

A Translation of Intersection and Union Types for the $\lambda\mu$ -Calculus

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We introduce an intersection and union type system for the $\lambda\mu$ -calculus, which includes a restricted version of the traditional union-elimination rule. Strong normalisation of terms typable by the system is shown via a translation of intersection and union types into intersection and product types, relying on the result of strong normalisation of terms typable by van Bakel, Barbanera and de'Liguoro's system.

1 Introduction

The Curry-Howard correspondence for classical logic is now widely known and has yielded various term systems, e.g. the calculi in [BB96, Par92]. Parigot's $\lambda\mu$ -calculus is one of such systems, and since it is a syntactical extension of the usual λ -calculus, the type-free version of the calculus, called pure $\lambda\mu$ -calculus in [Par92], has also been studied. As a type assignment system for type-free $\lambda\mu$ -terms, van Bakel, Barbanera and de'Liguoro [vBBdL11] recently introduced an intersection type system to develop model theory of the calculus. The system includes not only intersection types but also product types, and so looks involved at first sight. However, it can be naturally understood in the light of the negative translation used in [SR98], and indeed the simply typed part of the $\lambda\mu$ -calculus is interpreted by the systems with intersection and product types in [vBBdL11, vBBdL13].

Another approach to providing a type assignment system for type-free $\lambda\mu$ -terms is to employ a system with intersection and union types. In this approach, simple types inhabited by some terms correspond to implicational formulas that are provable in classical logic, and union types are used for continuations to have more than one type. There are two intersection and union type systems for the $\lambda\mu$ -calculus in the literature [Lau04, vB11]. In this paper we introduce another intersection and union type system where, unlike in the systems of [Lau04, vB11], union-introduction and elimination rules correspond to the usual or-introduction and elimination rules in natural deduction.

To clarify the relation between the two kinds of type systems, we introduce a translation from intersection and union types to intersection and product types. Using the translation, we prove strong normalisation of terms typable by the systems with intersection and union types.

2 The $\lambda\mu$ -calculus

First we introduce the syntax of Parigot's pure $\lambda\mu$ -calculus [Par92].

Definition 2.1 (Grammar of $\lambda\mu$). *The sets of terms and commands are defined inductively by the following grammar:*

$$\begin{aligned} M, N &::= x \mid \lambda x.M \mid MN \mid \mu\alpha.C && \text{(terms)} \\ C &::= [\alpha]M && \text{(commands)} \end{aligned}$$

where x and α range over denumerable sets of λ -variables and μ -variables, respectively.

The notions of free and bound variables are defined as usual, with both λ and μ as binders. The sets of free λ -variables and μ -variables of a term M are denoted by $FV_\lambda(M)$ and $FV_\mu(M)$, respectively. We identify α -convertible terms.

Definition 2.2 (Reduction system of $\lambda\mu$). *The reduction rules are:*

$$\begin{aligned} (\beta) \quad & (\lambda x.M)N \rightarrow M[x := N] \\ (\mu) \quad & (\mu\alpha.C)N \rightarrow \mu\alpha.C[\alpha \leftarrow N] \end{aligned}$$

where $[x := N]$ is usual capture-free substitution, and $[\alpha \leftarrow N]$ in the rule (μ) replaces inductively each occurrence in C of the form $[\alpha]P$ by $[\alpha](PN)$.

The reduction relation $\longrightarrow_{\beta,\mu}$ is defined by the contextual closure of the rules (β) and (μ) . A term M is said to be *strongly normalising* if there is no infinite β, μ -reduction sequence out of M . The set of strongly normalising terms is denoted by $SN^{\beta,\mu}$.

3 An intersection and union type system for the $\lambda\mu$ -calculus

In this section we introduce an intersection and union type system for the $\lambda\mu$ -calculus. The types we define here are of three kinds, following the definition in [Lau04] (without the empty intersection).

Definition 3.1. *The sets $\mathcal{T}_A, \mathcal{T}_I$ and \mathcal{T}_U of three kinds of types are defined inductively by the following grammar:*

$$\begin{aligned} \mathcal{T}_A: \quad & A, B ::= \varphi \mid I \rightarrow U \quad (\text{arrow types}) \\ \mathcal{T}_I: \quad & I, J ::= U \mid I \cap J \quad (\text{intersection types}) \\ \mathcal{T}_U: \quad & U, V ::= A \mid U \cup V \quad (\text{union types}) \end{aligned}$$

where φ ranges over a denumerable set of type variables. We identify types modulo associativity and commutativity of \cap and \cup , and use \equiv to denote the equivalence.

