Mathematical Methods: Tutorial sheet 7

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29 November 2007

Assessed questions: 1, 4(a) and 5(a,b,f). Due: Monday 10 December 2007.

More on convergence

1. **(Exam question Q5C1452005)** Use the ϵ-N method to prove rigorously that $x^n \to 0$ as $n \to \infty$ if $|x| < 1$. That is, given $\epsilon > 0$ find a number $N(\epsilon)$ (which depends on ϵ) for which $m > N(\epsilon) \Rightarrow |x^m| < \epsilon$.

   Solution: x^n is decreasing and $|x^N| = |x|^N = \epsilon$ when $N = \log_{|x|} \epsilon$. So $N(\epsilon) = \log_{|x|} \epsilon$ – or anything bigger – does the job.

2. Prove from first principles (i.e. use the ϵ-N method) that

   $$\lim_{n\to\infty} \cos^2(1/n) = 1$$

   by first:

   - using a geometrical argument to show that $\sin\theta < \theta$ for $0 < \theta < \pi/2$ (θ measured in radians).
   - deducing that $cos^2\theta > 1 - \theta^2$ for $0 < \theta < \pi/2$

   Solution:

   - Draw a sector of a circle with radius 1 and angle θ at the centre. The length of the arc is θ. Drop a perpendicular from one radius to the other; its length is $\sin\theta$ which is less than the arc length.
   - $\cos^2\theta = 1 - \sin^2\theta$ and so $\cos^2\theta > 1 - \theta^2$

   Thus, remembering that $|\cos\theta| \le 1$, $|\cos^2\theta - 1| < |\theta^2|$. (Actually easier, just use $|\cos^2\theta - 1| = |\sin^2\theta| < |\theta^2|$.) Now proceed as in part a of question 2 on exercise sheet 5, with $\alpha = 2$.

   You can also use the fact that $1 - \cos\theta = 2\sin^2(\theta/2) < \theta^2/2$ to show similarly that

   $$\lim_{n\to\infty} \cos(1/n) = 1$$

3. The 'limit superior' of a sequence $a_1, a_2, \ldots$ is written $\limsup_{n\to\infty} a_n$ and defined as follows:

- let $u_n = \sup_{n \le k < \infty} a_k$
- then $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} u_n$

Similarly, the 'limit inferior' $\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf_{n \le k < \infty} a_k$.

(a) Show that the sequence $u_1, u_2, \ldots$ is non-increasing, i.e. $u_1 \ge u_2 \ge u_3 \ldots$.

(b) Prove that a bounded sequence has a lim sup and a lim inf which are both finite.

(c) If a sequence is unbounded but neither diverges to $+\infty$ or $-\infty$, does it have a lim sup or a lim inf?

(d) If a sequence $a_1, a_2, \ldots$ has

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = x$$

does $\lim_{n\to\infty} a_n$ exist and, if so, what is it and why?

Solution:

(a) For all $n \in \mathbb{N}$, $u_n = \sup\{a_n, a_{n+1}, \ldots\} = \max\Big(a_n, \sup\{a_{n+1}, a_{n+2}, \ldots\}\Big) \ge \sup\{a_{n+1}, a_{n+2}, \ldots\} = u_{n+1}$.

(b) If $a_1, a_2, \ldots$ is bounded, then the non-increasing sequence $u_1, u_2, \ldots$ is bounded and so converges (to u say) by the Fundamental Axiom. Hence $\limsup a_n$ exists and is equal to u. Similarly for $\liminf a_n$.

(c) To prove $\{u_n, u_{n+1}, \ldots\}$ is bounded below for all $n \in \mathbb{N}$ it is sufficient to find *any* constant L *s.t.* $a_k > L$ for *infinitely many* k. In other words, we only need a real number L which is less than infinitely many sequence elements – *i.e. there are always some bigger than L no matter how far along the sequence we go.* A lower bound on the *whole* sequence is *not* necessary since the suprema u_n will clearly be bigger than L, being bigger than the particular a_k chosen by definition.

For example, the sequence defined by $a_n = 1 + 1/n$ if n is odd and $-n$ if n is even has no lower bound but has lim sup equal to 1, the limit of the sequence of upper bounds $u_n = 2, 4/3, 4/3, 6/5, 6/5, \ldots$.

So the answer is 'yes'!

(d) Let $l_n = \inf_{n \le k < \infty} a_k$. Then, $\forall n \in \mathbb{N}$, $l_n \le a_n \le u_n$. Therefore $a_1, a_2, \ldots$ converges and $\lim_{n\to\infty} a_n = x$ by the sandwich theorem.

4. Given that the sequences $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ converge to limits a and b respectively, show that:

   (a) The sequence $c_n = a_n + b_n$ converges to $a + b$;

   (b) If $a > 0$, the sequence $d_n = 1/a_n$ converges to $1/a$.

