Reasoning with Probabilities

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Introduction to Artificial Intelligence
2nd Part

The main reference

Stuart Russell and Peter Norvig
Artificial Intelligence: a modern approach
Chapter 14

Today

Today

- Bayes' rule
- Conditional independence
- Back to the Wumpus World
- Bayesian Networks

Bayes' Rule

Holiday

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How worried should you be?

Conditional probability and Bayes' Rule

Definition of conditional probability:

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Bayes' Rule

$$P(a|b) = \frac{P(b|a)P(a)}{P(b)}$$

Bayes' Rule

Useful for assessing diagnostic probability from causal probability:

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E.g., let c be cold, s be sore throat:

$$P(c|s) = \frac{P(s|c)P(c)}{P(s)} = \frac{0.9 \times 0.001}{0.005} = 0.18$$

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$$\mathbf{P}(C|s) = \alpha \langle P(s|c)P(c), P(s|\neg c)P(\neg c) \rangle$$

Bayes' Rule

$$\mathbf{P}(X|Y) = \alpha\, \mathbf{P}(Y|X)\mathbf{P}(X)$$

Holiday solved

Upon your return from a holiday on an exotic island, your doctor has bad news and good news. The bad news is that you've been diagnosed a serious disease and the test is 99% accurate. The good news is that the disease is very rare (1 in 10.000 get it).

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Notice:posterior probability of disease still quite small!

Back to joint distributions: combining evidence

Start with the joint distribution:

	toothache		$\neg$ *toothache*	
	catch	$\neg$ *catch*	*catch*	$\neg$ *catch*
cavity	.108	.012	.072	.008
$\neg$ *cavity*	.016	.064	.144	.576

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$$\mathbf{P}(Cavity | toothache \wedge catch) = \alpha \langle 0.108, 0.016 \rangle = \langle 0.871, 0.129 \rangle$$

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- It doesn't scale up to a large number of variables
- Absolute Independence is very rare
- Can we use Bayes' rule?

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$$\mathbf{P}(Cavity|toothache \wedge catch) =$$
$$\alpha \, \mathbf{P}(toothache \wedge catch|Cavity)\mathbf{P}(Cavity)$$

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Still not good: with n evidence variables 2^n possible combinations for which we would need to know the conditional probabilities

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We can't use absolute independence:

- Toothache and Catch are **not** independent: If the probe catches in the tooth then it is likely the tooth has a cavity, which means that toothache is likely too.
- But they are independent **given** the presence or the absence of cavity! Toothache depends on the state of the nerves in the tooth, catch depends on the dentist's skills, to which toothache is irrelevant

Conditional Independence

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Catch is conditionally independent of *Toothache* given *Cavity*:

Conditional independence

$$\mathbf{P}(Catch | Toothache, Cavity) = \mathbf{P}(Catch | Cavity)$$

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Equivalent statements:

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Equivalent statements:

$\mathbf{P}(Toothache|Catch, Cavity) = \mathbf{P}(Toothache|Cavity)$

$\mathbf{P}(Toothache, Catch|Cavity) =$

$\mathbf{P}(Toothache|Cavity)\mathbf{P}(Catch|Cavity)$

Conditional independence contd.

Write out full joint distribution using chain rule:

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I.e., $2 + 2 + 1 = 5$ independent numbers (equations 1 and 2 remove 2)

Conditional independence contd.

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Conditional independence is our most basic and robust form of knowledge about uncertain environments.

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Bayes' Rule and conditional independence

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\end{aligned}$$

Bayes' Rule and conditional independence

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This is an example of a naive Bayes model:

$$\mathbf{P}(Cause, Effect_1, \ldots, Effect_n) = \mathbf{P}(Cause)\prod_i \mathbf{P}(Effect_i|Cause)$$

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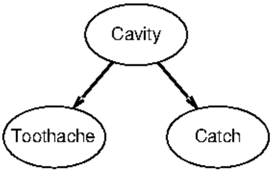

