Electronic Notes in Theoretical Computer Science 70 No. 1 (2002)
URL: http://www.elsevier.nl/locate/entcs/volume70.html 18 pages

Simple Easy Terms

F. Alessi [1]

Dipartimento di Matematica ed Informatica
Università di Udine, Via delle Scienze 208
33100 Udine, Italy

S. Lusin [2]

Dipartimento di Informatica
Università di Venezia
via Torino 153
30170 Venezia, Italy

Abstract

We illustrate the use of intersection types as a semantic tool for proving easiness result on λ-terms. We single out the notion of *simple easiness* for λ-terms as a useful semantic property for building filter models with special purpose features. Relying on the notion of *easy intersection type theory*, given λ-terms M and E, with E simple easy, we successfully build a filter model which equates interpretation of M and E, hence proving that simple easiness implies easiness. We finally prove that a class of λ-terms generated by $\omega_2\omega_2$ are simple easy, so providing alternative proof of easiness for them.

Introduction

Intersection types were introduced in the late 70's by Dezani and Coppo [10,12,6], to overcome the limitations of Curry's type discipline. They are a very expressive type language which allows to describe and capture various properties of λ-terms. For instance, they have been used in [26] to give the first type theoretic characterization of *strongly normalizable* terms and in [13] to capture *persistently normalizing terms* and *normalizing terms*. See [14] for a more complete account of this line of research.

Intersection types have a very significant realizability semantics with respect to applicative structures. This is a generalization of Scott's natural

[1] Email: alessi@dimi.uniud.it
[2] Email: slusin@oink.dsi.unive.it

semantics [28] of simple types. According to this interpretation types denote subsets of the applicative structure, an arrow type $A \to B$ denotes the sets of points which map all points belonging to the interpretation of A to points belonging to the interpretation of B, and an intersection type $A \cap B$ denotes the intersections of the interpretation of A and the interpretation of B. Building on this, intersection types have been used in [6] to give a proof of the completeness of the natural semantics of Curry's simple type assignment system in applicative structures, introduced in [28].

But intersection types have also an alternative semantics based on *duality* which is related to Abramsky's *Domain Theory in Logical Form* [1]. Actually it amounts to the application of that paradigm to the special case of ω-algebraic complete lattice models of pure lambda calculus, [11]. Namely, types correspond to *compact elements*: the type Ω denoting the least element, intersections denoting *joins* of compact elements, and arrow types denoting *step functions* of compact elements. A typing judgment then can be interpreted as saying that a given term belongs to a pointed compact open set in a ω-algebraic complete lattice model of λ-calculus. By duality, type theories give rise to *filter λ-models*. Intersection type assignment systems can then be viewed as *finitary logical* definitions of the interpretation of λ-terms in such models, where the meaning of a λ-term is the set of types which are deducible for it.

This duality lies at the heart of the success of intersection types as a powerful tool for the analysis of λ-models, see *e.g.* [2,6,11,13,3,16,20,15,25,18,27].

In this paper we face the issue of easiness proofs of λ-terms from the semantic point of view (we recall that a closed term P is *easy* if, for any other closed term M, the theory $\lambda\beta + \{M = P\}$ is consistent).

Actually the mainstream of easiness proofs is based on the use of syntactic tools (see [22], [23], [21], [8], [9], [7], [24], for easiness results on the λ-terms dealt with in the present paper and other general easiness results obtained via syntactic tools).

Instead, very little literature can be found on easiness issues handled by semantic tools, and we can summarize it in short lines.

A semantic proof of the easiness of $\omega_2\omega_2$ ($\omega_2 = \lambda x.xx$) appeared in [5] with a proof based on non-standard $\mathcal{P}(\omega)$ models. [19] builds extensional filter models equating $\omega_2\omega_2$ to arbitrary closed terms. The third reference is the main inspiration of the present paper: in [4] a strengthened version of intersection types theories, namely the *easy* ones, were introduced and successfully used for proving semantically easiness of the terms $\omega_2\omega_2$ and $\omega_3\omega_3\mathsf{I}$ ($\omega_3 = \lambda x.xxx$, $\mathsf{I} = \lambda x.x$), by exhibiting, for any M, suitable filter models which identify the interpretation of M with the interpretation of the given easy term.

In this paper we go in the direction of [4]. We introduce the notion of *simple easiness*: roughly speaking, an unsolvable term E is simple easy if, for each filter model $\mathcal{F}^\triangledown$ built on an easy intersection type theory $\Sigma^\triangledown$, any type C in $\Sigma^\triangledown$, we can expand $\Sigma^\triangledown$ to a new easy intersection type theory $\Sigma^{\triangledown'}$ such that the interpretation of E in $\mathcal{F}^{\nabla'}$ is the sup of the old interpretation of E in $\mathcal{F}^{\nabla}$ and the filter generated by C.

As a first consequence of this fact, if one starts from a filter model where the interpretation of E is the least element, then $[\![E]\!]$ can possibly become any filter.

A second consequence is that simple easiness is a stronger notion than easiness. A simple easy term E is easy, since, given an arbitrary closed term M, it is possible to build (in a canonical way) a non-trivial filter model which equates the interpretation of E and M.

Besides of that, simple easiness is interesting in itself, since it has to do with minimal sets of axioms which are needed in order to give the easy term a certain type.

The question whether easiness implies simple easiness is open.

We will prove that the terms R_n are simple easy where $R_0 = (\omega_2\omega_2)$ and $R_{n+1} = R_nR_n$. For $\omega_2\omega_2$, our simple easiness result can be viewed as a strengthened version of the easiness result of [4] for $\omega_2\omega_2$. Instead simple easiness of R_n for $n > 0$ is totally new.

The present paper is organized as follows. In Section 1 we present easy intersection type theories and type assignment systems for them. We prove some meta-theoretic properties including a Generation Theorem. In Section 2 we introduce λ-models based on spaces of filters in easy intersection type theories. In Section 3 we introduce the notion of simple easiness and prove that simple easiness implies easiness. Sections 4 and 5 contain respectively the simple easiness proofs for $\omega_2\omega_2$ and the generalization to R_n.

1 Intersection Type Assignment Systems

Intersection types are syntactical objects built inductively by closing a given set $\mathbb{C}$ of *type atoms* (constants) which contains the universal type Ω under the *function type* constructor$\to$ and the *intersection* type constructor $\cap$.

Definition 1.1 [Intersection Type Language]
Let $\mathbb{C}$ be a countable set of constants such that $\Omega \in \mathbb{C}$. The *intersection type language* over $\mathbb{C}$, denoted by $\mathbb{T} = \mathbb{T}(\mathbb{C})$ is defined by the following abstract syntax:

$$\mathbb{T} = \mathbb{C} \mid \mathbb{T} \to \mathbb{T} \mid \mathbb{T} \cap \mathbb{T}.$$

Notice that the most general form of an intersection type is a finite intersection of arrow types and type constants.