   Solution:

   (a) Given $\epsilon > 0, \exists N_1, N_2$ $s.t.$ $\forall n > N_1, |a_n - a| < \epsilon$ and $\forall n > N_2, |b_n - b| < \epsilon/2$. Thus, for $n > \max(N_1, N_2)$,

   $$|a_n + b_n - a - b| \leq |a_n - a| + |b_n - b| < \epsilon$$

   (b) Given $\epsilon > 0, \exists N_1$ $s.t.$ $\forall n > N_1, |a_n - a| < \epsilon$.
   Also, $\exists N_2$ $s.t.$ $|a_n - a| < |a|/2$ for $n > N_2$.
   Thus $|1/a_n - 1/a| = |a_n - a|/|a_n a| < 2|a_n - a|/|a^2| < 2\epsilon/|a^2|$ for $n > \max(N_1, N_2)$.

5. Investigate the convergence properties (either converges or diverges for different values of the parameter x, if present) of each of the following series $\sum\limits_{n=1}^{\infty} a_n$, where

   (a) $a_n = \frac{1}{3n+2}$

   (b) $a_n = \frac{1}{1+n^2}$

   (c) $a_n = n!x^n$

   (d) $a_n = \left(\frac{x}{n}\right)^n$

   (e) $a_n = \sin\frac{\pi}{n}$

   (f) $a_n = \frac{\sin nx}{n^2}$

   Note: $2x/\pi < \sin x < x \quad (0 < x < \pi/2)$

   Solution: Let the partial sum $S_n = \sum_{i=1}^{n} a_i$

   (a) $a_i > 1/4i$ for $i > 2$ and so $S_n > a_1 + a_2 + 0.25\sum_{i=3}^{n} 1/i$ which diverges. So series diverges by comparison test.

   (b) $a_i < 1/i^2$ so $S_n < \sum_{i=1}^{n} 1/i^2$ which converges. Hence series converges.

   (c) $a_{n+1}/a_n = (n+1)x > 1$ for $n \geq 1/x$. So series diverges for all $x > 0$ by D'Alembert's ratio test. Similarly for $x < 0$, when series oscillates.

   (d) $\frac{|x|}{n+1}\left(\frac{n}{n+1}\right)^n < \frac{|x|}{n+1} < 1$ for $n > |x|$. So series converges absolutely, and hence converges, for all x.

   (e) For $n > 2, a_n > 2/n$ so series diverges by comparison test, as in part (a).

(f) For $|a_n| < 1/n^2$ so series converges absolutely, and hence converges, by comparison test, as in part (b).

6. Consider the quadratic equation

$$x^2 - 2x - 1 = 0$$

which has roots $1 \pm \sqrt{2}$. [Usually in 'real' problems we don't know the answer, but will know a valid range of values, e.g. in the interval [2,3].] The aim of this exercise is to compute $\sqrt{2}$ iteratively. Define the recurrence:

$$a_{n+1} = \sqrt{2a_n + 1}, \quad a_1 = 1$$

(a) Show that, if the recurrence converges, its limit is one of the roots of the quadratic equation above;

(b) Prove by induction (if you know how) that $0 < a_n < 3$ for all $n \in \mathbb{N}$ [if you don't know induction, just show that $0 < a_n < 3 \Rightarrow 0 < a_{n+1} < 3$];

(c) Hence show that if the iteration converges, its limit is the positive root, $l = 1 + \sqrt{2}$;

(d) Calculate $a_{n+1}^2 - l^2 = (a_{n+1} + l)(a_{n+1} - l)$ and show that

$$a_{n+1} - l = \frac{2(a_n - l)}{a_{n+1} + l}$$

(e) Hence show that $a_n \to l$ as $n \to \infty$;

(f) Compute $\{a_n - 1 \mid 1 \leq n \leq 6\}$ and compare with $\sqrt{2}$.

Solution:

(a) If it converges, then in the limit we can set $a_n = a_{n+1} = l$ (no need for an ϵ-argument here, although to be strictly rigorous it can be done easily). Thus $l = \sqrt{2l + 1}$ and squaring gives $l^2 = 2l + 1$ which is the given quadratic.

(b) $a_1 = 1$ and so $0 < a_1 < 3$ is true. Suppose $0 < a_n < 3$ holds for $n \geq 1$. Then

$$\sqrt{2 \times 0 + 1} \leq a_{n+1} \leq \sqrt{2 \times 3 + 1} = \sqrt{7}$$

Thus certainly $0 \leq a_{n+1} \leq 3$. Actually, we have all $a_n > 1$.