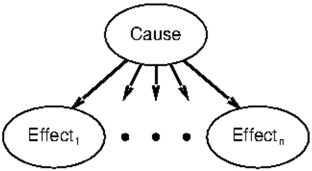

The Wumps World

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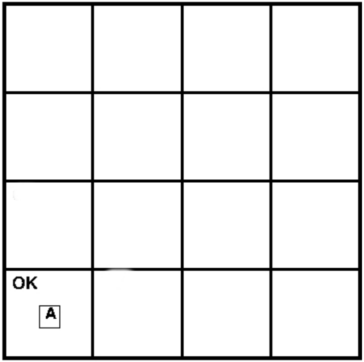

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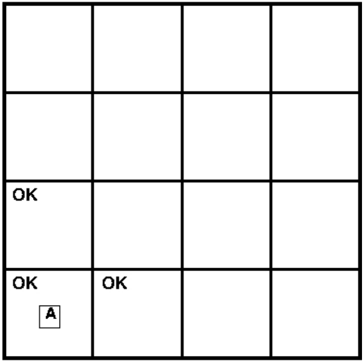

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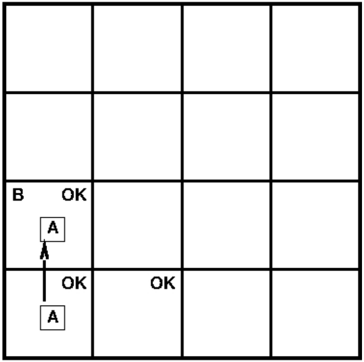

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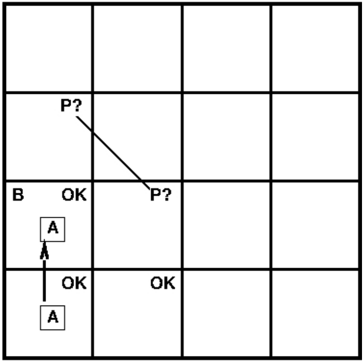

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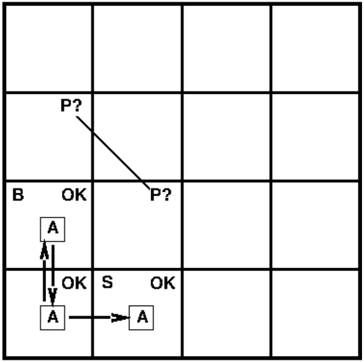

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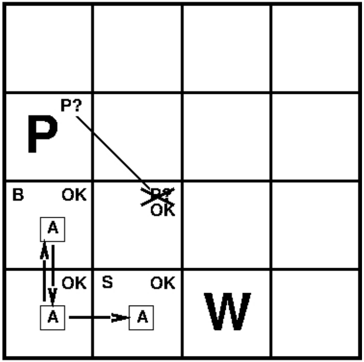

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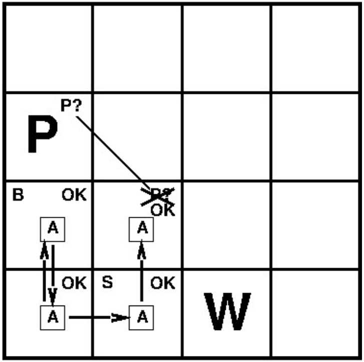

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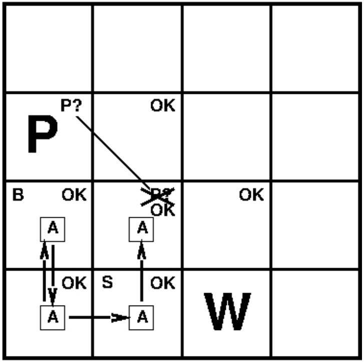

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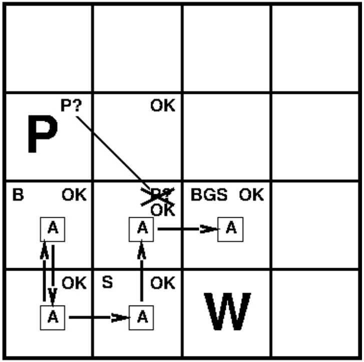

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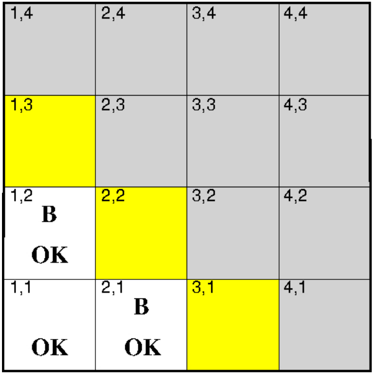