$\frac{}{\Gamma, x: I \vdash x: I \mid \Delta} \text{ (Ax)}$	$\frac{\Gamma \vdash M: U \mid \alpha: U, \Delta}{\Gamma \vdash \mu\alpha.[\alpha]M: U \mid \Delta} \text{ } (\mu_1)$	$\frac{\Gamma \vdash M: V \mid \alpha: U, \gamma: V, \Delta}{\Gamma \vdash \mu\alpha.[\gamma]M: U \mid \gamma: V, \Delta} \text{ } (\mu_2)$
$\frac{\Gamma, x: I \vdash M: U \mid \Delta}{\Gamma \vdash \lambda x.M: I \rightarrow U \mid \Delta} \text{ } (\rightarrow I)$	$\frac{\Gamma \vdash M: I \rightarrow U \mid \Delta \quad \Gamma \vdash N: I \mid \Delta}{\Gamma \vdash MN: U \mid \Delta} \text{ } (\rightarrow E)$	
$\frac{\Gamma \vdash M: I \mid \Delta \quad \Gamma \vdash M: J \mid \Delta}{\Gamma \vdash M: I \cap J \mid \Delta} \text{ } (\cap I)$	$\frac{\Gamma \vdash M: I \cap J \mid \Delta}{\Gamma \vdash M: I \mid \Delta} \text{ } (\cap E)$	$\frac{\Gamma \vdash M: I \cap J \mid \Delta}{\Gamma \vdash M: J \mid \Delta} \text{ } (\cap E)$
$\frac{\Gamma \vdash M: U \mid \Delta}{\Gamma \vdash M: U \cup V \mid \Delta} \text{ } (\cup I)$		$\frac{\Gamma \vdash M: V \mid \Delta}{\Gamma \vdash M: U \cup V \mid \Delta} \text{ } (\cup I)$
$\frac{\Gamma \vdash M: U \cup V \mid \Delta \quad \Gamma, x: U \vdash xN: I \mid \Delta \quad \Gamma, x: V \vdash xN: I \mid \Delta}{\Gamma \vdash MN: I \mid \Delta} \text{ } (\cup E)$		
where $x \notin FV_\lambda(N)$		

Figure 1: Type assignment system $\lambda\mu_{\cap\cup}$

The type assignment system $\lambda\mu_{\cup}$ is defined by the rules in Figure 1. A *type environment*, ranged over by Γ , is defined as a finite set of pairs $\{x_1 : I_1, \dots, x_n : I_n\}$ where the λ -variables are pairwise distinct. The typing context $\Gamma, x : I$ denotes the union $\Gamma \cup \{x : I\}$ where x does not appear in Γ . Similarly for typing contexts with μ -variables $\{\alpha_1 : U_1, \dots, \alpha_n : U_n\}$, ranged over by Δ , except that the types are restricted to union types. We write $\Gamma \vdash_{\cup} M : I \mid \Delta$ if $\Gamma \vdash M : I \mid \Delta$ is derivable with the rules in Figure 1.

The rule ($\cup E$) in Figure 1 is a rather restricted version of the traditional union-elimination rule that would have the form:

$$\frac{\Gamma \vdash M : U \cup V \mid \Delta \quad \Gamma, x : U \vdash N : I \mid \Delta \quad \Gamma, x : V \vdash N : I \mid \Delta}{\Gamma \vdash N[x := M] : I \mid \Delta}$$

This general version causes the subject-reduction problem as in the case of an intersection and union type system for λ -terms (cf. [BDCdL95]). Though our rule ($\cup E$) might look too restrictive, the system $\lambda\mu_{\cup}$ is more general than the intersection and union type systems proposed in [Lau04] and [vB11] in the sense that if a term is typable in one of their systems without the empty intersection then it is typable in $\lambda\mu_{\cup}$ (cf. Section 5). An example of a judgement that is derivable in $\lambda\mu_{\cup}$ but not in the systems in [Lau04, vB11] is $x : \varphi_1 \vdash x : \varphi_1 \cup \varphi_2 \mid$.

Example 3.2. The term $(\mu\alpha.[\alpha](\lambda y.\mu\gamma.[\alpha]y))z$ is typable in the system $\lambda\mu_{\cup}$ as follows. Let $A \equiv \varphi_1 \rightarrow \varphi_2$ and $\Gamma = \{z : \varphi_1 \cap A\}$, and let D_1 be the following derivation:

$$\frac{\frac{\frac{\frac{\frac{\frac{\Gamma, y : A \vdash y : A \mid \gamma : \varphi_3, \alpha : A \cup (A \rightarrow \varphi_3)}{\Gamma, y : A \vdash y : A \cup (A \rightarrow \varphi_3) \mid \gamma : \varphi_3, \alpha : A \cup (A \rightarrow \varphi_3)} (\cup 1)}{\Gamma, y : A \vdash \mu\gamma.[\alpha]y : \varphi_3 \mid \alpha : A \cup (A \rightarrow \varphi_3)} (\mu_2)}{\Gamma \vdash \lambda y.\mu\gamma.[\alpha]y : A \rightarrow \varphi_3 \mid \alpha : A \cup (A \rightarrow \varphi_3)} (\rightarrow I)}{\Gamma \vdash \lambda y.\mu\gamma.[\alpha]y : A \cup (A \rightarrow \varphi_3) \mid \alpha : A \cup (A \rightarrow \varphi_3)} (\cup 1)}{\Gamma \vdash \mu\alpha.[\alpha](\lambda y.\mu\gamma.[\alpha]y) : A \cup (A \rightarrow \varphi_3) \mid} (\mu_1)$$

Let D_2 be the following derivation:

$$\frac{\frac{\frac{\frac{\Gamma, x : A \vdash x : A \mid} (\text{Ax})}{\Gamma, x : A \vdash xz : \varphi_2 \mid} (\cup 1)}{\Gamma, x : A \vdash xz : \varphi_2 \cup \varphi_3 \mid} (\cup 1)}{\Gamma, x : A \vdash z : \varphi_1 \cap A \mid} (\cap E) \quad \frac{\Gamma, x : A \vdash z : \varphi_1 \cap A \mid} {\Gamma, x : A \vdash z : \varphi_1 \mid} (\rightarrow E)}$$