Notation
Upper case Roman letters i.e. $A, B, \ldots$, will denote arbitrary types. Greek letters will denote constants in $\mathbb{C}$. When writing intersection types we shall use the following convention: the constructor $\cap$ takes precedence over the constructor $\to$ and it associates to the right. Moreover $A^n \to B$ will be short

for $\underbrace{A \to \cdots \to A}_{n} \to B$. I, J, K etc. will denote non-empty finite sets.

Much of the expressive power of intersection type disciplines comes from the fact that types can be endowed with a *preorder relation* $\leq$, which induces the structure of a meet semi-lattice with respect to $\cap$, the top element being Ω. The notion we give of *easy* intersection type theory differs from the original one of [4] in that:

- we consider just extensional structures (where any constant is equivalent to an intersection of arrow types);
- we allow equivalence axioms $\psi \sim A$ of a slightly more general shape.

A part from these two (minor) points the present definition coincide with that of [4].

Definition 1.2 [Easy intersection type theories]
Let $\mathbb{T} = \mathbb{T}(\mathbb{C})$ be an intersection type language. The *easy intersection type theory* (*eitt* for short) $\Sigma(\mathbb{C}, \bigtriangledown)$ over $\mathbb{T}$ is the set of all judgments $A \leq B$ derivable from $\bigtriangledown$, where $\bigtriangledown$ is a collection of axioms and rules such that (we write $A \sim B$ for $A \leq B$ & $B \leq A$):

(i) $\bigtriangledown$ contains the set $\overline{\bigtriangledown}$ of axioms and rules:

(refl) $A \leq A$	(idem) $A \leq A \cap A$
(incl$_L$) $A \cap B \leq A$	(incl$_R$) $A \cap B \leq B$
(mon) $\dfrac{A \leq A' \quad B \leq B'}{A \cap B \leq A' \cap B'}$	(trans) $\dfrac{A \leq B \quad B \leq C}{A \leq C}$
(Ω) $A \leq \Omega$	(Ω-η) $\Omega \leq \Omega \to \Omega$
($\to$-$\cap$) $(A \to B) \cap (A \to C) \leq A \to B \cap C$	(η) $\dfrac{A' \leq A \quad B \leq B'}{A \to B \leq A' \to B'}$

(ii) further axioms can be of the following two shapes only:

$$\psi \leq \psi',$$
$$\psi \sim \bigcap_{h \in H}(\xi_h \to E_h).$$

where $\psi, \psi', \xi_h \in \mathbb{C}$, $A \in \mathbb{T}$, and $\psi, \psi' \not\sim \Omega$;

(iii) $\bigtriangledown$ does not contain further rules;

(iv) for each $\psi \not\sim \Omega$ there is exactly one axiom in $\bigtriangledown$ of the shape $\psi \sim A$;

(v) Let $\bigtriangledown$ contain $\psi \sim \bigcap_{h \in H}(\xi_h \to E_h)$ and $\psi' \sim \bigcap_{k \in K}(\xi'_k \to E'_k)$. Then $\bigtriangledown$ contains also $\psi \leq \psi'$ iff for each $k \in K$, there exists $h_k \in H$ such that ${\xi'}_k \leq \xi_{h_k}$ and $E_{h_k} \leq {E'}_k$ are both in $\bigtriangledown$.

Notice that:
(a) since $\Omega \sim \Omega \to \Omega \in \Sigma(\mathbb{C}, \bigtriangledown)$ by (Ω) and (Ω-η), it follows that all atoms

in $\mathbb{C}$ are equivalent to suitable (intersections of) arrow types;
(b) $\cap$ (modulo $\sim$) is associative and commutative;
(c) in the last clause of the above definition E'_k and E_{h_k} must be constant types for each $k \in K$.

Notation
When we consider an eitt $\Sigma(\mathbb{C}, \triangledown)$, we will write $\mathbb{C}^\triangledown$ for $\mathbb{C}$, $\mathbb{T}^\triangledown$ for $\mathbb{T}(\mathbb{C})$ and $\Sigma^\triangledown$ for $\Sigma(\mathbb{C}, \triangledown)$. Moreover $A \leq_\triangledown B$ will be short for $(A \leq B) \in \Sigma^\triangledown$ and $A \sim_\triangledown B$ for $A \leq_\triangledown B \leq_\triangledown A$. We will consider syntactic equivalence "$\equiv$" of types up to associativity and commutativity of $\cap$. We will write $\bigcap_{i \leq n} A_i$ for $A_1 \cap \ldots \cap A_n$. Similarly we will write $\bigcap_{i \in I} A_i$, where I denotes always a finite non-empty set.

A nice feature of easy intersection type structures is the possibility of performing smooth induction proofs based on the number of arrows in the types.

In view of this aim next definition and lemma work.

Definition 1.3
The mapping $\# : \mathbb{T}^\triangledown \to \mathbb{N}$ is defined inductively on types as follows:

$$\begin{aligned} \#(A) &= 0 \qquad \text{if } A \in \mathbb{C}^\triangledown; \\ \#(A \to B) &= \#(A) + 1; \\ \#(A \cap B) &= \max\{\#(A), \#(B)\}. \end{aligned}$$

Lemma 1.4
For all $A \in \mathbb{T}^\triangledown$ with $\#(A) \geq 1$ there is $B \in \mathbb{T}^\triangledown$ such that $A \sim_\triangledown B$, $B \equiv \bigcap_{i \in I}(C_i \to D_i)$, and $\#(B) = \#(A)$.

Proof. Let $A \equiv (\bigcap_{j \in J}(C'_j \to D'_j)) \cap (\bigcap_{h \in H} \psi_h)$, where $C'_j, D'_j \in \mathbb{T}^\triangledown$, $\psi_h \in \mathbb{C}^\triangledown$. For each $h \in H$ there are $I^{(h)}$, $\xi_i^{(h)} \in \mathbb{C}^\triangledown$, $E_i^{(h)} \in \mathbb{T}^\triangledown$, such that $\psi_h \sim_\triangledown \bigcap_{i \in I^{(h)}}(\xi_i^{(h)} \to E_i^{(h)})$. We can choose

$$B \equiv (\bigcap_{j \in J}(C'_j \to D'_j)) \cap (\bigcap_{h \in H}(\bigcap_{i \in I^{(h)}}(\xi_i^{(h)} \to E_i^{(h)}))).$$

□

Before giving the crucial notion of *intersection-type assignment system*, we introduce bases and some related definitions.

