COURSE 336
PERFORMANCE ANALYSIS
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Solution to assessed exercises

Solutions

A small shop has space for one customer only. If the shop is empty, then in the next time-unit, either a customer arrives with probability α or the shop remains empty. If a customer has arrived, then the shop is full, and could either stay full in the next time-unit or the customer could leave the shop empty with probability β.

1. Describe the behaviour of the shop as a Discrete Time Markov Chain with two states $1, 2$ representing the two states of the shop: empty and full. Define the one-step transition probability matrix $\mathbf{Q}$.

   Solution 1.1 *The one-step transition probability matrix* $\mathbf{Q}$ *is defined as follows:*

$$\mathbf{Q} = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

2. Write down the state probability vector at time 0, assuming that initially the Markov Chain is in the empty state. Calculate $q_{12}^{(2)}$ and $q_{11}^{(2)}$.

   Solution 1.2 *The state probability vector when the state is empty (for sure) is* $(1\ 0)$.

   We calculate $q_{12}^{(2)}$ *as follows:*

$$\begin{aligned} q_{12}^{(2)} &= P(X_2 = 2|X_0 = 1) \\ &= P(X_2 = 2|X_1 = 1)P(X_1 = 1|X_0 = 1) + P(X_2 = 2|X_1 = 2)P(X_1 = 2|X_0 = 1) \\ &= \alpha(1-\alpha) + \alpha(1-\beta) \end{aligned}$$

   Similarly, we calculate $q_{11}^{(2)}$ *as follows:*

$$\begin{aligned} q_{11}^{(2)} &= P(X_2 = 1|X_0 = 1) \\ &= P(X_2 = 1|X_1 = 1)P(X_1 = 1|X_0 = 1) + P(X_2 = 1|X_1 = 2)P(X_1 = 2|X_0 = 1 \\ &= (1-\alpha)^2 + \alpha\beta) \end{aligned}$$

   Alternatively, we simply calculate $\mathbf{Q}^2$

$$\mathbf{Q}^2 = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix} \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix} = \begin{pmatrix} (1-\alpha)^2 + \alpha\beta & \alpha(1-\alpha) + \alpha(1-\beta) \\ \beta(1-\alpha) + \beta^2 & \alpha\beta + (1-\beta)^2 \end{pmatrix}$$

   Reading from the matrix above we have: $q_{12}^{(2)} = \alpha(1-\alpha) + \alpha(1-\beta)$ *and* $q_{11}^{(2)} = (1-\alpha)^2 + \alpha\beta$.

3. Write down the state probability vector at time 0, assuming that initially the Markov Chain is in the full state. Calculate $q_{12}^{(2)}$ and $q_{11}^{(2)}$.

   Solution 1.3 *The state probability vector when the state is full is* (0 1). *The rest is identical to the previous part.*

4. Given the matrix $M = \begin{pmatrix} 1 & \alpha \\ 1 & -\beta \end{pmatrix}$ show that $\mathbf{Q}M = M \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}$. where $\omega = 1 - \alpha - \beta$. Hence show that, for $n \geq 0$

$$\mathbf{Q}^n M = M \begin{pmatrix} 1 & 0 \\ 0 & \omega^n \end{pmatrix}.$$

   Solution 1.4

$$\begin{aligned}
\mathbf{Q}M = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 1 & -\beta \end{pmatrix} &= \begin{pmatrix} (1-\alpha)+\alpha & (1-\alpha)\alpha - \alpha\beta \\ \beta + (1-\beta) & \alpha\beta - (1-\beta)\beta \end{pmatrix} \\
&= \begin{pmatrix} 1 & \alpha(1-\alpha-\beta) \\ 1 & -\beta(1-\alpha-\beta) \end{pmatrix}
\end{aligned}$$

$$M \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} = \begin{pmatrix} 1 & \alpha\omega \\ 1 & -\beta\omega \end{pmatrix}$$

   Now, observing that $\mathbf{Q} = M \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} M^{-1}$, we prove by induction that for all $n \geq 0$, $\mathbf{Q}^n = M \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}^n M^{-1}$, which is an equivalent statement.

   Base case For $n = 0$ (or 1), the result holds trivially (or from what is given).

   inductive step Assume that for $n \geq 1$

$$\mathbf{Q}^{n-1} = M \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}^{n-1} .M^{-1}$$

   then, since $\mathbf{Q}^n = \mathbf{Q}^{n-1}\mathbf{Q}$ we have

$$\begin{aligned}
\mathbf{Q}^n &= (M\mathbf{Q}^{n-1}M^{-1})\mathbf{Q} \\
&= (M\mathbf{Q}^{n-1}M^{-1})(M\mathbf{Q}M^{-1}) \\
&= M\mathbf{Q}^n\mathbf{I}\mathbf{Q}M^{-1} \\
&= M\mathbf{Q}^n M^{-1}
\end{aligned}$$

   where $\mathbf{I}$ is the identity matrix.

