2) =$$

   $$r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] = r_1r_2e^{i(\theta_1+\theta_2)}$$

(c) Show that

i. $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$
Solution:

$$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$$

by de Moivre's Theorem. Equating the real parts,

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta = 4\cos^3\theta - 3\cos\theta$$

ii. $8\cos^4\theta = \cos 4\theta + 4\cos 2\theta + 3$
Note: You may wish to use the fact that $2\cos\theta = e^{i\theta} + e^{-i\theta}$.
Solution:

$$16\cos^4\theta = e^{i4\theta} + 4e^{i3\theta}e^{-i\theta} + 6e^{i2\theta}e^{-i2\theta} + 4e^{i\theta}e^{-i3\theta} + e^{-i4\theta} =$$

$$2[\cos 4\theta + 4\cos 2\theta + 3]$$

(d) Find *all* the roots (for z) of the equation $z^4 = 8(\sqrt{3} + i)$, and *write them in Cartesian form*, i.e. in the form $a + ib$ for real numbers a and b.

Solution: In polar form, $z^4 = 16e^{i\pi/6} = 16e^{i\pi(2n+1/6)}$ for $n = 0, 1, 2, \ldots$. Thus, $z = 2e^{i\pi(n/2+1/24)}$ are distinct roots for $n = 0, 1, 2, 3$. This gives:

$$\begin{aligned}
z_1 &= 2\cos\pi/24 + i2\sin\pi/24 \\
z_2 &= 2\cos 13\pi/24 + i2\sin 13\pi/24 = -2\cos 11\pi/24 + i2\sin 11\pi/24 \\
z_3 &= 2\cos 25\pi/24 + i2\sin 25\pi/24 = -2\cos\pi/24 - i2\sin\pi/24 \\
z_4 &= 2\cos 37\pi/24 + i2\sin 37\pi/24 = 2\cos 11\pi/24 - i2\sin 11\pi/24
\end{aligned}$$

Alternatively: *any* root multiplied by $1, -1, i, -i$ is OK.

7. **(Exam question Q6C1452006)**

(a) Express the complex number $12 + 5i$ in polar form, i.e. in the form $re^{i\theta}$ for certain positive real numbers r and θ.

Solution: Modulus is $r = \sqrt{144 + 25} = 13$. Argument is $\arctan 5/12 = \phi$, say. Number is then $13e^{i\phi}$

(b) The cube roots of unity, i.e. complex numbers which give result 1 when cubed, are $1, \omega$ and ω'.

i. Obtain representations of these three roots in both Cartesian and polar form.
Solution: Roots are $e^{2\pi ki/3}$ for $k = 0, 1, 2$, corresponding to $1, \omega, \omega'$. 1 is already in both Cartesian and polar form. $\omega = \cos 2\pi i/3 + i\sin 2\pi i/3 = -1/2 + \sqrt{3}i/2$, $\omega' = \cos 4\pi i/3 + i\sin 4\pi i/3 = -1/2 - \sqrt{3}i/2$.

ii. Verify that the roots not equal to 1 are the squares of each other.
Solution: The square of the second root ($k = 1$) is $(e^{2\pi i/3})^2 = e^{4\pi i/3}$, i.e. the third root. The square of the third root ($k = 2$) is $(e^{4\pi i/3})^2 = e^{8\pi i/3} = e^{2\pi i/3}$, i.e. the second root.

iii. Verify that the roots not equal to 1 add up to -1.
Solution: Either note that ω, ω' are complex conjugates with real part $-1/2$ so sum to -1, or observe that the sum of all 3 roots is the coefficient of z^2 in the equation $z^3 - 1 = 0$, i.e. 0.

(c) Assuming that $e^{i\theta} = \cos\theta + i\sin\theta$ for real number θ, prove that

i.
$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$
for all real n.
Solution: $\cos n\theta + i\sin n\theta = e^{in\theta} = \left(e^{i\theta}\right)^n = (\cos\theta + i\sin\theta)^n$

ii. Show that $\sin 4\theta = 4\cos\theta\sin\theta - 8\cos\theta\sin^3\theta$
Solution:
$$\cos 4\theta + i\sin 4\theta = (\cos\theta + i\sin\theta)^4$$
Equating the imaginary parts,
$$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta = 4\cos\theta\sin\theta(1-\sin^2\theta) - 4\cos\theta\sin^3\theta$$
so the result follows.

(d) A rock star is worried about the quality of the output signal from one of his amplifiers. Therefore he monitors the signal to look for spurious noise using Fourier analysis. In the course of estimating certain coefficients, he needs to evaluate the integral
$$I = \int_0^\pi e^{-\theta}\cos 2n\theta \mathrm{d}\theta$$
for integer values n. Using the fact that $\cos 2n\theta$ is the real part of $e^{2in\theta}$, or otherwise, show that $I = \frac{1-e^{-\pi}}{1+4n^2}$.
Solution: $I = Re\left(\int_0^\pi e^{(2in-1)\theta}\mathrm{d}\theta\right) = Re\left[-\frac{e^{(2in-1)\theta}}{1-2in)}\right]_0^\pi = Re\left(\frac{(1+2in)(1-e^{(2in-1)\pi})}{1+4n^2}\right)$ so result follows.

