Towards a computational model of embodied mathematical language

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Abstract

We outline two theories of mathematical language acquisition and development, and discuss how a computational model of these theories may help to bridge the gap between automated theory formation and situated embodied agents. Finally, we briefly describe a simple theoretical case study of how such a model could work in the arithmetic domain.

Introduction

It is surprising that little work has been carried out into the way in which humans develop mathematical language, both on an individual and social level. A better understanding of the processes by which we learn to represent, store, communicate, use and develop mathematical ideas would have great educational potential as well as implications for other language acquisition, philosophy and psychology of mathematics, and robotics. The deficiency of work in this area led cognitive scientists Lakoff and Núñez to lament in 2001 that (prior to their work) "there was still no discipline of mathematical idea analysis" (Lakoff and Núñez, 2001, p.*XI*). A philosophical counterpart to Lakoff and Núñez's work is Lakatos's work in the philosophy of mathematics (Lakatos, 1976). Both theories reject the "romantic" or "deductivist" style in which mathematics is presented as an ever-increasing set of universal, absolute, certain truths which exist independently of humans, arguing instead that mathematics uses non-absolute, defeasible reasoning.

Lakoff and Núñez's theory of embodied mathematics

Lakoff and Núñez present the thesis that the human embodied mind brings mathematics into being (Lakoff and Núñez, 2001). That is, human mathematics is grounded in bodily experience of a physical world, and mathematical entities inherit properties which objects in the world have, such as being stable over time. They review studies which suggest that babies are able to distinguish one (small) number from another, to know the size of a small collection of objects (although not necessarily link size to order, so "3" is seen as different to, but not necessarily as bigger than, "4"), and to perform very simple arithmetic (see also (Butterworth, 1999)). For the sake of their argument, these abilities are called innate arithmetic. In order to form more complex mathematical ideas, we need to be able to form two types of conceptual metaphor between innate arithmetic and the more complex arithmetic of natural numbers. Firstly, we need to be able to make *grounding metaphors*. These allow us to project from everyday experiences onto abstract concepts. For instance, we make the metaphor between putting physical objects into groups, and the abstract concept of addition. Lakoff and Núñez identify four grounding metaphors for arithmetic: forming collections, putting objects together, using measuring sticks, and moving through space. The second type of metaphor that we need to be able to make is a *linking metaphor*. This consists of blending different metaphors and yields sophisticated ideas, such as mapping points on a line to numbers, algebraic equations to geometrical figures, or numbers to sets. Lakoff and Núñez argue that much of the abstraction of higher mathematics is the consequence of this type of systematic layering of metaphor upon metaphor and they show where mathematical concepts and laws come from, in terms of these metaphors.

The importance of the environment and our interaction with it in the development of mathematical ideas and capabilities is supported by work in mathematics education and psychology. For instance, Dienes developed a theory of embodied mathematical knowledge and situated cognition, claimed that the environment is "of outstanding importance" in learning mathematics (Dienes, 1973), and, in his theory of the acquisition of mathematics (discussed in (Taylor, 1976)), argued that interaction with the environment is a fundamental aspect of three of the six stages. Piaget gave experience in the environment and action central roles in the developmental process (Piaget, 2001). Choat provides another example in his argument that "all mathematical knowledge originates from contact with objects which constitute the environment" (Choat, 1980, p. 38).

Lakatos's theory of social mathematics

Lakatos charts the evolution of meaning of mathematical terms via dialectic. His influences include Hegel's dialectic, in which the *thesis* corresponds to a naïve mathematical conjecture and proof; the *antithesis* to a mathematical counterexample; and the *synthesis* to a refined theorem and proof (described in these terms in (Lakatos, 1976, pp.144-145)). Another influence is Plato, and some of the reasoning which Lakatos describes can be compared to that in Plato's *Republic*, in which arguments are not deductive: the meaning of terms in the arguments changes over time, and therefore a term in a premise of an argument may not mean the same as the same term in the argument's conclusion. For instance, Simonides proposes that "it is right to give back what is owed". This initial statement is questioned by Socrates with the counterexample of someone borrowing weapons from a friend who subsequently goes insane, in which case it would not be right to return the weapons. The discussion in *The Republic* then turns to what it means to give back what is owed, with Polemarchus suggesting that people owe their friends good deeds, and their enemies bad ones. The dialogue later turns to what the concept of *doing right* means, and leads into Plato's treatment of justice. Another example is the change of meaning of the mathematical term "set" which evolved, in response to Russell's paradox and other problems, from Cantor's "collection of objects" to Zermelo-Fraenkel's definition: "given the set S, and any meaningful property P, it is possible to form the set of all members of S which satisfy P". Lakatos calls this type of reasoning *monster-barring*, and gives examples from mathematics. Once the validity of a counterexample has been questioned, the focus of an argument switches from the *truth* of the conjecture to the *meaning* of its terms, which is negotiated by participants in a discussion according to their motivations and beliefs.