Let D_3 be the following derivation:

$$\frac{\frac{\frac{\frac{\Gamma, x : A \rightarrow \varphi_3 \vdash x : A \rightarrow \varphi_3 \mid} (\text{Ax})}{\Gamma, x : A \rightarrow \varphi_3 \vdash xz : \varphi_3 \mid} (\cup 1)}{\Gamma, x : A \rightarrow \varphi_3 \vdash xz : \varphi_2 \cup \varphi_3 \mid} (\cup 1)}{\Gamma, x : A \rightarrow \varphi_3 \vdash z : \varphi_1 \cap A \mid} (\cap E) \quad \frac{\Gamma, x : A \rightarrow \varphi_3 \vdash z : \varphi_1 \cap A \mid} {\Gamma, x : A \rightarrow \varphi_3 \vdash z : A \mid} (\rightarrow E)}$$

Then we have

$$\frac{\frac{\frac{\frac{\vdots D_1}{\Gamma \vdash \mu\alpha.[\alpha](\lambda y.\mu\gamma.[\alpha]y) : A \cup (A \rightarrow \varphi_3) \mid} (\cup 1)}{\Gamma, x : A \vdash xz : \varphi_2 \cup \varphi_3 \mid} (\cup 1)}{\Gamma, x : A \rightarrow \varphi_3 \vdash xz : \varphi_2 \cup \varphi_3 \mid} (\cup 1)}{\Gamma \vdash (\mu\alpha.[\alpha](\lambda y.\mu\gamma.[\alpha]y))z : \varphi_2 \cup \varphi_3 \mid} (\cup E) \quad \frac{\frac{\frac{\vdots D_2}{\Gamma, x : A \vdash xz : \varphi_2 \cup \varphi_3 \mid} (\cup 1)}{\Gamma, x : A \rightarrow \varphi_3 \vdash xz : \varphi_2 \cup \varphi_3 \mid} (\cup 1)}{\Gamma \vdash (\mu\alpha.[\alpha](\lambda y.\mu\gamma.[\alpha]y))z : \varphi_2 \cup \varphi_3 \mid} (\cup E)}$$

Note that this term is not typable without using the rules for \cup .

The next lemma shows that the system $\lambda\mu_{\cup}$ is closed under the weakening rule.

Lemma 3.3.

1. If $\Gamma \vdash_{\cup} t : I \mid \Delta$ and x is a fresh λ -variable then $\Gamma, x : J \vdash_{\cup} t : I \mid \Delta$.
2. If $\Gamma \vdash_{\cup} t : I \mid \Delta$ and α is a fresh μ -variable then $\Gamma \vdash_{\cup} t : I \mid \alpha : U, \Delta$.

Proof. By induction on the derivations. □

The rule (UE) can be generalised to the following form.

Lemma 3.4. *The following rule is admissible in the system $\lambda\mu_{\cup}$.*

$$\frac{\Gamma \vdash M : U_1 \cup \dots \cup U_n \mid \Delta \quad \Gamma, x : U_1 \vdash xN : I \mid \Delta \quad \dots \quad \Gamma, x : U_n \vdash xN : I \mid \Delta}{\Gamma \vdash MN : I \mid \Delta} \text{(UE)}^n$$

where $n \geq 2$ and $x \notin \text{FV}_{\lambda}(N)$.

Proof. By induction on n . If $n > 2$ then we have, by the induction hypothesis,

$$\frac{\frac{\Gamma' \vdash x : U_1 \cup \dots \cup U_{n-1} \mid \Delta}{\Gamma' \vdash x : U_1 \cup \dots \cup U_{n-1} \mid \Delta} \text{(Ax)} \quad \begin{array}{c} \vdots D_1 \\ \Gamma', x' : U_1 \vdash x'N : I \mid \Delta \end{array} \quad \dots \quad \begin{array}{c} \vdots D_{n-1} \\ \Gamma', x' : U_{n-1} \vdash x'N : I \mid \Delta \end{array}}{\Gamma' \vdash xN : I \mid \Delta} \text{(UE)}^{n-1}$$

where $\Gamma' = \Gamma, x : U_1 \cup \dots \cup U_{n-1}$ and each D_i is obtained from the assumption by renaming x to x' . Hence, by the rule (UE), we can derive $\Gamma \vdash MN : I \mid \Delta$ from $\Gamma \vdash M : U_1 \cup \dots \cup U_n \mid \Delta$, $\Gamma' \vdash xN : I \mid \Delta$ and $\Gamma, x : U_n \vdash xN : I \mid \Delta$. □

4 The type system of van Bakel, Barbanera and de'Liguoro

In this section we briefly recall the intersection type system in [vBBdL13], which also uses product types.

Definition 4.1. *The sets \mathcal{T}_D of term types and \mathcal{T}_C of continuation-stack types are defined inductively by the following grammar:*

$$\begin{aligned} \mathcal{T}_D : \delta &::= v \mid \omega \rightarrow v \mid \kappa \rightarrow v \mid \delta \wedge \delta \\ \mathcal{T}_C : \kappa &::= \delta \times \omega \mid \delta \times \kappa \mid \kappa \wedge \kappa \end{aligned}$$

where v and ω are type constants. Elements of $\mathcal{T}_D \cup \mathcal{T}_C$ are ranged over by σ, τ, ρ .