5. Verify that the inverse of the matrix M is $\frac{1}{\alpha+\beta}\begin{pmatrix} \beta & \alpha \\ 1 & -1 \end{pmatrix}$. Then show that:

$$\mathbf{Q}^n = \frac{1}{\alpha+\beta}\begin{pmatrix} \beta+\alpha\omega^n & \alpha(1-\omega^n) \\ \beta(1-\omega^n) & \alpha+\beta\omega^n \end{pmatrix}.$$

Solution 1.5 *We have to show that:*

$$M\frac{1}{\alpha+\beta}\begin{pmatrix} \beta & \alpha \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In fact:

$$\begin{aligned} M\tfrac{1}{\alpha+\beta}\begin{pmatrix} \beta & \alpha \\ 1 & -1 \end{pmatrix} &= \tfrac{1}{\alpha+\beta}M\begin{pmatrix} \beta & \alpha \\ 1 & -1 \end{pmatrix} \\ &= \tfrac{1}{\alpha+\beta}\begin{pmatrix} \alpha+\beta & 0 \\ 0 & \alpha+\beta \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

$\mathbf{Q}^n = M\begin{pmatrix} 1 & 0 \\ 0 & \omega^n \end{pmatrix}.M^{-1}$ *and we have*

$$M\begin{pmatrix} 1 & 0 \\ 0 & \omega^n \end{pmatrix}. = \begin{pmatrix} 1 & \alpha\omega^n \\ 1 & -\beta\omega^n \end{pmatrix}..$$

Thus, we conclude

$$\begin{aligned} \begin{pmatrix} 1 & \alpha\omega^n \\ 1 & -\beta\omega^n \end{pmatrix}.\tfrac{1}{\alpha+\beta}\begin{pmatrix} \beta & \alpha \\ 1 & -1 \end{pmatrix} &= \tfrac{1}{\alpha+\beta}\begin{pmatrix} \beta+\alpha\omega^n & \alpha-\alpha\omega^n \\ \beta-\beta\omega^n & \alpha+\beta\omega^n \end{pmatrix}. \\ &= \tfrac{1}{\alpha+\beta}\begin{pmatrix} \beta+\alpha\omega^n & \alpha(1-\omega^n) \\ \beta(1-\omega^n) & \alpha+\beta\omega^n \end{pmatrix}. \end{aligned}$$

6. If $-1 < \omega < 1$ and $\mathbf{Q}^\infty$ is the limit of $\mathbf{Q}^n$ as $n \to \infty$, show that the rows of $\mathbf{Q}^\infty$ are the same $\vec{p} = (p_1, p_2)$ and satisfy $\vec{p} = \vec{p}.\mathbf{Q}$.

   For which values of α, β does $\mathbf{Q}^n$ not converge as $n \to \infty$? What property does the Markov chain exhibit? What is the significance of case $\omega = 1$?

Solution 1.6

$$\begin{aligned} \lim_{n\to\infty} \mathbf{Q}^n &= \tfrac{1}{\alpha+\beta}\lim_{n\to\infty}\begin{pmatrix} \beta+\alpha\omega^n & \alpha(1-\omega^n) \\ \beta(1-\omega^n) & \alpha+\beta\omega^n \end{pmatrix} \\ &= \begin{pmatrix} \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \\ \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{pmatrix} \end{aligned}$$

It remains to show that $(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta})\mathbf{Q} = (\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta})$*:*

$$\begin{aligned}
(\tfrac{\beta}{\alpha+\beta}, \tfrac{\alpha}{\alpha+\beta})\mathbf{Q} &= (\tfrac{\beta(1-\alpha)}{\alpha+\beta} + \tfrac{\alpha\beta}{\alpha+\beta}, \tfrac{\alpha\beta}{\alpha+\beta} + \tfrac{\alpha(1-\beta)}{\alpha+\beta}) \\
&= \tfrac{\beta-\alpha\beta+\alpha\beta}{\alpha+\beta}, \tfrac{\alpha+\alpha\beta-\alpha\beta}{\alpha+\beta} \\
&= (\tfrac{\beta}{\alpha+\beta}, \tfrac{\alpha}{\alpha+\beta})
\end{aligned}$$

If $\alpha = \beta = 1$*, then the matrix does not converge and the chain is periodic. If* $\omega = 1$ *then* $\mathbf{Q}^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *and the chain is no longer irreducible.*