Definition 1.5 [Bases]

(i) A $\triangledown$-*basis* is a (possibly infinite) set of statements of the shape $x : B$, where $B \in \mathbb{T}^\triangledown$, with all variables distinct.

(ii) $x \in \Gamma$ is short for $\exists A \in \mathbb{T}^\triangledown$. $(x{:}A) \in \Gamma$ and $\Gamma, x{:}A$ is short for $\Gamma \cup \{x{:}A\}$ when $x \notin \Gamma$.

(iii) Let Γ and Γ' be ∇-bases. The ∇-basis $\Gamma \uplus \Gamma'$ is defined as follows:

$$\begin{aligned} \Gamma \uplus \Gamma' = & \{x : A \cap B \mid x : A \in \Gamma \text{ and } x : B \in \Gamma'\} \\ & \cup \{x : A \mid x : A \in \Gamma \text{ and } x \notin \Gamma'\} \\ & \cup \{x : B \mid x : B \in \Gamma' \text{ and } x \notin \Gamma\}. \end{aligned}$$

Accordingly we define:

$$\Gamma \Subset \Gamma' \ \Leftrightarrow \ \exists \Gamma''. \, \Gamma \uplus \Gamma'' = \Gamma'.$$

Definition 1.6 [The type assignment system]
The *intersection type assignment system* relative to the eitt Σ^∇, notation $\lambda\cap^\nabla$, is a formal system for deriving judgements of the form $\Gamma \vdash^\nabla M : A$, where the *subject* M is an untyped λ-term, the *predicate* A is in $\mathbb{T}^\nabla$, and Γ is a ∇-basis. Its axioms and rules are the following:

$$(\text{Ax}) \ \frac{(x{:}A) \in \Gamma}{\Gamma \vdash^\nabla x{:}A} \qquad\qquad (\text{Ax-}\Omega) \ \Gamma \vdash^\nabla M : \Omega$$

$$(\to \text{I}) \ \frac{\Gamma, x{:}A \vdash^\nabla M : B}{\Gamma \vdash^\nabla \lambda x.M : A \to B} \qquad (\to \text{E}) \ \frac{\Gamma \vdash^\nabla M : A \to B \quad \Gamma \vdash^\nabla N : A}{\Gamma \vdash^\nabla MN : B}$$

$$(\cap \text{I}) \ \frac{\Gamma \vdash^\nabla M : A \quad \Gamma \vdash^\nabla M : B}{\Gamma \vdash^\nabla M : A \cap B} \qquad (\leq_\nabla) \ \frac{\Gamma \vdash^\nabla M : A \quad A \leq_\nabla B}{\Gamma \vdash^\nabla M : B}$$

As usual we consider λ-terms modulo α-conversion. Notice that intersection elimination rules

$$(\cap E) \ \frac{\Gamma \vdash^\nabla M : A \cap B}{\Gamma \vdash^\nabla M : A} \quad \frac{\Gamma \vdash^\nabla M : A \cap B}{\Gamma \vdash^\nabla M : B}.$$

can be immediately proved to be derivable in all $\lambda\cap^\nabla$. A first simple proposition, which can be proved straightforwardly by induction on the structure of derivations is the following.

Proposition 1.7

(i) *If* $x \notin FV(M)$ *and* $\Gamma, x{:}B \vdash^\nabla M : A$, *then* $\Gamma \vdash^\nabla M : A$;

(ii) *If* $\Gamma \vdash^\nabla M : A$ *and* $\Gamma \Subset \Gamma'$, *then* $\Gamma' \vdash^\nabla M : A$.

We end this section by stating a Generation Theorem (its proof is quite similar to that given in [4]), or for the type assignment system $\lambda\cap^\nabla$.

Theorem 1.8 (Generation Theorem)

(i) *Assume* $A \not\sim_\nabla \Omega$. $\Gamma \vdash^\nabla x : A$ *iff* $(x : B) \in \Gamma$ *and* $B \leq_\nabla A$ *for some* $B \in \mathbb{T}^\nabla$.

(ii) $\Gamma \vdash^\nabla MN : A$ *iff* $\Gamma \vdash^\nabla M : B \to A$, *and* $\Gamma \vdash^\nabla N : B$ *for some* $B \in \mathbb{T}^\nabla$.

(iii) $\Gamma \vdash^\nabla \lambda x.M : A$ *iff* $\Gamma, x : B_i \vdash^\nabla M : C_i$ *and* $\bigcap_{i \in I}(B_i \to C_i) \leq_\nabla A$, *for some* I *and* $B_i, C_i \in \mathbb{T}^\nabla$.

(iv) $\Gamma \vdash^{\triangledown} \lambda x.M : B \to C$ *iff* $\Gamma, x{:}B \vdash^{\triangledown} M : C$.

2 Filter Models

In this section we discuss how to build λ-models out of type theories. We start with the definition of *filter* for eitt's. Then we show how to turn the space of filters into an applicative structure. Finally we will define a notion of interpretation of λ-terms and show that we get λ-models (*filter models*).

Filter models arise naturally in the context of those generalizations of Stone duality that are used in discussing domain theory in logical form (see [1], [11], [29]). This approach provides a conceptually independent semantics to intersection types, the *lattice semantics*. Types are viewed as *compact elements* of domains. The type Ω denotes the least element, intersections denote joins of compact elements, and arrow types allow to internalize the space of continuous endomorphisms. Following the paradigm of Stone duality, type theories give rise to filter models, where the interpretation of λ-terms can be given through a finitary logical description.

Definition 2.1

(i) A $\triangledown$-filter (or a filter over $\mathbb{T}^{\triangledown}$) is a set $X \subseteq \mathbb{T}^{\triangledown}$ such that:
  - $\Omega \in X$;
  - if $A \leq_{\triangledown} B$ and $A \in X$, then $B \in X$;
  - if $A, B \in X$, then $A \cap B \in X$;

(ii) $\mathcal{F}^{\triangledown}$ denotes the set of $\triangledown$-filters over $\mathbb{T}^{\triangledown}$;

(iii) if $X \subseteq \mathbb{T}^{\triangledown}$, $\uparrow X$ denotes the $\triangledown$-filter generated by X;

(iv) a $\triangledown$-filter is *principal* if it is of the shape $\uparrow \{A\}$, for some type A. We shall denote $\uparrow \{A\}$ simply by $\uparrow A$.

It is well known that $\mathcal{F}^{\triangledown}$ is a ω-algebraic cpo, whose compact (or finite) elements are the filters of the form $\uparrow A$ for some type A and whose bottom element is $\uparrow \Omega$.