The relations \leq_D and \leq_C on \mathcal{T}_D and \mathcal{T}_C , respectively, are defined by the rules in Figure 2, where \leq_A denotes either \leq_D or \leq_C .

Lemma 4.2. $(\delta_1 \wedge \dots \wedge \delta_n) \times (\kappa_1 \wedge \dots \wedge \kappa_n) \leq_C (\delta_1 \times \kappa_1) \wedge \dots \wedge (\delta_n \times \kappa_n)$.

Proof. By $\delta_1 \wedge \dots \wedge \delta_n \leq_D \delta_i$ and $\kappa_1 \wedge \dots \wedge \kappa_n \leq_C \kappa_i$, we have $(\delta_1 \wedge \dots \wedge \delta_n) \times (\kappa_1 \wedge \dots \wedge \kappa_n) \leq_C \delta_i \times \kappa_i$ for each $i \in \{1, \dots, n\}$. Hence $(\delta_1 \wedge \dots \wedge \delta_n) \times (\kappa_1 \wedge \dots \wedge \kappa_n) \leq_C (\delta_1 \times \kappa_1) \wedge \dots \wedge (\delta_n \times \kappa_n)$. □

The type assignment system $\lambda\mu_{\wedge \times}$ is defined by the rules in Figure 3. We write $\Gamma \vdash_{\wedge \times} M : \delta \mid \Delta$ if $\Gamma \vdash M : \delta \mid \Delta$ is derivable with the rules of Figure 3.

In [vBBdL13], it was shown that strongly normalising terms are characterised by means of typability in the system $\lambda\mu_{\wedge \times}$. The following is one direction of the characterisation theorem.

Theorem 4.3 ([vBBdL13]). *If $\Gamma \vdash_{\wedge \times} M : \delta \mid \Delta$ for some Γ, δ and Δ , then $M \in \text{SN}^{\beta, \mu}$.*

$$\begin{array}{c}
\overline{\sigma \leq_A \sigma} \quad \overline{\sigma \wedge \tau \leq_A \sigma} \quad \overline{\sigma \wedge \tau \leq_A \tau} \quad \overline{v \leq_D \omega \rightarrow v} \quad \overline{\omega \rightarrow v \leq_D v} \quad \overline{\delta_1 \times \delta_2 \times \omega \leq_C \delta_1 \times \omega} \\
\\
\overline{(\delta_1 \times \omega) \wedge (\delta_2 \times \kappa) \leq_C (\delta_1 \wedge \delta_2) \times \kappa} \quad \overline{(\delta_1 \times \kappa_1) \wedge (\delta_2 \times \kappa_2) \leq_C (\delta_1 \wedge \delta_2) \times (\kappa_1 \wedge \kappa_2)} \\
\text{where } \kappa_1 \neq \omega \text{ and } \kappa_2 \neq \omega \\
\\
\frac{\sigma \leq_A \rho \quad \rho \leq_A \tau}{\sigma \leq_A \tau} \quad \frac{\sigma \leq_A \tau_1 \quad \sigma \leq_A \tau_2}{\sigma \leq_A \tau_1 \wedge \tau_2} \\
\\
\frac{\delta_1 \leq_D \delta_2}{\delta_1 \times \omega \leq_C \delta_2 \times \omega} \quad \frac{\delta_1 \leq_D \delta_2 \quad \kappa_1 \leq_C \kappa_2}{\delta_1 \times \kappa_1 \leq_C \delta_2 \times \kappa_2} \quad \frac{\kappa_2 \leq_C \kappa_1}{\kappa_1 \rightarrow v \leq_D \kappa_2 \rightarrow v}
\end{array}$$

Figure 2: Relations \leq_D and \leq_C

$$\begin{array}{c}
\frac{}{\Gamma, x : \delta \vdash x : \delta \mid \Delta} \text{ (Ax)} \quad \frac{\Gamma \vdash M : \kappa \rightarrow v \mid \alpha : \kappa, \Delta}{\Gamma \vdash \mu\alpha.[\alpha]M : \kappa \rightarrow v \mid \Delta} \text{ } (\mu) \quad \frac{\Gamma \vdash M : \kappa' \rightarrow v \mid \alpha : \kappa, \gamma : \kappa', \Delta}{\Gamma \vdash \mu\alpha.[\gamma]M : \kappa \rightarrow v \mid \gamma : \kappa', \Delta} \text{ } (\mu) \\
\\
\frac{\Gamma, x : \delta \vdash M : \kappa \rightarrow v \mid \Delta}{\Gamma \vdash \lambda x.M : \delta \times \kappa \rightarrow v \mid \Delta} \text{ (Abs)} \quad \frac{\Gamma \vdash M : \delta \times \kappa \rightarrow v \mid \Delta \quad \Gamma \vdash N : \delta \mid \Delta}{\Gamma \vdash MN : \kappa \rightarrow v \mid \Delta} \text{ (App)} \\
\text{where } \kappa \text{ is either a type in } \mathcal{T}_C \text{ or } \omega \quad \text{where } \kappa \text{ is either a type in } \mathcal{T}_C \text{ or } \omega \\
\\
\frac{\Gamma \vdash M : \delta \mid \Delta \quad \Gamma \vdash M : \delta' \mid \Delta}{\Gamma \vdash M : \delta \wedge \delta' \mid \Delta} \text{ } (\wedge) \quad \frac{\Gamma \vdash M : \delta \mid \Delta \quad \delta \leq_D \delta'}{\Gamma \vdash M : \delta' \mid \Delta} \text{ } (\leq)
\end{array}$$

Figure 3: Type assignment system $\lambda\mu_{\wedge \times}$

5 A translation of intersection and union types

Now we introduce a translation of intersection and union types into intersection and product types, extending the translation of simple types in [vBBdL13]. The translation is defined along the structure of the three kinds of types in Definition 3.1. The aim is to prove strong normalisation of terms typable in $\lambda\mu_{\cap \cup}$, using this translation and the strong normalisation result of terms typable in $\lambda\mu_{\wedge \times}$.

