Course: Operational Semantics
First Assessed Exercise
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1) The terms of the Lambda Calculus are defined by the following abstract syntax:

$$\begin{array}{rcl} x & \in & \mathcal{V} \\ M & \in & \Lambda \\ \\ M & ::= & x \mid (\lambda x.M) \mid (M_1 M_2) \end{array}$$

The set of Curry types is defined by the following abstract syntax:

$$\begin{array}{rcl} \varphi & \in & \Phi \\ \sigma & \in & \mathcal{T} \\ \\ \sigma & ::= & \varphi \mid (\sigma_1 \rightarrow \sigma_2) \end{array}$$

A *statement* is a pair of term M and type σ and is written as $M : \sigma$. A *basis* B is a set of statements about variables, like $\{x_1{:}\sigma_1, x_2{:}\sigma_2, \ldots, x_n{:}\sigma_n\}$, where each x_i occurs at most once. Curry type assignment for terms in Λ derives formulae of the shape $B \vdash M : \sigma$, and is defined via the following three rules:

$$\frac{}{B \vdash x : \sigma}\,(x{:}\sigma \in B) \qquad \frac{B \cup \{x{:}\sigma\} \vdash M : \tau}{B \vdash (\lambda x.M) : (\sigma \rightarrow \tau)} \qquad \frac{B \vdash M : (\sigma \rightarrow \tau) \quad B \vdash N : \sigma}{B \vdash (MN) : \tau}$$

Substitution on types, $[\alpha/\varphi]$, the replacement of all occurrences of the type variable φ by the type α, is defined by:

$$\begin{array}{rcll} \varphi[\alpha/\varphi] & \underset{\text{def}}{=} & \alpha & \\ \varphi'[\alpha/\varphi] & \underset{\text{def}}{=} & \varphi', & \text{if } \varphi \neq \varphi' \\ (\sigma_1 \rightarrow \sigma_2)[\alpha/\varphi] & \underset{\text{def}}{=} & (\sigma_1[\alpha/\varphi] \rightarrow \sigma_2[\alpha/\varphi]). & \end{array}$$

Show, for all terms M, term variables x, types σ, α and type variables φ, that

$$\text{if } \{x_1{:}\sigma_1, \ldots, x_n{:}\sigma_n\} \vdash M : \sigma, \text{ then } \{x_1{:}\sigma_1[\alpha/\varphi], \ldots, x_n{:}\sigma_n[\alpha/\varphi]\} \vdash M : \sigma[\alpha/\varphi].$$

Answer:

(*Ax*) : Then $M \equiv x$, and $x{:}\sigma \in \{x_1{:}\sigma_1, \ldots, x_n{:}\sigma_n\}$. Notice that then $x{:}\sigma[\alpha/\varphi] \in \{x_1{:}\sigma_1[\alpha/\varphi], \ldots, x_n{:}\sigma_n[\alpha/\varphi]\}$, so, by rule (*Ax*), $\{x_1{:}\sigma_1[\alpha/\varphi], \ldots, x_n{:}\sigma_n[\alpha/\varphi]\} \vdash_{\mathrm{C}} x{:}\sigma[\alpha/\varphi]$.

($\rightarrow$*I*) : Then there are M', σ, ρ such that $M \equiv \lambda x.M'$, $\sigma = \rho \rightarrow \mu$, and $\{x_1{:}\sigma_1, \ldots, x_n{:}\sigma_n, x{:}\rho\} \vdash_{\mathrm{C}} M'{:}\mu$. Since this statement is derived in a sub-derivation, by induction, we know that,

$$\{x_1{:}\sigma_1[\alpha/\varphi], \ldots, x_n{:}\sigma_n[\alpha/\varphi], x{:}\rho[\alpha/\varphi]\} \vdash_{\mathrm{C}} M'{:}\mu[\alpha/\varphi].$$

To this result we can apply rule ($\rightarrow$*I*), to obtain

$$\{x_1{:}\sigma_1[\alpha/\varphi], \ldots, x_n{:}\sigma_n[\alpha/\varphi]\} \vdash_{\mathrm{C}} \lambda x.M'{:}\rho[\alpha/\varphi] \rightarrow \mu[\alpha/\varphi].$$

Since, by definition of substitutions, $\rho[\alpha/\varphi] \rightarrow \mu[\alpha/\varphi] = (\rho \rightarrow \mu)[\alpha/\varphi] = \sigma[\alpha/\varphi]$, we get

$$\{x_1{:}\sigma_1[\alpha/\varphi], \ldots, x_n{:}\sigma_n[\alpha/\varphi]\} \vdash_{\mathrm{C}} \lambda x.M'{:}\sigma[\alpha/\varphi].$$

$(\to E)$: Then M_1, M_2, τ such that $M \equiv M_1 M_2$,

$$\{x_1{:}\sigma_1, \ldots, x_n{:}\sigma_n\} \ \vdash_{\mathrm{C}} M_1 : \tau{\to}\sigma, \text{ and } \{x_1{:}\sigma_1, \ldots, x_n{:}\sigma_n\} \ \vdash_{\mathrm{C}} M_2 : \tau.$$