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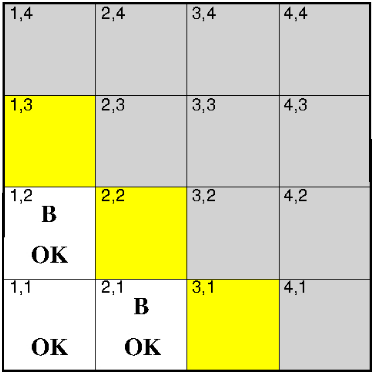

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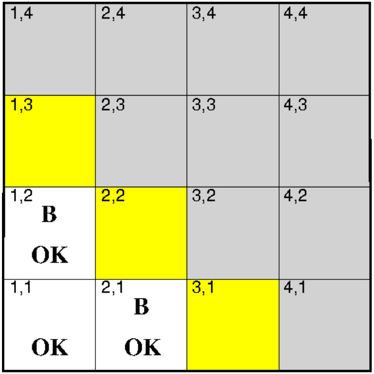

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Apply product rule:
$\mathbf{P}(B_{1,1}, B_{1,2}, B_{2,1} \mid P_{1,1}, \ldots, P_{4,4})\mathbf{P}(P_{1,1}, \ldots, P_{4,4})$
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$$\mathbf{P}(P_{1,1}, \ldots, P_{4,4}) = \prod_{i,j=1,1}^{4,4} \mathbf{P}(P_{i,j}) = 0.2^n \times 0.8^{16-n}$$

for n pits.

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Define $Unexplored$ = P_{ij}s other than $P_{1,3}$ and $Explored$

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We know the following facts:

$b = \neg b_{1,1} \wedge b_{1,2} \wedge b_{2,1}$

$explored = \neg p_{1,1} \wedge \neg p_{1,2} \wedge \neg p_{2,1}$

Query is $\mathbf{P}(P_{1,3} | explored, b)$

Define $Unexplored$ = P_{ij}s other than $P_{1,3}$ and $Explored$

Complexity

For inference by enumeration, we have

$$\mathbf{P}(P_{1,3} | explored, b) = \alpha \sum_{unexplored} \mathbf{P}(P_{1,3}, unexplored, explored, b)$$

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Complexity

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- There are 12 unknown squares
- The summation contains $2^{12} = 4096$ terms

Complexity

For inference by enumeration, we have

$$\mathbf{P}(P_{1,3}|explored, b) = \alpha \sum_{unexplored} \mathbf{P}(P_{1,3}, unexplored, explored, b)$$

- There are 12 unknown squares
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In general the summation grows exponentiatlly with the number of squares!

Complexity

For inference by enumeration, we have

$$\mathbf{P}(P_{1,3}|explored, b) = \alpha \sum_{unexplored} \mathbf{P}(P_{1,3}, unexplored, explored, b)$$

- There are 12 unknown squares
- The summation contains $2^{12} = 4096$ terms

In general the summation grows exponentiatlly with the number of squares!

And now?

Using conditional independence

Basic insight: observations are conditionally independent of other hidden squares given neighbouring hidden squares

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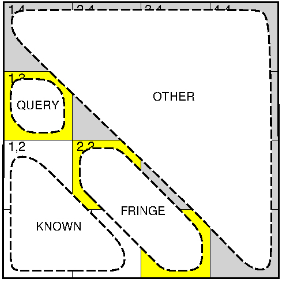
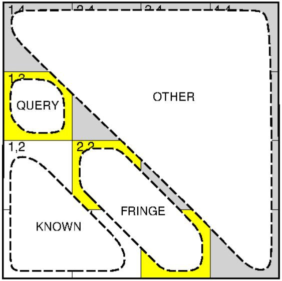


Using conditional independence

Basic insight: observations are conditionally independent of other hidden squares given neighbouring hidden squares



Define $Unexplored = Fringe \cup Other$

Using conditional independence

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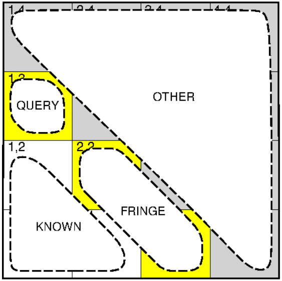