Definition 5.1. *The mappings $(\cdot)^D : \mathcal{T}_I \rightarrow \mathcal{T}_D$ and $(\cdot)^C : \mathcal{T}_U \rightarrow \mathcal{T}_C$ are defined inductively as follows:*

$$\begin{aligned}
\varphi^C &:= v \times \omega \\
(I \rightarrow U)^C &:= I^D \times U^C \\
(U \cup V)^C &:= U^C \wedge V^C \\
U^D &:= U^C \rightarrow v \\
(I \cap J)^D &:= I^D \wedge J^D
\end{aligned}$$

It can be easily verified that the above mappings are well-defined. We extend the mappings to typing contexts by $\Gamma^D := \{x : I^D \mid x : I \in \Gamma\}$ and $\Delta^C := \{\alpha : U^C \mid \alpha : U \in \Delta\}$.

Proving the preservation of derivability in $\lambda\mu_{\cap \cup}$ by the translation requires some observations on the system $\lambda\mu_{\wedge \times}$. We give a detailed proof of it in the appendix. Here we instead show the preservation of

derivability in the system that is obtained from $\lambda\mu_{\cup}$ by replacing the rules $(\rightarrow E)$ and $(\cup E)$ with the following one:

$$\frac{\Gamma \vdash M : (I_1 \rightarrow U_1) \cup \dots \cup (I_n \rightarrow U_n) \mid \Delta \quad \Gamma \vdash N : I_1 \mid \Delta \quad \dots \quad \Gamma \vdash N : I_n \mid \Delta}{\Gamma \vdash MN : U_1 \cup \dots \cup U_n \mid \Delta} (\rightarrow E)'$$

where $n \geq 1$. This rule is the same as one of the rules of the intersection and union type system in [vB11]. A similar rule also appeared in [Lau04]. We write $\Gamma \vdash_{\cup} M : \delta \mid \Delta$ if $\Gamma \vdash M : \delta \mid \Delta$ is derivable in this alternative system. By using the rule $(\cup E)^n$ in Lemma 3.4, we see that the rule $(\rightarrow E)'$ is derivable in the original system $\lambda\mu_{\cup}$, so $\Gamma \vdash_{\cup} M : \delta \mid \Delta$ implies $\Gamma \vdash_{\cup} M : \delta \mid \Delta$.

Theorem 5.2. *If $\Gamma \vdash_{\cup} M : I \mid \Delta$ then $\Gamma^D \vdash_{\wedge} M : I^D \mid \Delta^C$.*

Proof. By induction on the derivation of $\Gamma \vdash_{\cup} M : I \mid \Delta$. Here we consider some cases.

- $\frac{\Gamma \vdash M : V \mid \alpha : U, \gamma : V, \Delta}{\Gamma \vdash \mu\alpha.[\gamma]M : U \mid \gamma : V, \Delta} (\mu_2)$

By the induction hypothesis, we have $\Gamma^D \vdash_{\wedge} M : V^D \mid \alpha : U^C, \gamma : V^C, \Delta^C$ where $V^D \equiv V^C \rightarrow v$. Then by the rule (μ) , we obtain $\Gamma^D \vdash_{\wedge} \mu\alpha.[\gamma]M : U^C \rightarrow v \mid \gamma : V^C, \Delta^C$.

- $\frac{\Gamma, x : I \vdash M : U \mid \Delta}{\Gamma \vdash \lambda x.M : I \rightarrow U \mid \Delta} (\rightarrow I)$

By the induction hypothesis, we have $\Gamma^D, x : I^D \vdash_{\wedge} M : U^D \mid \Delta^C$ where $U^D \equiv U^C \rightarrow v$. Then by the rule (Abs) , we obtain $\Gamma^D \vdash_{\wedge} \lambda x.M : I^D \times U^C \rightarrow v \mid \Delta^C$ where $I^D \times U^C \rightarrow v \equiv (I \rightarrow U)^C \rightarrow v \equiv (I \rightarrow U)^D$.

- $\frac{\Gamma \vdash M : U \mid \Delta}{\Gamma \vdash M : U \cup V \mid \Delta} (\cup I)$

By the induction hypothesis, we have $\Gamma^D \vdash_{\wedge} M : U^D \mid \Delta^C$ where $U^D \equiv U^C \rightarrow v$. From the definition of \leq_D , we have $U^C \rightarrow v \leq_D U^C \wedge V^C \rightarrow v$. Hence by the rule (\leq) , we obtain $\Gamma^D \vdash_{\wedge} M : U^C \wedge V^C \rightarrow v \mid \Delta^C$ where $U^C \wedge V^C \rightarrow v \equiv (U \cup V)^C \rightarrow v \equiv (U \cup V)^D$.