Since these two statements are derived in a sub-derivation, we know that, by induction,

$$\{x_1{:}\sigma_1[\alpha/\varphi], \ldots, x_n{:}\sigma_n[\alpha/\varphi]\} \ \vdash_{\mathrm{C}} M_1 : \tau{\to}\sigma[\alpha/\varphi]$$

and

$$\{x_1{:}\sigma_1[\alpha/\varphi], \ldots, x_n{:}\sigma_n[\alpha/\varphi]\} \ \vdash_{\mathrm{C}} M_2 : \tau[\alpha/\varphi].$$

Since, by definition of substitution, $(\tau{\to}\sigma)[\alpha/\varphi] = \tau[\alpha/\varphi]{\to}\sigma[\alpha/\varphi]$, we also have

$$\{x_1{:}\sigma_1[\alpha/\varphi], \ldots, x_n{:}\sigma_n[\alpha/\varphi]\} \ \vdash_{\mathrm{C}} M_1 : \tau[\alpha/\varphi]{\to}\sigma[\alpha/\varphi]m$$

and we can apply rule $(\to E)$ to obtain $\{x_1{:}\sigma_1[\alpha/\varphi], \ldots, x_n{:}\sigma_n[\alpha/\varphi]\} \ \vdash_{\mathrm{C}} M_1 M_2 : \sigma[\alpha/\varphi]$. ∎

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2) Give the Natural Semantics of the statement $x := 5\,\texttt{;}\,\texttt{while}\ x{<}7\ \texttt{do}\ x := x{+}1$.

Answer:

$$\dfrac{\dfrac{}{\langle x := 5, s_1\rangle \to s_3} \qquad \dfrac{\dfrac{}{\langle x := x{+}1, s_3\rangle \to s_4} \qquad \dfrac{\dfrac{}{\langle x := x{+}1, s_4\rangle \to s_5} \qquad \dfrac{}{\langle \texttt{while}\ x{<}7\ \texttt{do}\ x := x{+}1, s_5\rangle \to s_2}\ (\mathcal{B}[\![x{<}7]\!]\,s_5 = \textbf{ff})}{\langle \texttt{while}\ x{<}7\ \texttt{do}\ x := x{+}1, s_4\rangle \to s_2}\ (\mathcal{B}[\![x{<}7]\!]\,s_4 = \textbf{tt})}{\langle \texttt{while}\ x{<}7\ \texttt{do}\ x := x{+}1, s_3\rangle \to s_2}\ (\mathcal{B}[\![x{<}7]\!]\,s_3 = \textbf{tt})}{\langle x := 5\,\texttt{;}\,\texttt{while}\ x{<}7\ \texttt{do}\ x := x{+}1, s_1\rangle \to s_2}$$

where $s_3 = s_1[x \mapsto 5]$, $s_4 = s_3[x \mapsto 6] = s_1[x \mapsto 6]$, $s_5 = s_4[x \mapsto 7] = s_1[x \mapsto 7]$.

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3) Show that, for the language `While`, '$S_1\,\texttt{;}\,\texttt{if}\ b\ \texttt{then}\ S_2\ \texttt{else}\ S_3$' is semantically equivalent to '$\texttt{if}\ b\ \texttt{then}\ S_1\,\texttt{;}\,S_2\ \texttt{else}\ S_1\,\texttt{;}\,S_3$', provided that b does not depend on the variables modified by running S_1.

Answer:

To show: $\langle S_1\,\texttt{;}\,\texttt{if}\ b\ \texttt{then}\ S_2\ \texttt{else}\ S_3, s_1\rangle \to s_2$ if and only if $\langle \texttt{if}\ b\ \texttt{then}\ S_1\,\texttt{;}\,S_2\ \texttt{else}\ S_1\,\texttt{;}\,S_3, s_1\rangle \to s_2$. Well, $\langle S_1\,\texttt{;}\,\texttt{if}\ b\ \texttt{then}\ S_2\ \texttt{else}\ S_3, s_1\rangle \to s_2$ implies $\langle S_1, s_1\rangle \to s_3$ and $\langle \texttt{if}\ b\ \texttt{then}\ S_2\ \texttt{else}\ S_3, s_3\rangle \to s_2$, for some s_3. Now, there are two cases:

$(\mathcal{B}[\![b]\!]\,s_3 = \textbf{tt})$: Then $\langle S_2, s_3\rangle \to s_2$, so also $\langle S_1\,\texttt{;}\,S_2, s_1\rangle \to s_2$, and, since $\mathcal{B}[\![b]\!]\,s_1 = \mathcal{B}[\![b]\!]\,s_3 = \textbf{tt}$, also $\langle \texttt{if}\ b\ \texttt{then}\ S_1\,\texttt{;}\,S_2\ \texttt{else}\ S_1\,\texttt{;}\,S_3, s_1\rangle \to s_2$.

$(\mathcal{B}[\![b]\!]\,s_3 = \textbf{ff})$: Then $\langle S_3, s_3\rangle \to s_2$, so also $\langle S_1\,\texttt{;}\,S_3, s_1\rangle \to s_2$, and, since $\mathcal{B}[\![b]\!]\,s_1 = \mathcal{B}[\![b]\!]\,s_3 = \textbf{ff}$, also $\langle \texttt{if}\ b\ \texttt{then}\ S_1\,\texttt{;}\,S_2\ \texttt{else}\ S_1\,\texttt{;}\,S_3, s_1\rangle \to s_2$.