(c) *If* the iteration converges, its limit must be ≥ 0 because every $a_n \geq 0$. (Again, from first principles, you would say every large enough n is maximum ϵ away from the limit, so the limit can't be less than 0 ...... but you don't need to say this unless it's *specfically asked for.*) This excludes the negative root $1 - \sqrt{2}$.

(d) By definition of the recurrence and because $l^2 = 2l + 1$,

$$a_{n+1}^2 - l^2 = (\sqrt{2a_n + 1})^2 - 2l - 1 = 2(a_n - l)$$

Hence,

$$(a_{n+1} + l)(a_{n+1} - l) = 2(a_n - l)$$

(e) Since $a_{n+1} > 0$ and the positive root $l > 2.4$ (because $\sqrt{2} > 1.4$),

$$a_{n+1} - l < \frac{2}{2.4}(a_n - l)$$

and so $a_{n+1} - l$ converges to 0 by the ratio test. Thus $a_n \to l$ as $n \to \infty$.

(f) $a_1 = 1, a_2 = \sqrt{2+1} = \sqrt{3} = 1.732051, a_3 = \sqrt{2\sqrt{3}+1} = 2.11284, a_4 = \sqrt{2 \times 2.11284 + 1} = 2.28598, a_5 = 2.36050, a_6 = 2.39186$ so the set is

$$\{0, 1.732051, 1.11284, 1.28598, 1.36050, 1..39186\}$$

and the next 5 elements are:

$$\{1.40494, 1.41037, 1.41262, 1.41355, 1.41394\}$$

7. **Compare question 2 on Logic Exercises 7.**
Let L be the signature consisting of constant-symbols written as the underlined decimal numbers (hence including the positive integer-symbols $\underline{1}, \underline{2}, \underline{3}, \ldots$), binary relation symbols $<, >, \leq, \geq$ and binary function symbols $+, \times$. Let N be the structure whose domain consists of the real numbers $\mathbb{R}$, with the symbols of L interpreted in the natural way. The formula $\exists v(x = v \times v)$ expresses that x is non-negative, for example.

(a) In the same kind of way, write down a first-order L-formula expressing that the sequence of real numbers $a_1, a_2, \ldots$ converges to a real number as n tends to infinity.

(b) Try to express the same statement in the same signature in a different way.

(c) Using the rules for negating quantified expressions in predicate logic, define the condition for a sequence $a_1, a_2, \ldots$ *not* to converge – i.e. find the negation of the usual definition of convergence. What does this condition mean in terms of the existence (or otherwise) of 'tram-lines'?

Solution:

(a) $\exists l(\forall \epsilon(\exists N(\forall n(n > N \to (a_n - l) \times (a_n - l) < \epsilon \times \epsilon))))$.

(b) Is there a different way?

(c) $\forall l(\exists\epsilon(\forall N(\exists n(\neg(n > N \rightarrow (a_n - l) \times (a_n - l) < \epsilon \times \epsilon))))) =$
$\forall l(\exists\epsilon(\forall N(\exists n(n > N \wedge \neg((a_n - l) \times (a_n - l) < \epsilon \times \epsilon)))))$
So for every real number l, there is a number ϵ for which there are infinitely many a_n lying outside the tramlines at $l \pm \epsilon$ — there is no 'last' such a_n.

8. Given real functions g, f which are continuous on (a, b) and on the image of g, $\{z = g(x) \mid x \in (a, b)\}$, respectively, prove rigorously that the function h defined by $h(x) = f(g(x))$ is continuous on (a, b).

   Hint: You have to prove that, given a point $x_0 \in (a, b)$, $\forall\epsilon > 0, \exists\delta > 0$ s.t. $|x - x_0| < \delta \Rightarrow |h(x) - h(x_0)| < \epsilon$ and find δ as a function of ϵ, using corresponding quantities δ_f, δ_g relating to the continuity of f, g.

   Solution: For h to be continuous, as x get very close to x_0, then $h(x)$ must get very close to $h(x_0)$ – this is what the hint says in a rigorous way.

   Given a point $x_0 \in (a, b)$ and a real number $\epsilon > 0, \exists\delta_f > 0$ s.t.

$$|g(x) - g(x_0)| < \delta_f \Rightarrow |f(g(x)) - f(g(x_0))| < \epsilon$$

   since f is continuous at $g(x_0)$.

   But $\exists\delta_g > 0$ s.t. $|x - x_0| < \delta_g \Rightarrow |g(x) - g(x_0)| < \delta_f$ since g is continuous at x_0. So if we choose $\delta \leq \delta_g$, we have

$$|x - x_0| < \delta \Rightarrow |g(x) - g(x_0)| < \delta_f \Rightarrow |h(x) - h(x_0)| < \epsilon$$

   as required.