The interface between automated theory formation and situated embodied agents

A computational model of the embodied and social mathematics described above may also help to bridge the gap between automated theory formation and situated embodied agents. Despite forty years of research into automating the formation of mathematical theories, there is still no automated theory formation system which works at the pre-axiomatic stage or takes cognitively plausible knowledge as input. Conversely, although the subsumption architecture framework proposed by Brooks has proven itself in allowing the creation of reactive robots that can deal with the natural complexity of the real world, the architecture has proved somewhat limited in the complexity of the tasks to which it can be applied.

To allow robots, or embodied agents, to undertake more complex tasks, a return has been seen to the older sense-model-plan-act approach but with the robustness to the natural world being being built in at the modelling level through the use of powerful statistical techniques (Thrun, 2002). Recent work has proposed approaches which can build up concepts and rules about the world based on experience gained from interacting with a stochastic domain (Pasula et al., 2006; Shanahan, 2005). Being able to reason at a high level about these rules and concepts would be a powerful tool for an embodied agent learning about its environment, especially if such reasoning resulted in testable hypotheses that the embodied agent could try out in its world. Grounding a system of mathematics via embodied interaction with an environment would also relate to the symbol grounding problem; enabling us to provide an account of how mathematical language acquires meaning, and what this meaning might be.

A computational model of mathematical language acquisition and development

We are currently drawing from these ideas to produce a computational model of mathematical language acquisition and development. Such a model must comprise both an embodied level where mathematical ideas can be seen as hidden rules which hold for, or are inspired by, a physical world (based on Lakoff and Núñez's work), and an abstract level where these ideas are explored and sometimes changed (based on Lakatos's theory). We have already developed a computational model of Lakatos's theory and used our model to evaluate his theory (Pease et al., 2002; Pease et al., 2004). We envisage a 4-stage model in which the interaction between the embodied agent and the reasoning software would work in a simple arithmetic domain as described below.

An embodied agent is equipped with innate arithmetic capabilities such as ability to distinguish small numbers, subitizing, and perception of simple arithmetic relationships, as well as cognitive capacities including grouping, ordering, pairing, memory and metaphorizing (see (Lakoff and Núñez, 2001, pp. 51-52)), as well as ability to select and abstract common properties (see (Liebeck, 1984)). In the first stage the agent is able to interact with its environment, for example, by moving objects around into different piles and configurations, and to abstract properties of the group, such as its size. The agent may remember, or store, the results of adding a first pile to a second pile, and the results of adding the second pile to the first. This embodied interaction would lead to a set of concepts and facts about the environment which would then be passed as input to a theory formation system (which can be achieved with methods similar to those proposed by (Pasula et al., 2006; Shanahan, 2005) as described above). In the second stage this theory formation system would abstract and generalise rules which are descriptive of the patterns it finds. For example, it might generate the *commutative axiom of addition* (for natural numbers a and b, $a+b=b+a$). The system would then explore the search space which the axioms define, by generating further concepts, making conjectures empirically, such as *whenever we subtract 1 from a number then we get another number*, and *all numbers can be written as the sum of two numbers*, and passing these to a theorem prover. In stage three, conjectures and theorems would then be passed back to the embodied agent for evaluation. For instance, the agent might evaluate relevance by testing whether a theorem can be instantiated within the world, or interestingness in terms of whether the theorem provides a new description of known behaviour or describes previously unknown behaviour. The agent might note that the two conjectures above hold for every collection of objects except for the collection of one object. It might then extend its concept of collection to including the empty collection, by performing the operation of removing one object from a collection of just that object and labelling the result a collection. Finally, in stage four, the same theory formation program would be used to analyse the information about the theorems and axioms and used to modify the axiom set. If one axiom had only been used to generate uninteresting theorems then this may be rejected at this stage. Conversely, for instance, having the "number" zero in the system might suggest further conjectures which would justify its inclusion in the theory. If any of the theorems contradicted each other then the axioms used would need to be modified or rejected.

We would evaluate our model based on whether it could reinvent concepts such as "zero" or axioms in a cognitively plausible way, and whether it recognised the interestingness of such pivotal concepts.

Conclusion

The theories we discuss in embodied and social mathematics are early characterisations of ways in which people do mathematics. We hope to build a computational representation of the theories which, starting from cognitively plausible innate abilities, models how we interact with an environment and how we formulate, explore, evaluate and modify axioms which describe that world. Our goal is to both extend and evaluate the theories we have discussed. It will be particularly exciting to further investigate the role that embodied interaction with an environment plays in our human mathematical development.

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