Define $Unexplored = Fringe \cup Other$
$\mathbf{P}(b|P_{1,3}, Explored, Unexplored) = \mathbf{P}(b|P_{1,3}, Explored, Fringe)$

Using conditional independence

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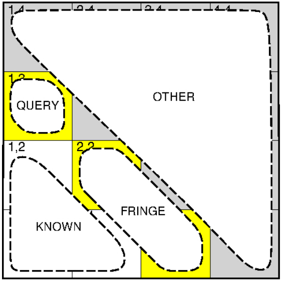
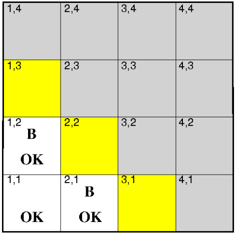


Define $\mathit{Unexplored} = \mathit{Fringe} \cup \mathit{Other}$

$\mathbf{P}(b|P_{1,3}, \mathit{Explored}, \mathit{Unexplored}) = \mathbf{P}(b|P_{1,3}, \mathit{Explored}, \mathit{Fringe})$

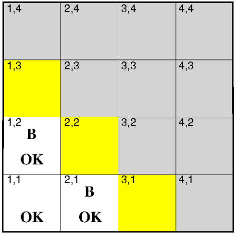
Manipulate query into a form where we can use this!

Using conditional independence



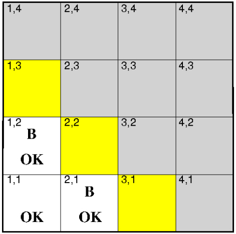
$\mathbf{P}(P_{1,3}|explored, b)$

Using conditional independence



$$\mathbf{P}(P_{1,3}|explored, b) = \alpha \sum_{unexplored} \mathbf{P}(P_{1,3}, unexplored, explored, b)$$

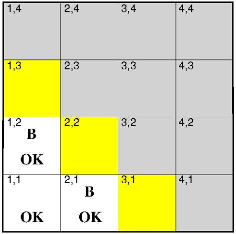
Using conditional independence



Inference by enumeration

$$\mathbf{P}(P_{1,3}|explored, b) = \alpha \sum_{unexplored} \mathbf{P}(P_{1,3}, unexplored, explored, b)$$

Using conditional independence



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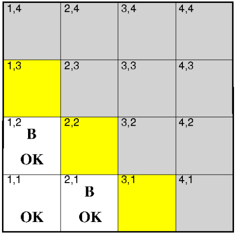
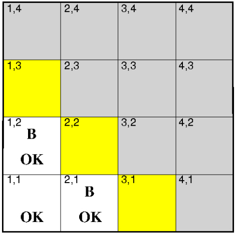


$$\alpha \sum_{unexplored} \mathbf{P}(P_{1,3}, unexplored, explored, b)$$

$$= \alpha \sum_{unexplored} \mathbf{P}(b|explored, P_{1,3}, unexplored) \times$$

$$\times \mathbf{P}(P_{1,3}, explored, unexplored)$$

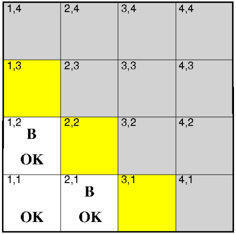
Using conditional independence



Product rule

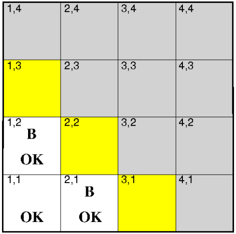
$$\alpha \sum_{unexplored} \mathbf{P}(P_{1,3}, unexplored, explored, b)$$
$$= \alpha \sum_{unexplored} \mathbf{P}(b|explored, P_{1,3}, unexplored)\times$$
$$\times \mathbf{P}(P_{1,3}, explored, unexplored)$$

Using conditional independence



$$\alpha \sum_{unexplored} \mathbf{P}(b|P_{1,3}, unexplored, explored)\mathbf{P}(P_{1,3}, unexplored, explored)$$

