COURSE 336
PERFORMANCE ANALYSIS
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Coursework 3
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Assessed
due date: see CATE

A small shop has space for one customer only. If the shop is empty, then in the next time-unit, either a customer arrives with probability α or the shop remains empty. If a customer has arrived, then the shop is full, and could either stay full in the next time-unit or the customer could leave the shop empty with probability β.

1. Describe the behaviour of the shop as a Discrete Markov Chain with two states $1, 2$ representing the two states of the shop: empty and full. Define the one step probability matrix $\mathbf{Q}$.

2. Write down the state probability at time 0, assuming that initially the Markov Chain is in the state empty. Calculate $q_{12}^{(2)}$ and $q_{11}^{(2)}$.

3. Write down the state probability at time 0, assuming that initially the Markov Chain is in the state full. Calculate $q_{12}^{(2)}$ and $q_{11}^{(2)}$.

4. Given the matrix $M = \begin{pmatrix} 1 & \alpha \\ 1 & -\beta \end{pmatrix}$ show that $\mathbf{Q}M = M \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}$. where $\omega = 1 - \alpha - \beta$. Hence show that, for $n \geq 0$

$$\mathbf{Q}^n M = M \begin{pmatrix} 1 & 0 \\ 0 & \omega^n \end{pmatrix}.$$

5. Verify that the inverse of the matrix M is $\frac{1}{\alpha+\beta} \begin{pmatrix} \beta & \alpha \\ 1 & -1 \end{pmatrix}$. Then show that:

$$\mathbf{Q}^n = \frac{1}{\alpha + \beta} \begin{pmatrix} \beta + \alpha\omega^n & \alpha(1-\omega^n) \\ \beta(1-\omega^n) & \alpha + \beta\omega^n \end{pmatrix}.$$

6. If $-1 < \omega < 1$ and $\mathbf{Q}^\infty$ is the limit of $\mathbf{Q}^n$ as $n \to \infty$, show that the rows of $\mathbf{Q}^\infty$ are the same $p = (p_1, p_2)$ and satisfy $p = p\mathbf{Q}$.

   For which values of α, β does $\mathbf{Q}^n$ not converge $n \to \infty$? What property does the Markov chain exhibit? What is the significance of case $\omega = 1$?