Next we endow the space of filters with the notions of application and of λ-term interpretation. Let $\mathsf{Env}_{\mathcal{F}^{\triangledown}}$ be the set of all mappings from the set of term variables to $\mathcal{F}^{\triangledown}$.

Definition 2.2

(i) Application $\cdot : \mathcal{F}^{\triangledown} \times \mathcal{F}^{\triangledown} \to \mathcal{F}^{\triangledown}$ is defined as

$$X \cdot Y = \{B \mid \exists A \in Y. A \to B \in X\}.$$

(ii) The interpretation function: $[\![\ ]\!]^{\triangledown} : \Lambda \times \mathsf{Env}_{\mathcal{F}^{\triangledown}} \to \mathcal{F}^{\triangledown}$ is defined by

$$[\![M]\!]^{\triangledown}_{\rho} = \{A \in \mathbb{T}^{\triangledown} \mid \exists \Gamma \models \rho.\ \Gamma \vdash^{\triangledown} M : A\},$$

where ρ ranges over $\mathsf{Env}_{\mathcal{F}^{\triangledown}}$ and $\Gamma \models \rho$ if and only $(x : B) \in \Gamma$ implies $B \in \rho(x)$.

(iii) The triple $\langle \mathcal{F}^\nabla, \cdot, [\![\ ]\!]^\nabla \rangle$ is called the *filter model* over Σ^∇.

Notice that previous definition is sound, since it is easy to verify that $X \cdot Y$ is a ∇-filter. Next we prove that $\mathcal{F}^\nabla$ is a λ-model. First we need a syntactic result, which is proved by induction on the derivation of judgments.

Theorem 2.3
For all I, and $A_i, B_i, C, D \in \mathbb{T}^\nabla$,

$$\bigcap_{i \in I}(A_i \to B_i) \leq_\nabla C \to D \ \Rightarrow\ \exists J \subseteq I. C \leq_\nabla \bigcap_{i \in J} A_i \ \&\ \bigcap_{i \in J} B_i \leq_\nabla D,$$

provided that $D \not\sim_\nabla \Omega$.

Theorem 2.4
The filter model $\langle \mathcal{F}^\nabla, \cdot, [\![\ ]\!]^\nabla \rangle$ is a λ-model, in the sense of Hindley-Longo [17], that is:

(i) $[\![x]\!]^\nabla_\rho = \rho(x)$;

(ii) $[\![MN]\!]^\nabla_\rho = [\![M]\!]^\nabla_\rho \cdot [\![N]\!]^\nabla_\rho$;

(iii) $[\![\lambda x.M]\!]^\nabla_\rho \cdot X = [\![M]\!]^\nabla_{\rho[X/x]}$;

(iv) $(\forall x \in FV(M).\ [\![x]\!]^\nabla_\rho = [\![x]\!]^\nabla_{\rho'}) \ \Rightarrow\ [\![M]\!]^\nabla_\rho = [\![M]\!]^\nabla_{\rho'}$;

(v) $[\![\lambda x.M]\!]^\nabla_\rho = [\![\lambda y.M[y/x]]\!]^\nabla_\rho$, *if* $y \notin FV(M)$;

(vi) $(\forall X \in \mathcal{F}^\nabla. [\![M]\!]^\nabla_{\rho[X/x]} = [\![N]\!]^\nabla_{\rho[X/x]}) \ \Rightarrow\ [\![\lambda x.M]\!]^\nabla_\rho = [\![\lambda x.N]\!]^\nabla_\rho$.

Moreover it is extensional, that is $[\![\lambda x.Mx]\!]^\nabla_\rho = [\![M]\!]^\nabla_\rho$ when $x \notin FV(M)$.

Proof. By Theorem 2.3 and Theorem 2.13 (iii) of [11], $[\mathcal{F}^\nabla \to \mathcal{F}^\nabla]$ is a retract of $\mathcal{F}^\nabla$, hence it is a λ-model. □

3 Simple easy terms

In this section we give the main notion of the paper, namely *simple easiness*. A term E is simple easy if, given any eitt Σ^∇ and a type Z in it, we can extend in a conservative way Σ^∇ to a new easy intersection type theory, say $\Sigma^{\nabla'}$, so that $[\![E]\!]^{\nabla'} = \uparrow Z \cup [\![E]\!]^\nabla$. On one hand, a consequence of this notion is that it is possible to build through a uniform technique, filter models that equate the interpretation of E with the interpretation of M, for M arbitrary. Therefore simple easiness implies easiness. On the other hand, simple easiness is interesting in itself: in fact when E is simple easy then for any Σ^∇ and type Z in it, we can enrich Σ^∇ with a set of new constants and axioms for them, which is *minimal*, in the sense that in the enriched intersection type theory, E can receive just Z (and its intersections with other types already derivable for E in Σ^∇), as new type with respect to the old types E could receive in Σ^∇.

Definition 3.1

(i) Let Σ^{∇} and $\Sigma^{\nabla'}$ be two easy intersection type theories. We define $\Sigma^{\nabla} \sqsubseteq \Sigma^{\nabla'}$ iff $\mathbb{C}^{\nabla} \subseteq \mathbb{C}^{\nabla'}$ and for all $A, B \in \mathbb{T}^{\nabla}$,

$$A \leq_{\nabla} B \Leftrightarrow A \leq_{\nabla'} B.$$

(ii) Let, for any $n \in \mathbb{N}$, $\Sigma^{\nabla_n} \sqsubseteq \Sigma^{\nabla_{n+1}}$. We define

$$\Sigma^{\nabla_*} = \Sigma(\bigcup_n \mathbb{C}^{\nabla_n}, \bigcup_n \nabla_n).$$

It is immediate to prove that in the definition above Σ^{∇_*} is an eitt and for each n, $\Sigma^{\nabla_n} \sqsubseteq \Sigma^{\nabla_*}$.

Definition 3.2

(i) A *pointed* eitt is a pair (Σ^{∇}, Z) with $Z \in \mathbb{T}^{\nabla}$.

(ii) *EITT* and *PEITT* denote respectively the class of eitts and pointed eitts.

(iii) A *filter scheme* is a mapping $\mathcal{S} : PEITT \to EITT$, such that for all (Σ^{∇}, Z)

$$\Sigma^{\nabla} \sqsubseteq \mathcal{S}(\Sigma^{\nabla}, Z).$$

We now give the central notion of *simple easy* term.

Definition 3.3
An unsolvable term E is *simple easy* if there exists a filter scheme $\mathcal{S}_E$ such that for all pointed eitt (Σ^{∇}, Z),

$$\vdash^{\nabla'} E : B \iff \exists C \in \mathbb{T}^{\nabla}. C \cap Z \leq_{\nabla'} B \;\&\; \vdash^{\nabla} E : C,$$

where $\Sigma^{\nabla'} = \mathcal{S}_E(\Sigma^{\nabla}, Z)$.