Using conditional independence

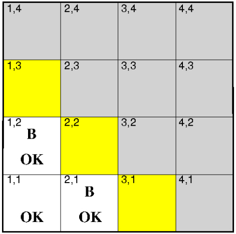


$$\alpha \sum_{unexplored} \mathbf{P}(b | P_{1,3}, unexplored, explored) \mathbf{P}(P_{1,3}, unexplored, explored)$$

$$= \alpha \sum_{fringe} \sum_{other} \mathbf{P}(b | explored, P_{1,3}, fringe, other) \times$$

$$\times \mathbf{P}(P_{1,3}, explored, fringe, other)$$

Using conditional independence



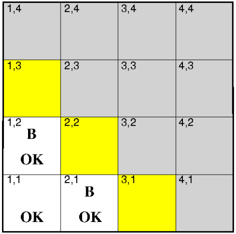
Distinguishing the unknown

$$\alpha \sum_{unexplored} \mathbf{P}(b|P_{1,3}, unexplored, explored)\mathbf{P}(P_{1,3}, unexplored, explored)$$

$$= \alpha \sum_{fringe} \sum_{other} \mathbf{P}(b|explored, P_{1,3}, fringe, other) \times$$

$$\times \mathbf{P}(P_{1,3}, explored, fringe, other)$$

Using conditional independence



$$\alpha \sum_{fringe} \sum_{other} \mathbf{P}(b|explored, P_{1,3}, fringe, other) \times$$
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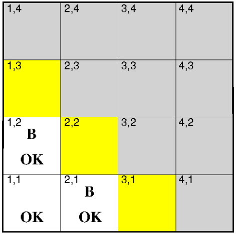

$$\alpha \sum_{fringe} \sum_{other} \mathbf{P}(b|explored, P_{1,3}, fringe, other) \times$$
$$\times \mathbf{P}(P_{1,3}, explored, fringe, other)$$

$$= \alpha \sum_{fringe} \sum_{other} \mathbf{P}(b|explored, P_{1,3}, fringe) \times$$
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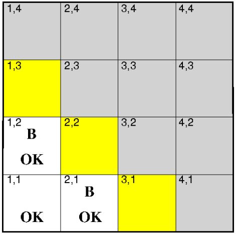
Using conditional independence



Conditional Independence

$$\alpha \sum_{fringe} \sum_{other} \mathbf{P}(b|explored, P_{1,3}, fringe, other) \times$$
$$\times \mathbf{P}(P_{1,3}, explored, fringe, other)$$

$$= \alpha \sum_{fringe} \sum_{other} \mathbf{P}(b|explored, P_{1,3}, fringe) \times$$
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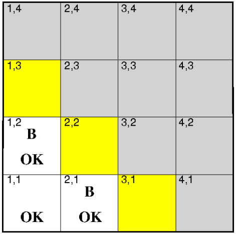
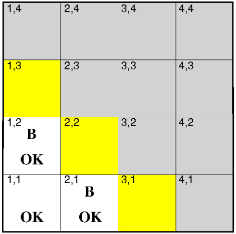


$$\alpha \sum_{fringe} \sum_{other} \mathbf{P}(b|explored, P_{1,3}, fringe) \times$$
$$\times \mathbf{P}(P_{1,3}, explored, fringe, other)$$

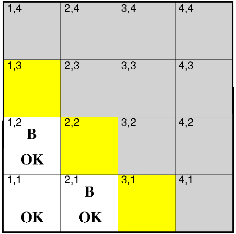
Using conditional independence



$$\alpha \sum_{fringe} \sum_{other} \mathbf{P}(b \mid explored, P_{1,3}, fringe) \times$$
$$\times \mathbf{P}(P_{1,3}, explored, fringe, other)$$

$$= \alpha \sum_{fringe} \mathbf{P}(b \mid explored, P_{1,3}, fringe) \times$$
$$\times \sum_{other} \mathbf{P}(P_{1,3}, explored, fringe, other)$$

Using conditional independence

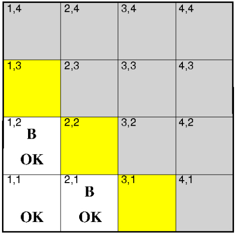


Pushing the sums inwards

$$\alpha \sum_{fringe} \sum_{other} \mathbf{P}(b|explored, P_{1,3}, fringe) \times$$
$$\times \mathbf{P}(P_{1,3}, explored, fringe, other)$$