- $\frac{\Gamma \vdash M : (I_1 \rightarrow U_1) \cup \dots \cup (I_n \rightarrow U_n) \mid \Delta \quad \Gamma \vdash N : I_1 \mid \Delta \quad \dots \quad \Gamma \vdash N : I_n \mid \Delta}{\Gamma \vdash MN : U_1 \cup \dots \cup U_n \mid \Delta} (\rightarrow E)'$

By the induction hypothesis, we have $\Gamma^D \vdash_{\wedge} M : ((I_1 \rightarrow U_1) \cup \dots \cup (I_n \rightarrow U_n))^D \mid \Delta^C$ and, for all $i \in \{1, \dots, n\}$, $\Gamma^D \vdash_{\wedge} N : I_i^D \mid \Delta^C$. Then by the rule (\wedge) , we have $\Gamma^D \vdash_{\wedge} N : I_1^D \wedge \dots \wedge I_n^D \mid \Delta^C$. Now

$$\begin{aligned} ((I_1 \rightarrow U_1) \cup \dots \cup (I_n \rightarrow U_n))^D &\equiv ((I_1 \rightarrow U_1) \cup \dots \cup (I_n \rightarrow U_n))^C \rightarrow v \\ &\equiv (I_1 \rightarrow U_1)^C \wedge \dots \wedge (I_n \rightarrow U_n)^C \rightarrow v \\ &\equiv (I_1^D \times U_1^C) \wedge \dots \wedge (I_n^D \times U_n^C) \rightarrow v \\ &\leq_D (I_1^D \wedge \dots \wedge I_n^D) \times (U_1^C \wedge \dots \wedge U_n^C) \rightarrow v \quad (\text{by Lemma 4.2}) \end{aligned}$$

Hence by the rules (\leq) and (App) , we obtain $\Gamma^D \vdash_{\wedge} MN : (U_1^C \wedge \dots \wedge U_n^C) \rightarrow v \mid \Delta^C$ where $(U_1^C \wedge \dots \wedge U_n^C) \rightarrow v \equiv (U_1 \cup \dots \cup U_n)^C \rightarrow v \equiv (U_1 \cup \dots \cup U_n)^D$. \square

Corollary 5.3. *If $\Gamma \vdash_{\cup} M : I \mid \Delta$ for some Γ, I and Δ , then $M \in \text{SN}^{\beta, \mu}$.*

Proof. By Theorems 4.3 and 5.2. □

The above corollary is enough to show strong normalisation of terms typable in the systems (without the type constants) of [Lau04] and [vB11]. We can also prove the same result for the full system $\lambda\mu_{\text{nu}}$. (See Appendix A.)

6 Conclusion

We have introduced an intersection and union type system with a restricted version of the traditional union-elimination rule. We have presented a translation of intersection and union types into intersection and product types, and proved via the translation strong normalisation of terms typable by the intersection and union type system. Since strong normalisation was not proved in [Lau04, vB11], this seems the first strong normalisation result of terms typable in an intersection and union type system for the $\lambda\mu$ -calculus.

It is expected that our type system for the $\lambda\mu$ -calculus enjoys the subject reduction property. This is plausible since, by the side condition of the union-elimination rule, the variable to be discharged can occur only once in each premiss, in which case known counter examples do not emerge. It is also expected that all strongly normalising terms in the $\lambda\mu$ -calculus are typable in our system. These problems are to be investigated in future work.

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A Appendix

In this appendix we prove the preservation of derivability in $\lambda\mu_{\cap\cup}$ by the translation in Section 5.

To simplify the argument, we first restrict the relations \leq_D and \leq_C to $\leq_D^\#$ and $\leq_C^\#$ defined by the rules in Figure 4 (where $\leq_A^\#$ denotes either $\leq_D^\#$ or $\leq_C^\#$). Clearly, $\sigma \leq_A^\# \tau$ implies $\sigma \leq_A \tau$.

$\overline{\sigma \leq_A^\# \sigma}$	$\overline{\sigma \wedge \tau \leq_A^\# \sigma}$	$\overline{\sigma \wedge \tau \leq_A^\# \tau}$
$\frac{\sigma \leq_A^\# \rho \quad \rho \leq_A^\# \tau}{\sigma \leq_A^\# \tau}$	$\frac{\sigma \leq_A^\# \tau_1 \quad \sigma \leq_A^\# \tau_2}{\sigma \leq_A^\# \tau_1 \wedge \tau_2}$	
$\frac{\delta_1 \leq_D^\# \delta_2 \quad \kappa_1 \leq_C^\# \kappa_2}{\delta_1 \times \kappa_1 \leq_C^\# \delta_2 \times \kappa_2}$		$\frac{\kappa_2 \leq_C^\# \kappa_1}{\kappa_1 \rightarrow v \leq_D^\# \kappa_2 \rightarrow v}$

Figure 4: Relations $\leq_D^\#$ and $\leq_C^\#$

Lemma A.1. *Let $\delta_1 \wedge \dots \wedge \delta_n \leq_D^\# \delta'_1 \wedge \dots \wedge \delta'_m$ where none of the δ_i ($i \in \{1, \dots, n\}$) and δ'_j ($j \in \{1, \dots, m\}$) is an intersection. Then, for each δ'_j of the form $\kappa' \rightarrow v$, there exists δ_i such that $\delta_i \equiv \kappa \rightarrow v$ and $\kappa' \leq_C^\# \kappa$.*

Proof. By induction on the definition of $\leq_D^\#$. □

The next lemma is the $\leq^\#$ analogue of Lemma 4.2.