Theorem 3.4
With the same notation of previous definition, we have $[\![E]\!]^{\nabla'} = \uparrow Z \sqcup [\![E]\!]^{\nabla}$.

Proof. ($\supseteq$) We have, taking $B = \Omega$ in the Definition 3.3, $\vdash^{\nabla'} E : Z$. Therefore $\uparrow Z \subseteq [\![E]\!]^{\nabla'}$. Since moreover $[\![E]\!]^{\nabla} \subseteq [\![E]\!]^{\nabla'}$, we get $[\![E]\!]^{\nabla'} \supseteq \uparrow Z \sqcup [\![E]\!]^{\nabla}$.
($\subseteq$) If $B \in [\![E]\!]^{\nabla'}$, then $\vdash^{\nabla'} E : B$, hence, by Definition 3.3, there exists $C \in \mathbb{T}^{\nabla}$ such that $C \in [\![E]\!]^{\nabla}$ and $C \cap Z \leq_{\nabla'} B$. We are done, since $C \cap Z \in \uparrow Z \sqcup [\![E]\!]^{\nabla'}$. □

Theorem 3.5
Let E be a simple easy term. Then E is easy.

Proof. Let M be an arbitrary closed λ-term. We prove that there exists a non-trivial filter model $\mathcal{F}^{\nabla}$ such that $[\![M]\!]^{\nabla} = [\![E]\!]^{\nabla}$. First a simple remark on interpretations of terms. Let $(\Sigma^{\nabla_n})_n$ be an ascending chain of easy intersection type theories, with $\Sigma^{\nabla_n} \sqsubseteq \Sigma^{\nabla_{n+1}}$ for each n. For each n, we can find a sequence of types $(A_p^{(n)})_p \subseteq \mathbb{T}^{\nabla_n}$ such that

$$\forall n, p. A_{p+1}^{(n)} \leq_{\nabla_n} A_p^{(n)} \;\&\; [\![M]\!]^{\nabla_n} = \bigcup_p \uparrow A_p^{(n)}.$$

Actually it is not restrictive to choose such sequences so that this further condition holds:

$$(*) \qquad \forall n. A^{(n)}_{n+1} \leq_{\nabla_{n+1}} A^{(n)}_n.$$

Given such sequences $(A^{(n)}_p)_p$, we define, for each n, $Z_n = A^{(n)}_n \in \mathbb{T}^{\nabla_n}$. We show that in the filter model $\mathcal{F}^{\nabla_*}$

$$(\dagger) \qquad [\![M]\!]^{\nabla_*} = \bigcup_n \uparrow Z_n.$$

$(\supseteq)$ is immediate since by definition of Z_n we have $Z_n \in [\![M]\!]^{\nabla_n} \subseteq [\![M]\!]^{\nabla_*}$.
As to $(\subseteq)$, let $A \in [\![M]\!]^{\nabla_*}$. Then there exists n such that $A \in [\![M]\!]^{\nabla_n} = \bigcup_p \uparrow A^{(n)}_p$. This implies that there exists p such that $A^{(n)}_p \leq_{\nabla_n} A$. For any $m \geq n, p$ it follows $Z_m \leq_{\nabla_m} A$, hence $A \in \bigcup_n \uparrow Z_n$.

We now exploit the equality $(\dagger)$ and define a filter model such that the interpretation of E is equal to $\bigcup_n \uparrow Z_n$. Here is the construction of the model.

step 0:
take the easy intersection type theory Σ^{∇_0} whose filter model is isomorphic to Scott D_∞ (see [3]):
- $\mathbb{C}^{\nabla_0} = \{\Omega, \omega\}$;
- $\nabla_0 = \overline{\nabla} \cup \{\omega \sim \Omega \to \omega\}$.

step $(n+1)$:
perform the following operations:
- compute $[\![M]\!]^{\nabla_n}$;
- take a sequence $(A^{(n)}_p)_p$ such that $[\![M]\!]^{\nabla_n} = \bigcup_p \uparrow A^{(n)}_p$ and condition $(*)$ above is satisfied;
- define the type Z_n as $A^{(n)}_n$;
- define $\Sigma^{\nabla_{n+1}} = \mathcal{S}_E(\Sigma^{\nabla_n}, Z_n)$;

final step:
take Σ^{∇_*}.

We will prove that $\mathcal{F}^{\nabla_*}$ identifies M and E, but before that we have to prove that $\mathcal{F}^{\nabla_*}$ is not trivial. For this aim we show that $[\![\mathsf{I}]\!]^{\nabla_*} \neq [\![\mathsf{K}]\!]^{\nabla_*}$, where $\mathsf{K} = \lambda xy.x$. Let $D \equiv (\omega \to \omega) \to (\omega \to \omega)$. Since $\vdash^{\nabla_*} \mathsf{I} : D$, we have that $D \in [\![\mathsf{I}]\!]^{\nabla_*}$. On the other hand, if $D \in [\![\mathsf{K}]\!]^{\nabla_*}$, then there should be n such that $D \in [\![\mathsf{K}]\!]^{\nabla_n}$. This would imply (by applying several times the Generation Theorem) $\omega \to \omega \leq_{\nabla_n} \omega$. Since we have $\Sigma^{\nabla_p} \sqsubseteq \Sigma^{\nabla_{p+1}}$ for any p, we should have $\omega \to \omega \leq_{\nabla_0} \omega$. Since $\omega \sim_{\nabla_0} \Omega \to \omega$, we should conclude, by Theorem 2.3, $\Omega \leq_{\nabla_0} \omega$, which is a contradiction. Therefore we cannot have $D \in [\![\mathsf{K}]\!]^{\nabla_*}$ and the model $\mathcal{F}^{\nabla_*}$ is non-trivial.

