Mathematical Methods: Tutorial sheet 8

Peter Harrison

1 December 2005

All questions are unassessed, but very useful!.

Fixpoints and continuity

1. Consider the quadratic equation

$$x^2 - 2x - 1 = 0$$

which has roots $1 \pm \sqrt{2}$. [Usually in 'real' problems we don't know the answer, but will know a valid range of values, e.g. in the interval [2,3].] The aim of this exercise is to compute $\sqrt{2}$ iteratively. Define the recurrence:

$$a_{n+1} = \sqrt{2a_n + 1}, \quad a_1 = 1$$

(a) Show that, if the recurrence converges, its limit is one of the roots of the quadratic equation above;

(b) Prove by induction (if you know how) that $0 < a_n < 3$ for all $n \in \mathbb{N}$ [if you don't know induction, just show that $0 < a_n < 3 \Rightarrow 0 < a_{n+1} < 3$];

(c) Hence show that if the iteration converges, its limit is the positive root, $l = 1 + \sqrt{2}$;

(d) Calculate $a_{n+1}^2 - l^2 = (a_{n+1} + l)(a_{n+1} - l)$ and show that

$$a_{n+1} - l = \frac{2(a_n - l)}{a_{n+1} + l}$$

(e) Hence show that $a_n \to l$ as $n \to \infty$;

(f) Compute $\{a_n - 1 \mid 1 \leq n \leq 6\}$ and compare with $\sqrt{2}$.

Solutions:

(a) If it converges, then in the limit we can set $a_n = a_{n+1} = l$ (no need for an ϵ-argument here, although to be strictly rigorous it can be done easily). Thus $l = \sqrt{2l + 1}$ and squaring gives $l^2 = 2l + 1$ which is the given quadratic.

(b) $a_1 = 1$ and so $0 < a_1 < 3$ is true. Suppose $0 < a_n < 3$ holds for $n \geq 1$. Then

$$\sqrt{2 \times 0 + 1} \leq a_{n+1} \leq \sqrt{2 \times 3 + 1} = \sqrt{7}$$

Thus certainly $0 \leq a_{n+1} \leq 3$. Actually, we have all $a_n > 1$.

(c) *If* the iteration converges, its limit must be ≥ 0 because every $a_n \geq 0$. (Again, from first principles, you would say every large enough n is maximum ϵ away from the limit, so the limit can't be less than 0 ...... but you don't need to say this unless it's *specfically asked for*.) This excludes the negative root $1 - \sqrt{2}$.

(d) By definition of the recurrence and because $l^2 = 2l + 1$,

$$a_{n+1}^2 - l^2 = (\sqrt{2a_n + 1})^2 - 2l - 1 = 2(a_n - l)$$

Hence,

$$(a_{n+1} + l)(a_{n+1} - l) = 2(a_n - l)$$

(e) Since $a_{n+1} > 0$ and the positive root $l > 2.4$ (because $\sqrt{2} > 1.4$),

$$a_{n+1} - l < \frac{2}{2.4}(a_n - l)$$

and so $a_{n+1} - l$ converges to 0 by the ratio test. Thus $a_n \to l$ as $n \to \infty$.

(f) $a_1 = 1, a_2 = \sqrt{2+1} = \sqrt{3} = 1.732051, a_3 = \sqrt{2\sqrt{3}+1} = 2.11284, a_4 = \sqrt{2 \times 2.11284 + 1} = 2.28598, a_5 = 2.36050, a_6 = 2.39186$ so the set is

$$\{0, 1.732051, 1.11284, 1.28598, 1.36050, 1..39186\}$$

and the next 5 elements are:

$$\{1.40494, 1.41037, 1.41262, 1.41355, 1.41394\}$$

2. Given real functions g, f which are continuous on (a, b) and on the image of g, $\{z = g(x) \mid x \in (a, b)\}$, respectively, prove rigorously that the function h defined by $h(x) = f(g(x))$ is continuous on (a, b).

   Hint: You have to prove that, given a point $x_0 \in (a, b)$, $\forall \epsilon > 0, \exists \delta > 0$ s.t. $|x - x_0| < \delta \Rightarrow |h(x) - h(x_0)| < \epsilon$ and find δ as a function of ϵ, using corresponding quantities δ_f, δ_g relating to the continuity of f, g.

   Solutions: For h to be continuous, as x get very close to x_0, then $h(x)$ must get very close to $h(x_0)$ – this is what the hint says in a rigorous way.

   Given a point $x_0 \in (a, b)$ and a real number $\epsilon > 0, \exists \delta_f > 0$ s.t.

$$|g(x) - g(x_0)| < \delta_f \Rightarrow |f(g(x)) - f(g(x_0))| < \epsilon$$

   since f is continuous at $g(x_0)$.

But $\exists \delta_g > 0$ s.t. $|x - x_0| < \delta_g \Rightarrow |g(x) - g(x_0)| < \delta_f$ since g is continuous at x_0. So if we choose $\delta \leq \delta_g$, we have

$$|x - x_0| < \delta \Rightarrow |g(x) - g(x_0)| < \delta_f \Rightarrow |h(x) - h(x_0)| < \epsilon$$

as required.

3. **Refer to question 3 on Logic Exercises 7 (no. 6 on cate).**
Let L be the signature consisting of constant-symbols written as the underlined decimal numbers (hence including the positive integer-symbols $\underline{1}, \underline{2}, \underline{3}, \ldots$), binary relation symbols $<, >, \leq, \geq$ and binary function symbols $+, \times$. Let N be the structure whose domain consists of the real numbers $\mathbb{R}$, with the symbols of L interpreted in the natural way. The formula $\exists v(x = v \times v)$ expresses that x is non-negative, for example.

   (a) In the same kind of way, write down a first-order L-formula expressing that the sequence of real numbers $a_1, a_2, \ldots$ converges to a real number as n tends to infinity.

   (b) Try to express the same statement in the same signature in a different way.

   (c) Using the rules for negating quantified expressions in predicate logic, define the condition for a sequence $a_1, a_2, \ldots$ *not* to converge – i.e. find the negation of the usual definition of convergence. What does this condition mean in terms of the existence (or otherwise) of 'tramlines'?

   Solutions:

   (a) $\exists l(\forall \epsilon(\exists N(\forall n(n > N \rightarrow (a_n - l) \times (a_n - l) < \epsilon \times \epsilon))))$.

   (b) Is there a different way?

   (c) $\forall l(\exists \epsilon(\forall N(\exists n(\neg(n > N \rightarrow (a_n - l) \times (a_n - l) < \epsilon \times \epsilon))))) =$
   $\forall l(\exists \epsilon(\forall N(\exists n(n > N \wedge \neg((a_n - l) \times (a_n - l) < \epsilon \times \epsilon)))))$
   So for every real number l, there is a number ϵ for which there are infinitely many a_n lying outside the tramlines at $l \pm \epsilon$ — there is no 'last' such a_n.