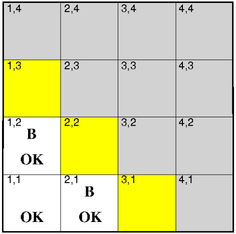
$$= \alpha \sum_{fringe} \mathbf{P}(b|explored, P_{1,3}, fringe) \times$$
$$\times \sum_{other} \mathbf{P}(P_{1,3}, explored, fringe, other)$$

Using conditional independence



$$\alpha \sum_{fringe} \mathbf{P}(b | explored, P_{1,3}, fringe) \times$$
$$\sum_{other} \mathbf{P}(P_{1,3}, explored, fringe, other)$$

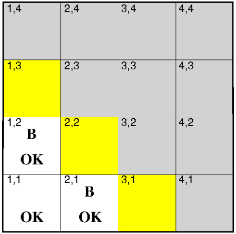
Using conditional independence



$$\alpha \sum_{fringe} \mathbf{P}(b|explored, P_{1,3}, fringe) \times \sum_{other} \mathbf{P}(P_{1,3}, explored, fringe, other)$$

$$= \alpha \sum_{fringe} \mathbf{P}(b|explored, P_{1,3}, fringe) \times \sum_{other} \mathbf{P}(P_{1,3}) P(explored) P(fringe) P(other)$$

Using conditional independence

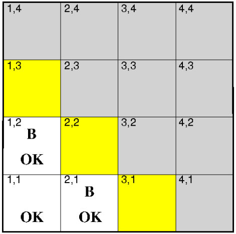


Independence

$$\alpha \sum_{fringe} \mathbf{P}(b|explored, P_{1,3}, fringe) \times \sum_{other} \mathbf{P}(P_{1,3}, explored, fringe, other)$$

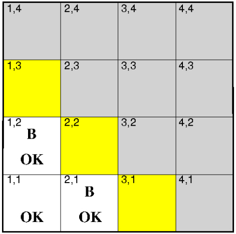
$$= \alpha \sum_{fringe} \mathbf{P}(b|explored, P_{1,3}, fringe) \times \sum_{other} \mathbf{P}(P_{1,3}) P(explored) P(fringe) P(other)$$

Using conditional independence



$$\alpha \sum_{fringe} \mathbf{P}(b|explored, P_{1,3}, fringe) \times$$
$$\times \sum_{other} \mathbf{P}(P_{1,3}) P(explored) P(fringe) P(other)$$

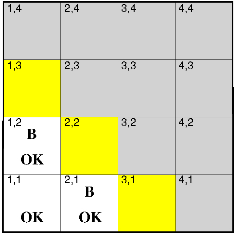
Using conditional independence



$$\alpha \sum_{fringe} \mathbf{P}(b|explored, P_{1,3}, fringe)\times$$
$$\times \sum_{other} \mathbf{P}(P_{1,3})P(explored)P(fringe)P(other)$$

$$= \alpha\, P(explored)\mathbf{P}(P_{1,3})\times$$
$$\times \sum_{fringe} \mathbf{P}(b|explored, P_{1,3}, fringe)P(fringe) \sum_{other} P(other)$$

Using conditional independence

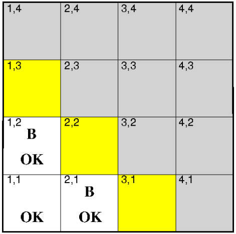


Reordering
and pushing sums inwards

$$\alpha \sum_{fringe} \mathbf{P}(b|explored, P_{1,3}, fringe) \times$$
$$\times \sum_{other} \mathbf{P}(P_{1,3}) P(explored) P(fringe) P(other)$$

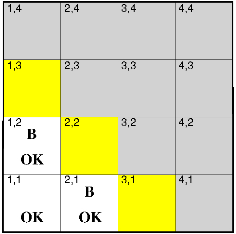
$$= \alpha\, P(explored) \mathbf{P}(P_{1,3}) \times$$
$$\times \sum_{fringe} \mathbf{P}(b|explored, P_{1,3}, fringe) P(fringe) \sum_{other} P(other)$$