In order to prove that $[\![M]\!]^{\nabla_*} = [\![E]\!]^{\nabla_*}$, in view of $(\dagger)$, it is sufficient to prove that

$$[\![E]\!]^{\nabla_*} = \bigcup_n \uparrow Z_n.$$

First we prove ($\supseteq$). By Theorem 3.4 and the definition of Σ^{∇_n}, we have that for all n, $Z_n \in [\![E]\!]^{\nabla_n}$, hence $Z_n \in [\![E]\!]^{\nabla_*}$ and the inclusion is proved.
We prove ($\subseteq$) by induction on n, by showing that $[\![E]\!]^{\nabla_n} \subseteq \uparrow Z_n$. If $n = 0$, then $[\![E]\!]^{\nabla_0} = \uparrow \Omega$, since $\mathcal{F}^{\nabla_0}$ is the Scott D_∞ model, where all unsolvable terms are equated to bottom. Suppose the thesis true for n and let $B \in [\![E]\!]^{\nabla_{n+1}}$. Then $\vdash^{\nabla_{n+1}} E : B$. This is possible only if there exists $C \in \mathbb{T}^{\nabla_n}$ such that $C \cap Z_{n+1} \leq_{\nabla_{n+1}} B$ and moreover $\vdash^{\nabla_n} E : C$. By induction we have $C \in \uparrow Z_n$, hence $Z_n \leq_{\nabla_n} C$. Since $Z_{n+1} \leq_{\nabla_{n+1}} Z_n$, we derive $Z_{n+1} \leq_{\nabla_{n+1}} C$, hence $Z_{n+1} \leq_{\nabla_{n+1}} Z_{n+1} \cap C \leq_{\nabla_{n+1}} B$. □

4 Simple easiness of $\omega_2\omega_2$

In this section we prove that $\omega_2\omega_2$ is simple easy, and as a by-product of Theorem 3.5 we obtain its easiness.

First we give a lemma which characterizes the types derivable for ω_2 and $\omega_2\omega_2$.

Lemma 4.1

(i) $\vdash^{\nabla} \omega_2 : A \to B$ *iff* $A \leq_{\nabla} A \to B$*;*

(ii) $\vdash^{\nabla} \omega_2\omega_2 : B$ *iff* $A \leq_{\nabla} A \to B$ *for some* $A \in \mathbb{T}^{\nabla}$ *such that* $\vdash^{\nabla} \omega_2 : A$*.*

(iii) *If* $\vdash^{\nabla} \omega_2\omega_2 : B$ *then there exists* $A \in \mathbb{T}^{\nabla}$ *such that* $\#(A) = 0$*,* $A \leq_{\nabla} A \to B$ *and* $\vdash^{\nabla} \omega_2 : A$*.*

Proof. Using Theorem 1.8 and Lemma 1.4. A direct proof can be found in [4]. □

The first step for proving simple easiness of $\omega_2\omega_2$ is to find its filter scheme.

Definition 4.2
Let be (Σ^{∇}, Z) be a pointed eitt. We define

$$\mathcal{S}_{(\omega_2\omega_2)}(\Sigma^{\nabla}, Z) = \Sigma^{\nabla'},$$

where:

- $\mathbb{C}^{\nabla'} = \mathbb{C}^{\nabla} \cup \{\chi\}$ (with $\chi \notin \mathbb{C}^{\nabla}$);
- $\nabla' = \nabla \cup \{\chi \sim \chi \to Z\}$.

Lemma 4.3

(i) $\mathcal{S}_{(\omega_2\omega_2)}(\Sigma^{\nabla}, Z)$ *is an easy intersection type theory;*

(ii) $\Sigma^{\nabla} \sqsubseteq \mathcal{S}_{(\omega_2\omega_2)}(\Sigma^{\nabla}, Z)$*.*

Proof. (i) is immediate by Definition 4.2. (ii) follows by induction on derivation of judgements. □

Next lemma is crucial for proving that $\mathcal{S}_{(\omega_2\omega_2)}$ is a filter scheme for $\omega_2\omega_2$.

Lemma 4.4
Let $\Sigma^{\nabla'} = \mathcal{S}_{(\omega_2\omega_2)}(\Sigma^{\nabla}, Z)$*. Then*

$$\vdash^{\nabla'} \omega_2\omega_2 : B \iff \exists C \in \mathbb{T}^{\nabla}.C \cap Z \leq_{\nabla'} B \;\&\; \vdash^{\nabla} \omega_2\omega_2 : C.$$

Proof. Throughout the proof we use the Generation Theorem and Theorem 2.3 without explicitely mentioning them each time.

($\Rightarrow$) Let $\vdash^{\nabla'} \omega_2\omega_2 : B$. Then there exists a type $P \in \mathbb{T}^{\nabla'}$ such that
(a) $P \equiv \bigcap_{i\in I}(\xi_i \to E_i) \cap \chi$;
(b) $\forall i \in I.\xi_i \in \mathbb{C}^{\nabla}$ & $E_i \in \mathbb{T}^{\nabla}$;
(c) $P \leq_{\nabla'} P \to B$;
(d) $\vdash^{\nabla'} \omega_2 : P$.
In fact, by Lemma 4.1(iii) it follows that there exists $T \in \mathbb{T}^{\nabla'}$ such that the following three properties hold:
(i) $\#(T) = 0$;
(ii) $T \leq_{\nabla'} T \to B$;
(iii) $\vdash^{\nabla'} \omega_2 : T$.
If we consider $T' \equiv T \cap \chi$, it is easy to prove that T' satisfies (i), (ii) and (iii) above. It must hold $T' \sim_{\nabla'} (\bigcap_{k\in K} \psi_k) \cap \chi$, with $\psi_k \in \mathbb{C}^{\nabla}$, $\psi_k \not\sim_{\nabla'} \Omega$ for all $k \in K$, since the unique possible shape for T' is an intersection of constants containing χ. Next, since for each $k \in K$, we have, from the axioms of ∇, $\psi_k \sim_{\nabla} \bigcap_{l\in L^{(k)}}(\xi_l^{(k)} \to E_l^{(k)})$, we can define $P \equiv \bigcap_{k\in K}(\bigcap_{l\in L^{(k)}}(\xi_l^{(k)} \to E_l^{(k)})) \cap \chi$. Then, by reindexing the types and using a unique intersection, we get the required syntactic shape for P as in (a).
Considering (a), (d), ($\leq_{\nabla}$) and Lemma 4.1(i), we have that for all $i \in I$, $\xi_i \leq_{\nabla'} \xi_i \to E_i$. Since $\Sigma^{\nabla} \sqsubseteq \Sigma^{\nabla'}$ and for each $i \in I$, $\xi_i, E_i \in \mathbb{T}^{\nabla}$, it follows that $\xi_i \leq_{\nabla} \xi_i \to E_i$, for all $i \in I$. By applying Lemma 4.1(i) and ($\cap$I), we get $\vdash^{\nabla} \omega_2 : \bigcap_{i\in I}(\xi_i \to E_i)$. Because of (c), there exists $I' \subseteq I$ such that $P \leq_{\nabla'} (\bigcap_{i\in I'} \xi_i) \cap \chi$ and $(\bigcap_{i\in I'} E_i) \cap Z \leq_{\nabla'} B$. Because of (d) and ($\leq_{\nabla'}$), it follows $\vdash^{\nabla'} \omega_2 : \bigcap_{i\in I'} \xi_i$. Let $\xi_i \equiv \bigcap_{m\in M^{(i)}}(\zeta_m^{(i)} \to D_m^{(i)})$. Then by ($\leq_{\nabla'}$), we have $\vdash^{\nabla'} \omega_2 : \zeta_m^{(i)} \to D_m^{(i)}$ for each $i \in I'$ and $m \in M^{(i)}$. By Lemma 4.1(i) it follows, for each $i \in I'$ and $m \in M^{(i)}$, $\zeta_m^{(i)} \leq_{\nabla'} \zeta_m^{(i)} \to D_m^{(i)}$. Exploiting again $\Sigma^{\nabla} \sqsubseteq \Sigma^{\nabla'}$, we have, for each $i \in I'$ and $m \in M^{(i)}$, $\zeta_m^{(i)} \leq_{\nabla} \zeta_m^{(i)} \to D_m^{(i)}$, hence, by Lemma 4.1(i), $\vdash^{\nabla} \omega_2 : \zeta_m^{(i)} \to D_m^{(i)}$, for each $i \in I'$ and $m \in M^{(i)}$. Therefore, by ($\cap$I), we have $\vdash^{\nabla} \omega_2 : \bigcap_{i\in I'}(\bigcap_{m\in M^{(i)}}(\zeta_m^{(i)} \to D_m^{(i)}))$, that is $\vdash^{\nabla} \omega_2 : \bigcap_{i\in I'} \xi_i$. Since $\vdash^{\nabla} \omega_2 : \bigcap_{i\in I}(\xi_i \to E_i)$, by ($\leq_{\nabla}$) we get $\vdash^{\nabla} \omega_2 : (\bigcap_{i\in I'} \xi_i) \to (\bigcap_{i\in I'} E_i)$. Therefore, applying ($\to$ E), we obtain $\vdash^{\nabla} \omega_2\omega_2 : \bigcap_{i\in I'} E_i$. Since we have proven $(\bigcap_{i\in I'} E_i) \cap Z \leq_{\nabla'} B$, we are done, by choosing $C \equiv \bigcap_{i\in I'} E_i$.
($\Leftarrow$) By Theorem 3.4 we have that $\vdash^{\nabla'} \omega_2\omega_2 : Z$. Since by hypothesis $\vdash^{\nabla} \omega_2\omega_2 : C$ and moreover $\Sigma^{\nabla} \sqsubseteq \Sigma^{\nabla'}$, we obtain $\vdash^{\nabla'} \omega_2\omega_2 : C$. By applying ($\leq_{\nabla'}$) we have $\vdash^{\nabla'} \omega_2\omega_2 : B$. $\square$