Lemma A.2. $(\delta_1 \wedge \dots \wedge \delta_n) \times (\kappa_1 \wedge \dots \wedge \kappa_n) \leq_C^\# (\delta_1 \times \kappa_1) \wedge \dots \wedge (\delta_n \times \kappa_n)$.

The type assignment system $\lambda\mu_{\wedge\lambda}^\#$ is defined by the same rules as $\lambda\mu_{\wedge\lambda}$ except that the rule (\leq) is replaced by the rule ($\leq^\#$) with $\leq_D^\#$ instead of \leq_D . (We also assume $\kappa \neq \omega$ in the rules (Abs) and (App).) We write $\Gamma \vdash_{\wedge\lambda}^\# M : \delta \mid \Delta$ if $\Gamma \vdash M : \delta \mid \Delta$ is derivable in the system $\lambda\mu_{\wedge\lambda}^\#$. Clearly, $\Gamma \vdash_{\wedge\lambda}^\# M : \delta \mid \Delta$ implies $\Gamma \vdash_{\wedge\lambda} M : \delta \mid \Delta$.

Lemma A.3. *If $\Gamma, x : \delta \vdash_{\wedge\lambda}^\# M : \delta'$ and $x \notin \text{FV}_\lambda(M)$ then $\Gamma \vdash_{\wedge\lambda}^\# M : \delta'$.*

Proof. By induction on the derivation of $\Gamma, x : \delta \vdash_{\wedge\lambda}^\# M : \delta'$. □

Lemma A.4 (Generation Lemma).

1. *If $\Gamma \vdash_{\wedge\lambda}^\# x : \delta$ then there exists $x : \delta' \in \Gamma$ such that $\delta' \leq_D^\# \delta$.*
2. *If $\Gamma \vdash_{\wedge\lambda}^\# MN : \delta \mid \Delta$ then there exist $\kappa_1, \dots, \kappa_n, \delta_1, \dots, \delta_n$ ($n \geq 1$) such that $(\kappa_1 \rightarrow v) \wedge \dots \wedge (\kappa_n \rightarrow v) \leq_D^\# \delta$ and, for all $i \in \{1, \dots, n\}$, $\Gamma \vdash_{\wedge\lambda}^\# M : \delta_i \times \kappa_i \rightarrow v \mid \Delta$ and $\Gamma \vdash_{\wedge\lambda}^\# N : \delta_i \mid \Delta$.*

Proof. By induction on the derivations. □

Now we prove the announced theorem.

Theorem A.5. *If $\Gamma \vdash_{\cap\cup} M : I \mid \Delta$ then $\Gamma^D \vdash_{\wedge\lambda}^\# M : I^D \mid \Delta^C$.*

Proof. By induction on the derivation of $\Gamma \vdash_{\cap\cup} M : I \mid \Delta$. Here we only consider the case where the last applied rule in the derivation is ($\cup E$). The other cases are proved similarly to those in the proof of Theorem 5.2 by checking that the derivations under consideration only use the rules of $\lambda\mu_{\wedge\lambda}^\#$.

$$\bullet \frac{\Gamma \vdash M : U \cup V \mid \Delta \quad \Gamma, x : U \vdash xN : I \mid \Delta \quad \Gamma, x : V \vdash xN : I \mid \Delta}{\Gamma \vdash MN : I \mid \Delta} \text{ (}\cup\text{E)}$$

where $x \notin \text{FV}_\lambda(N)$. By the induction hypothesis, we have $\Gamma^D \vdash_{\wedge \times}^\# M : (U \cup V)^D \mid \Delta^C$,

$\Gamma^D, x : U^D \vdash_{\wedge \times}^\# xN : I^D \mid \Delta^C$ and $\Gamma^D, x : V^D \vdash_{\wedge \times}^\# xN : I^D \mid \Delta^C$. Then by Lemma A.4(2), we have

- (a) there exist $\kappa_1, \dots, \kappa_n, \delta_1, \dots, \delta_n$ ($n \geq 1$) such that $(\kappa_1 \rightarrow v) \wedge \dots \wedge (\kappa_n \rightarrow v) \leq_D^\# I^D$ and, for all $i \in \{1, \dots, n\}$, $\Gamma^D, x : U^D \vdash_{\wedge \times}^\# x : \delta_i \times \kappa_i \rightarrow v \mid \Delta^C$ and $\Gamma^D, x : U^D \vdash_{\wedge \times}^\# N : \delta_i \mid \Delta^C$, and
- (b) there exist $\kappa'_1, \dots, \kappa'_{n'}, \delta'_1, \dots, \delta'_{n'}$ ($n' \geq 1$) such that $(\kappa'_1 \rightarrow v) \wedge \dots \wedge (\kappa'_{n'} \rightarrow v) \leq_D^\# I^D$ and, for all $k \in \{1, \dots, n'\}$, $\Gamma^D, x : V^D \vdash_{\wedge \times}^\# x : \delta'_k \times \kappa'_k \rightarrow v \mid \Delta^C$ and $\Gamma^D, x : V^D \vdash_{\wedge \times}^\# N : \delta'_k \mid \Delta^C$.