Bounds and Limits

8. What is the *least upper bound* (**sup**remum) and *greatest lower bound* (**inf**imum) of the following sets of numbers:

(a) $\{x \in \mathbb{N} \mid 1 \le x^2 \le 29\}$
(b) $\{x \in \mathbb{Q} \mid 1 \le x^2 \le 29\}$
(c) $\{x \in \mathbb{R} \mid 1 \le x^2 \le 29\}$

in each case, are the infimum and supremum in the given set?

Solution:

(a) The set is $\{1, 2, 3, 4, 5\}$ which is finite, so the inf is 1 (minimum element), the sup is 5 (maximum element).

(b) The set consists of all rationals in $[-\sqrt{29}, -1]$ and $[1, \sqrt{29}]$, i.e. it is $\mathbb{Q} \cap ([-\sqrt{29}, -1] \cup [1, \sqrt{29}])$. The inf is $-\sqrt{29}$, the sup is $\sqrt{29}$, neither is in the set since they are not rational.

(c) Same as previous, but both inf and sup are in the set.

9. (a) Prove *rigorously* that, for $\alpha > 0$, the sequence $a_n = n^{-\alpha}$ converges to zero as n tends to infinity. In other words, use ϵ and N, an appropriate value of which should be given as a function of ϵ.

(b) Hence use the trapping or 'sandwich' theorem to show that

$$\frac{n!}{n^n} \to 0$$

as $n \to \infty$

(c) What happens to

$$\frac{n!}{n^p}$$

as $n \to \infty$, where p is a fixed integer?

Solution:

(a) Pick $\epsilon > 0$. Then we want an N s.t. $a_N = N^{-\alpha} \leq \epsilon$ so that then all $a_n < \epsilon$ for $n > N$ since a_n is decreasing. Thus we want $N^\alpha \geq 1/\epsilon$ so any integer greater than $\epsilon^{-1/\alpha}$ will do the job, e.g. choose

$$N = \lceil \epsilon^{-1/\alpha} \rceil$$

(b)

$$\frac{n!}{n^n} = 1 \times (1 - \frac{1}{n}) \times (1 - \frac{2}{n}) \times \ldots \times \frac{1}{n} < \frac{1}{n}$$

But $\frac{1}{n} > 0 \ \forall n > 0$ and the given sequence is trapped.

(c) Diverges since, for $n > p$,

$$\frac{n!}{n^p} > (n-p)! \frac{(n-p+1)^p}{n^p} = (n-p)! \left(1 - \frac{p-1}{n}\right)^p > (n-p)!(1/2)^p$$

for $n > 2(p-1)$.

10. Investigate the convergence properties (either converges or diverges) of each of the following sequences $a_n, n = 1, 2, \ldots$

(a) $a_n = \frac{3n^2+2n+4}{5n^2-7n+1}$

(b) $a_n = \frac{3n^3+2n+4}{5n^2-7n+1}$

(c) $a_n = n + (-1)^n n^2$

(d) $a_n = \frac{1}{n^2} + \frac{1}{(n+1)^2} + \ldots + \frac{1}{(2n)^2}$

Solution:

(a) Divide top and bottom by n^2 to get: $a_n = \frac{3+2/n+4/n^2}{5-7/n+1/n^2}$. As $n \to \infty$, $a_n \to 3/5$

(b) Similarly, $a_n = \frac{3n+2/n+4/n^2}{5-7/n+1/n^2} > 3n/5$ so diverges.

(c) $n^2 - n \leq |a_n| \leq n^2 + n$ (because $||x| - |y|| \leq |x \pm y| \leq |x| + |y|$) so a_n diverges as fast as n^2, but alternating sign makes it oscillate.

(d) $a_n < n/n^2 = 1/n$ so a_n converges by trapping.

11. Consider the decimal $D = 0.d_1d_2d_3 \ldots \in [0,1)$, as in lectures, and the sequence of *finite decimals* $a_n = 0.d_1d_2 \ldots d_n,\ n = 1, 2, \ldots$. (D is also equivalent to a finite decimal if $d_k = 0\ \ \forall k > K$ for some integer $K > 0$.) Show that:

(a) Each a_n is rational for $n > 0$.

(b) The sequence a_n is increasing and bounded above.

(c) D is a real number.

Conversely, let E be a real number in $[0,1)$. Then successively choose the rational decimal numbers $b_n = 0.e_1 \ldots e_n,\ n = 1, 2, \ldots$ such that $b_n \leq E$ and $b_n + 10^{-n} > E$ – i.e. b_n is the biggest such decimal. Prove that

$$\lim_{n\to\infty} b_n = E$$

and hence that E is a decimal.

Solution:

(a) Obvious since a_n is a finite decimal and so a sum of inverse powers of 10, each of which is rational.

(b) Obvious since we add a multiple of 10^{-n} to get a_n from a_{n-1} $(n > 1)$. (Strictly, we should say 'non-decreasing' as the multiple may be zero.) An upper bound is 1.

(c) D is real by the fundamental axiom.

E is clearly the supremum of the sequence b_n (simple proof by contradiction since $E - b_n \leq 10^{-n}$ for arbitrarily large n). b_n is increasing and bounded above and therefore has a limit by the fundamental axiom. That limit is E because $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $|b_N - E| < \epsilon$ since E is the supremum. Hence $|b_n - E| < \epsilon\ \forall n \geq N$.