Theorem 4.5
$\omega_2\omega_2$ *is simple easy.*

Proof. It follows immediately by Definition 3.3, Lemmas 4.3 and 4.4. □

By previous theorem and Theorem 3.5 we get via semantics the well known result on easiness of $\omega_2\omega_2$ (see e.g. [5] for another semantic proof).

Corollary 4.6 *$\omega_2\omega_2$ is easy.*

5 Generalizing simple easiness to R_n

In this section we generalize the results of previous section, proving that a class of λ-terms generated by $\omega_2\omega_2$ is simple easy. More in details, consider the terms so defined inductively:

$$R_0 = \omega_2\omega_2;$$
$$R_{n+1} = R_n R_n.$$

Relying on semantic proof of easiness for $\omega_2\omega_2$, it is not difficult to prove via semantics that for any n, R_n is an easy term: in fact we know from previous sections that there exists a filter model $\mathcal{F}^{\nabla_*}$ which identifies R_0 to an arbitrary M'. Take $M' = \mathsf{K}M$. Then we have

$$\begin{aligned}[\![R_1]\!]^{\nabla_*} &= [\![R_0 R_0]\!]^{\nabla_*}\\ &= [\![\mathsf{K}MR_0]\!]^{\nabla_*}\\ &= [\![M]\!]^{\nabla_*}.\end{aligned}$$

Thus we have that for any M, we can build a model which identifies R_1 with M. So going on inductively, we can prove that R_n is easy for any n.

Nevertheless proving simple easiness of R_n is a rather more difficult task, and it is the aim of the present section.

We start fixing some notations. From now on $\tilde{\sigma}$ stands for a (non-empty) sequence of types $[\sigma_1, \ldots, \sigma_n]$. Given A and B types, $\rho(\tilde{\sigma}, B)$ will be short for the type

$$\sigma_1 \cap (\sigma_1 \to \sigma_2) \cap \ldots \cap (\sigma_1 \to \sigma_2 \ldots \to \sigma_n) \cap (\sigma_1 \to \sigma_2 \ldots \to \sigma_n \to B).$$

For each $0 \leq p \leq n$, we write $\rho^{(p)}(\tilde{\sigma}, B)$ as short for $\rho([\sigma_{p+1}, \ldots, \sigma_n], B)$. Notice that

- $\rho(\tilde{\sigma}, B) = \rho^{(0)}(\tilde{\sigma}, B)$,
- $A \to \rho([\sigma_1, \ldots, \sigma_n, \sigma_{n+1}], B) \sim A \to \rho([\sigma_1, \ldots, \sigma_n], \sigma_{n+1} \cap (\sigma_{n+1} \to B))$,
- $\rho^{(p)}(\tilde{\sigma}) \sim \sigma_{p+1} \to \rho^{(p+1)}(\tilde{\sigma})$,
- $\rho^{(n)}(\tilde{\sigma}, B) \equiv B$.

Let τ be a fresh constant. We define a set of axioms $\mathcal{A}(\tilde{\sigma}, \tau)$ as follows:

$$\mathcal{A}(\tilde{\sigma}) = \{\sigma_j \sim \tau \to \sigma_j \mid 1 \leq j \leq n\} \cup \{\tau \sim \sigma_1 \to \sigma_1\}.$$

Before going on we have to remark the auxiliary character of the set of axioms $\mathcal{A}$. In Definition 5.1 below, in defining the theory $\Sigma^{\nabla^{(n)}}$, the central role is played by the axiom $\chi \sim \rho(\chi, \tilde{\sigma}, Z)$, which allows to give the easy term the type Z. But since we consider extensional structures, we need to find also suitable axioms for each new constant that is introduced. For this aim we introduce $\mathcal{A}$.

We now define the sequence of filter scheme $\mathcal{S}_{R_n}$.

Definition 5.1
For any $n > 0$ and (Σ^∇, Z) pointed eitt, we define $\mathcal{S}_{R_n}(\Sigma^\nabla, Z) = \Sigma^{\nabla^{(n)}}$, where:
- $\mathbb{C}^{\nabla^{(n)}} = \mathbb{C}^\nabla \cup \{\chi, \sigma_1, \ldots, \sigma_n, \tau\}$;
- $\nabla^{(n)} = \nabla \cup \mathcal{A}(\tilde{\sigma}, \tau) \cup \{\chi \sim \chi \to \rho(\tilde{\sigma}, Z)\}$.