Using conditional independence



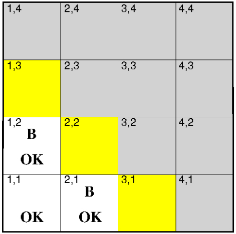
$$\alpha\, P(explored)\mathbf{P}(P_{1,3})\times$$
$$\times \sum_{fringe} \mathbf{P}(b|explored, P_{1,3}, fringe)P(fringe) \sum_{other} P(other)$$

Using conditional independence



$$\alpha\, P(explored)\mathbf{P}(P_{1,3})\times$$
$$\times \sum_{fringe} \mathbf{P}(b|explored, P_{1,3}, fringe)P(fringe) \sum_{other} P(other)$$
$$= \alpha'\, \mathbf{P}(P_{1,3}) \sum_{fringe} \mathbf{P}(b|explored, P_{1,3}, fringe)P(fringe)$$

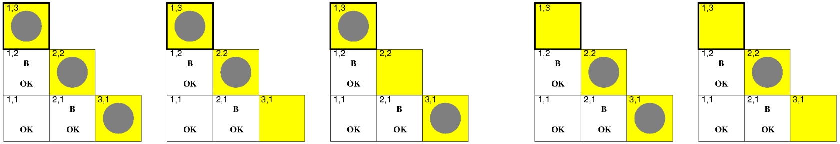
Using conditional independence



Simplifying

$$\alpha\, P(explored)\mathbf{P}(P_{1,3})\times$$
$$\times \sum_{fringe} \mathbf{P}(b|explored, P_{1,3}, fringe)P(fringe) \sum_{other} P(other)$$
$$= \alpha'\, \mathbf{P}(P_{1,3}) \sum_{fringe} \mathbf{P}(b|explored, P_{1,3}, fringe)P(fringe)$$

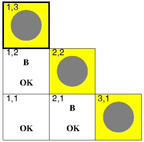
Using conditional independence contd.



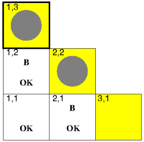
0.2 x 0.2 = 0.04 | 0.2 x 0.8 = 0.16 | 0.8 x 0.2 = 0.16 | 0.2 x 0.2 = 0.04 | 0.2 x 0.8 = 0.16

$\mathbf{P}(b|explored, P_{1,3}, fringe)$

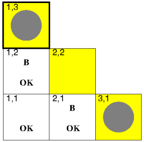
Using conditional independence contd.



0.2 x 0.2 = 0.04



0.2 x 0.8 = 0.16



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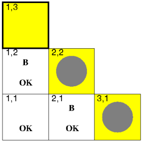

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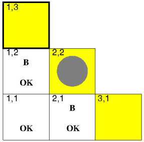
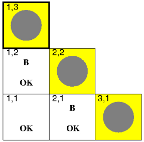


0.2 x 0.8 = 0.16

$\mathbf{P}(b|explored, P_{1,3}, fringe)$

- $= 1$ when the frontier is consistent with the observations
- $= 0$ otherwise

Using conditional independence contd.



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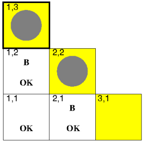

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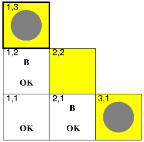

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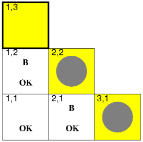

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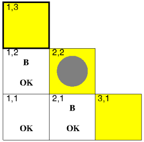


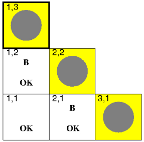
0.2 x 0.8 = 0.16

$\mathbf{P}(b|explored, P_{1,3}, fringe)$

- $= 1$ when the frontier is consistent with the observations
- $= 0$ otherwise

We can sum over the *possible configurations* for the frontier variables that are consistent with the known facts.

Using conditional independence contd.



