COURSE 336
PERFORMANCE ANALYSIS
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Solutions 5
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1. In an M/M/1 queue, assume that customers arrive as a Poisson process with parameter one per 12 minutes, and that the service time is exponential at rate one service per 8 minutes. Calculate the following quantities:

   (a) L the average number of customers in the system;

   (b) W the average amount of time that a customer spends in the system;

   (c) W_Q the average amount of time a customer spends in the queue, waiting to start service.

   If there are no customers waiting to be served on arrival, what is the value of the queueing time random variable?

   Solution

   Since $\lambda = \frac{1}{12}$ and $\mu = \frac{1}{8}$ minute^{-1}, then

   (a) $L = \frac{\lambda}{\mu-\lambda}$, then $L = \dfrac{\frac{1}{12}}{\frac{1}{24}} = 2$; thus the average number of customers in the queue is 2.

   (b) $W = \frac{1}{\mu-\lambda}$, so $W = 24$; thus the average time spent in the system is 24 minutes.

   (c) $W_Q = W - \frac{1}{\mu}$, so $W_Q = 16$; thus the average time spent in the queue waiting to start service is 16 minutes.

   Intuitively, if the queue is empty, the only time waiting is the service time. Formally, observing that $W = W_Q + \frac{1}{\mu}$ and $W_Q = 0$, we derive $W = \frac{1}{\mu} = 8$ minutes.

2. Consider an M/M/1 queue where the jobs' willingness to join the queue is influenced by the queue size. More precisely, a job which finds i other jobs in the system joins the queue with probability $1/(i+1)$ and departs immediately otherwise ($i = 0, 1, \ldots$). Draw the state diagram for this system with arrival rate λ and mean service time $1/\mu$. Write down and solve the balance equations for the equilibrium probabilities π_i and show that a steady-state distribution always exists. Find the utilisation of the server, the throughput , the average number of jobs in the system and the average reponse time for a job that decides to join. Note that the form of

equilibrium probabilities is the same as that of the M/M/∞ queue.

Solution Diagram is the same as for the M/M/1 queue drawn in lectures, with rates from states $i+1$ to i all having rate μ and the rate from i to $i+1$ having rate $\lambda \times \frac{1}{i+1}$ since a fraction $\frac{i}{i+1}$ of arrivals do not join the queue at all ($i \geq 0$).

Balance equation for contour round states $1, 2, \ldots, i$ is therefore:

$$p_i \lambda/(i+1) = p_{i+1}\mu$$

so that

$$p_i = \rho^i/i! p_0$$

where $\rho = \lambda/\mu$. Normalising, $p_0 = e^{-\rho}$, so steady state always exists.

Utilisation $= 1 - p_0 = 1 - e^{-\rho}$.

Throughput $= (1 - e^{-\rho})\mu$ since service rate is constant.

Mean queue length $L = \sum_{i=1}^{\infty} ip_i = p_0 \sum_{i=1}^{\infty} i\rho^i/i!$
$= p_0\rho \sum_{i=1}^{\infty} \rho^{i-1}/(i-1)! = \rho$

By Little's result, mean response time $W = L/(1-e^{-\rho})\mu = \rho/(1-e^{-\rho})\mu = \lambda/(1-e^{-\rho})\mu^2$

3. Consider an M/M/∞ queue with discouraged arrivals but with the following birth-and-death coefficients:

$$\lambda(i) = \frac{\lambda}{(i+1)^b} \quad \text{and} \quad \mu(i) = i^c \mu$$

where c is the "pressure-coefficient" – a constant that indicates the degree to which the service rate of the system is affected by the system state – and b is the "discouraging coefficien". Obtain the equilibrium probabilities π_i for this birth-and-death process. What is the equilibrium distribution when $b + c = 1$?

Solution As in the previous question,

$$p_i \lambda/(i+1)^b = p_{i+1}\mu(i+1)^c$$

i.e.

$$p_{i+1} = p_i \rho/(i+1)^{b+c}$$

Thus

$$p_i = \rho^i/(i!)^{b+c} p_0$$

$$p_0 = \left[\sum_{k=0}^{\infty} \rho^k/(k!)^{b+c} \right]^{-1}$$

Same as previous question / IS queue when $b + c = 1$.

4. Let D be a random interval between two consecutive departures from an M/M/1 queue at equilibrium. If, just after the first departure, the queue was not empty, then D coincides with the service time of the next job. If the queue was empty, then D consists of the period up to the next arrival, as well as the service time of the next job. Assuming that the probability of a non-empty system just after departure is the same as the utilisation, show that D is exponentially distributed with parameter λ.

   Solution
   Recall that $\rho = \frac{\lambda}{\mu}$ is the utilisation of the server. Write X and Y for the random variables corresponding to the interarrival time and the service time respectively. Let D have probability distribution function F. Then:

$$
\begin{aligned}
F(t) &= P(D \le t) \;=\; \rho P(Y \le t) + (1-\rho)P(X+Y \le t) \\
&= \rho(1-e^{-\mu t}) + (1-\rho)\int_0^t P(X \le t-u)\mu e^{-\mu u} du \\
&\quad \text{(since } X, Y \text{ are independent)} \\
&= \rho(1-e^{-\mu t}) + (1-\rho)\int_0^t \mu e^{-\mu u} - \mu e^{-\lambda t} e^{-(\mu-\lambda)u} du \\
&= \rho(1-e^{-\mu t}) + (1-\rho)\left[1 - e^{-\mu t} - e^{-\lambda t}(\mu/(\mu-\lambda))(1-e^{-(\mu-\lambda)t})\right] \\
&= 1 - e^{-\mu t} - e^{-\lambda t}(1-e^{-(\mu-\lambda)t}) \\
&= 1 - e^{-\lambda t}
\end{aligned}
$$

   as required.

5. Let X_1 and X_2 be two independent exponential random variables with parameters λ_1 and λ_2. Prove that the random variable $Z = \min(X_1, X_2)$ is an exponential random variable with parameter $\lambda_1 + \lambda_2$.

   Solution For the random variable Z we are interested in the time until either X_1 or X_2 occurs, so:

$$
\begin{aligned}
P(Z \le t) &= 1 - P(Z > t) \\
&= 1 - P(\min(X_1, X_2) > t) \\
&= 1 - P(X_1 > t, X_2 > t) \\
&= 1 - P(X_1 > t)P(X_2 > t) \quad \text{by independence} \\
&= 1 - (e^{-\lambda_1 t})(e^{-\lambda_2 t}) \\
&= 1 - e^{-(\lambda_1+\lambda_2)t}
\end{aligned}
$$