Let $I^D \equiv (U_1 \cap \dots \cap U_m)^D \equiv U_1^D \wedge \dots \wedge U_m^D \equiv (U_1^C \rightarrow v) \wedge \dots \wedge (U_m^C \rightarrow v)$. Then, from (a) and Lemma A.1, we have

- (c) for each $U_j^D (\equiv U_j^C \rightarrow v)$ ($j \in \{1, \dots, m\}$), there exists κ_i ($i \in \{1, \dots, n\}$) such that
 - (c1) $U_j^C \leq_C^\# \kappa_i$,
 - (c2) $\Gamma^D, x : U^D \vdash_{\wedge \times}^\# x : \delta_i \times \kappa_i \rightarrow v \mid \Delta^C$, and
 - (c3) $\Gamma^D, x : U^D \vdash_{\wedge \times}^\# N : \delta_i \mid \Delta^C$.

Similarly, from (b) and Lemma A.1, we have

- (d) for each $U_j^D (\equiv U_j^C \rightarrow v)$ ($j \in \{1, \dots, m\}$), there exists κ'_k ($k \in \{1, \dots, n'\}$) such that
 - (d1) $U_j^C \leq_C^\# \kappa'_k$,
 - (d2) $\Gamma^D, x : V^D \vdash_{\wedge \times}^\# x : \delta'_k \times \kappa'_k \rightarrow v \mid \Delta^C$, and
 - (d3) $\Gamma^D, x : V^D \vdash_{\wedge \times}^\# N : \delta'_k \mid \Delta^C$.

Now, fix $j \in \{1, \dots, m\}$, and take κ_i and κ'_k satisfying (c1)-(c3) and (d1)-(d3), respectively. From (c2), we have $U^D \leq_D^\# \delta_i \times \kappa_i \rightarrow v$ by Lemma A.4(1). Since $U^D \equiv U^C \rightarrow v$, we have $\delta_i \times \kappa_i \leq_C^\# U^C$ by Lemma A.1. Similarly, from (d2), we have $\delta'_k \times \kappa'_k \leq_C^\# V^C$. Hence $(\delta_i \times \kappa_i) \wedge (\delta'_k \times \kappa'_k) \leq_C^\# U^C \wedge V^C$, and so

$$\begin{aligned} U^C \wedge V^C \rightarrow v &\leq_D^\# (\delta_i \times \kappa_i) \wedge (\delta'_k \times \kappa'_k) \rightarrow v \\ &\leq_D^\# (\delta_i \wedge \delta'_k) \times (\kappa_i \wedge \kappa'_k) \rightarrow v \end{aligned} \quad \text{(by Lemma A.2)}$$

By the induction hypothesis, we have $\Gamma^D \vdash_{\wedge \times}^\# M : (U \cup V)^D \mid \Delta^C$ where $(U \cup V)^D \equiv (U \cup V)^C \rightarrow v \equiv U^C \wedge V^C \rightarrow v$. So we have $\Gamma^D \vdash_{\wedge \times}^\# M : (\delta_i \wedge \delta'_k) \times (\kappa_i \wedge \kappa'_k) \rightarrow v \mid \Delta^C$ by the rule $(\leq^\#)$. On the other hand, from (c3) and (d3), we have $\Gamma^D \vdash_{\wedge \times}^\# N : \delta_i \mid \Delta^C$ and $\Gamma^D \vdash_{\wedge \times}^\# N : \delta'_k \mid \Delta^C$ by Lemma A.3. Thus we have the following derivation:

$$\frac{\begin{array}{c} \vdots \\ \vdots \\ \Gamma^D \vdash M : (\delta_i \wedge \delta'_k) \times (\kappa_i \wedge \kappa'_k) \rightarrow v \mid \Delta^C \end{array} \quad \frac{\begin{array}{c} \vdots \\ \vdots \\ \Gamma^D \vdash N : \delta_i \mid \Delta^C \quad \Gamma^D \vdash N : \delta'_k \mid \Delta^C \end{array}}{\Gamma^D \vdash N : \delta_i \wedge \delta'_k \mid \Delta^C} (\wedge)}{\Gamma^D \vdash MN : \kappa_i \wedge \kappa'_k \rightarrow v \mid \Delta^C} \text{ (App)}$$

Finally, from (c1) and (d1), we have $U_j^C \leq_C^\# \kappa_i \wedge \kappa'_k$, and so $\kappa_i \wedge \kappa'_k \rightarrow v \leq_D^\# U_j^C \rightarrow v \equiv U_j^D$. Hence by the rule $(\leq^\#)$, we obtain $\Gamma^D \vdash_{\wedge \times}^\# MN : U_j^D \mid \Delta^C$.

This argument holds for each U_j^D ($j \in \{1, \dots, m\}$), so we have $\Gamma^D \vdash_{\wedge \times}^\# MN : U_1^D \wedge \dots \wedge U_m^D \mid \Delta^C$. Hence $\Gamma^D \vdash_{\wedge \times}^\# MN : I^D \mid \Delta^C$. \square

Corollary A.6. *If $\Gamma \vdash_{\cap \cup} M : I \mid \Delta$ for some Γ, I and Δ , then $M \in \text{SN}^{\beta, \mu}$.*

Proof. By Theorems 4.3 and A.5, and the fact that $\Gamma^D \vdash_{\wedge \times}^\# M : I^D \mid \Delta^C$ implies $\Gamma^D \vdash_{\wedge \times} M : I^D \mid \Delta^C$. \square