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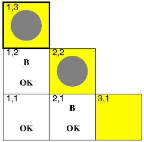

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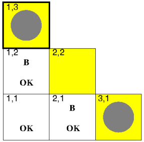

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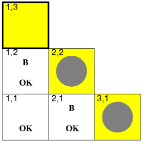

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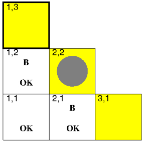
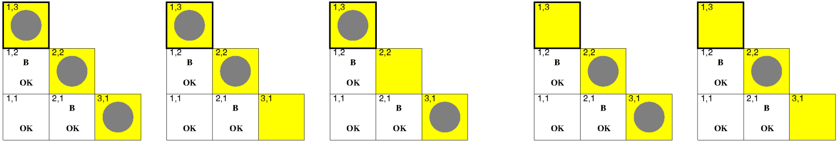


0.2 x 0.8 = 0.16

$\mathbf{P}(P_{1,3}|explored, b) =$

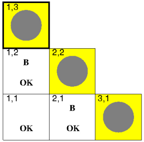
Using conditional independence contd.



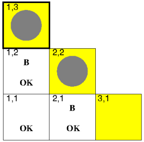
0.2 x 0.2 = 0.04 | 0.2 x 0.8 = 0.16 | 0.8 x 0.2 = 0.16 | 0.2 x 0.2 = 0.04 | 0.2 x 0.8 = 0.16

$$\mathbf{P}(P_{1,3}|explored, b) = \alpha' \langle 0.2(0.04 + 0.16 + 0.16),\ 0.8(0.04 + 0.16) \rangle$$

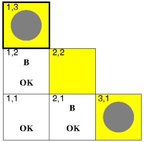
Using conditional independence contd.



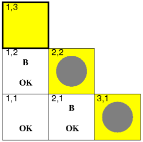
0.2 x 0.2 = 0.04



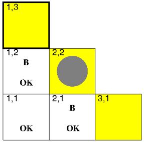
0.2 x 0.8 = 0.16



0.8 x 0.2 = 0.16



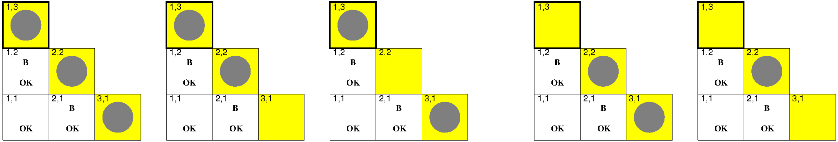
0.2 x 0.2 = 0.04



0.2 x 0.8 = 0.16

$$\mathbf{P}(P_{1,3}|explored, b) =$$
$$\alpha' \langle 0.2(0.04 + 0.16 + 0.16),\ 0.8(0.04 + 0.16) \rangle$$
$$\approx \langle 0.31, 0.69 \rangle$$

Using conditional independence contd.

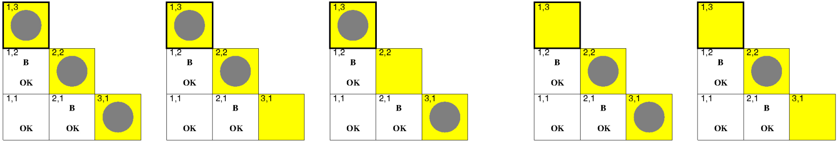


0.2 x 0.2 = 0.04 0.2 x 0.8 = 0.16 0.8 x 0.2 = 0.16 0.2 x 0.2 = 0.04 0.2 x 0.8 = 0.16

$\mathbf{P}(P_{1,3}|explored, b) =$
$\alpha' \langle 0.2(0.04 + 0.16 + 0.16),\ 0.8(0.04 + 0.16) \rangle$
$\approx \langle 0.31, 0.69 \rangle$

$\mathbf{P}(P_{2,2}|explored, b) \approx \langle 0.86, 0.14 \rangle$

Using conditional independence contd.



0.2 x 0.2 = 0.04 | 0.2 x 0.8 = 0.16 | 0.8 x 0.2 = 0.16 | 0.2 x 0.2 = 0.04 | 0.2 x 0.8 = 0.16

$$\mathbf{P}(P_{1,3}|explored, b) =$$
$$\alpha' \langle 0.2(0.04 + 0.16 + 0.16),\ 0.8(0.04 + 0.16) \rangle$$
$$\approx \langle 0.31, 0.69 \rangle$$

$$\mathbf{P}(P_{2,2}|explored, b) \approx \langle 0.86, 0.14 \rangle$$