Lemma 5.2

(i) $\mathcal{S}_{R_n}(\Sigma^\nabla, Z)$ *is an easy intersection type theory;*

(ii) $\Sigma^\nabla \sqsubseteq \mathcal{S}_{R_n}(\Sigma^\nabla, Z)$.

Proof. (i) follows immediately by Definition 4.2. (ii) follows by induction on derivation of judgements. □

Next two lemmata are very useful. Their proofs are long but not difficult, relying on the Generation Theorem and Theorem 2.3.

Lemma 5.3
Let $n > 0$. Let $A \in \mathbb{T}^\nabla$, $A \not\sim_\nabla \Omega$. Then for any $1 \leq j \leq n$,

(i) $A \not\leq_{\nabla^{(n)}} \sigma_j$;

(ii) $A \not\leq_{\nabla^{(n)}} \tau$;

(iii) $\sigma_j \not\leq_{\nabla^{(n)}} A$;

(iv) $\tau \not\leq_{\nabla^{(n)}} A$.

Lemma 5.4
Let $n > 0$ and $0 \leq p \leq n$. Then

(i) $\vdash^{\nabla^{(n)}} R_p : \rho^{(p)}(\tilde{\sigma}, Z)$;

(ii) $\not\vdash^{\nabla^{(n)}} R_p : \sigma_j$, *for* $j \leq p$;

(iii) $\not\vdash^{\nabla^{(n)}} R_p : \tau$.

Next theorem is the key result for proving simple easiness of R_n. Its consequence, as expected, will be that $\mathcal{S}_{R_n}$ are filter schemes for R_n.

Theorem 5.5 *Let $n > 0$ and $0 \leq p \leq n$. Then*

$$\vdash^{\nabla^{(n)}} R_p : B \iff \exists C \in \mathbb{T}^\nabla. \vdash^\nabla R_p : C \;\&\; C \cap \rho^{(p)}(\tilde{\sigma}, Z) \leq_{\nabla^{(n)}} B.$$

Proof. ($\Rightarrow$) We reason by induction on p. If $p = 0$, then the proof follows exactly the lines of the proof of Theorem 4.4, by replacing Z with $\rho^{(p)}(\tilde{\sigma}, Z)$. Suppose now the thesis true for p. We prove the thesis for $p + 1$. Let $\vdash^{\nabla^{(n)}} R_{p+1} : B$. Then there exists $A' \in \mathbb{T}^{\nabla^{(n)}}$ such that $\vdash^{\nabla^{(n)}} R_p : A' \cap (A' \to B)$.

Since $\rho^{(p)}(\tilde{\sigma}, Z) \leq_{\nabla^{(n)}} \sigma_{p+1}$, by Lemma 5.4(i) and $(\leq_{\nabla^{(n)}})$, we get $\vdash^{\nabla^{(n)}} R_p : \sigma_{p+1}$. Hence we can define $A \in \mathbb{T}^{\nabla^{(n)}}$ as $A \equiv A' \cap (A' \to B) \cap \sigma_{p+1}$ so

that
- $\vdash^{\nabla^{(n)}} R_p : A$ and
- $A \leq_{\nabla^{(n)}} \sigma_{p+1} \cap (A \to B)$.

By induction there exists $C' \in \mathbb{T}^{\nabla}$ such that:

(i) $\vdash^{\nabla} R_p : C'$;

(ii) $C' \cap \rho^{(p)}(\tilde{\sigma}, Z) \leq_{\nabla^{(n)}} A$.

Let $C' \sim_{\nabla} \bigcap_{i \in I}(D_i \to E_i)$, with $D_i \in \mathbb{T}^{\nabla}$ for each $i \in I$. By (ii) and (trans) it follows

$$(\dagger) \quad C' \cap \rho^{(p)}(\tilde{\sigma}, Z) \leq_{\nabla^{(n)}} A \to B.$$

Notice that

$$\rho^{(p)}(\tilde{\sigma}, Z) \sim_{\nabla^{(n)}} (\tau \to \sigma_{p+1}) \cap (\sigma_{p+1} \to \rho^{(p+1)}(\tilde{\sigma}, Z)).$$

Moreover we cannot have $A \leq_{\nabla^{(n)}} \tau$. If so, we could deduce $\vdash^{\nabla^{(n)}} R_p : \tau$, contradicting Lemma 5.4(iii). So, when applying Theorem 2.3 to ($\dagger$), we conclude that there exists $I' \subseteq I$ such that:

(a) $A \leq_{\nabla^{(n)}} (\bigcap_{i \in I'} D_i) \cap \sigma_{p+1}$;

(b) $(\bigcap_{i \in I'} E_i) \cap \rho^{(p+1)}(\tilde{\sigma}, Z) \leq_{\nabla^{(n)}} B$.

(a) along with (ii), implies $C' \cap \rho^{(p)}(\tilde{\sigma}, Z) \leq_{\nabla^{(n)}} \bigcap_{i \in I'} D_i$. Let $K, T_k, U_k \in \mathbb{T}^{\nabla}$, be such that $\bigcap_{i \in I'} D_i \equiv \bigcap_{k \in K}(T_k \to U_k)$. By (trans) for all $k \in K$ we have $C \cap \rho^{(p)}(\tilde{\sigma}, Z) \leq_{\nabla^{(n)}} T_k \to U_k$, that is

$$(c) \quad \bigcap_{i \in I}(D_i \to E_i) \cap (\tau \to \sigma_{p+1}) \cap (\sigma_{p+1} \to \rho^{(p+1)}(\tilde{\sigma}, Z)) \leq_{\nabla^{(n)}} T_k \to U_k.$$

Since $T_k \in \mathbb{T}^{\nabla}$, by Lemma 5.3(i) and (ii), we can have neither $T_k \leq_{\nabla^{(n)}} \tau$ nor $T_k \leq_{\nabla^{(n)}} \sigma_{p+1}$. So, when applying Theorem 2.3 to (c), we obtain that there exists $I_k \subseteq I$, such that $T_k \leq_{\nabla^{(n)}} \bigcap_{i \in I_k} D_i$ and $\bigcap_{i \in I_k} E_i \leq_{\nabla^{(n)}} U_k$. By standard computations we get $\bigcap_{i \in I}(D_i \to E_i) \leq_{\nabla^{(n)}} T_k \to U_k$ for all $k \in K$, hence

$$C' \leq_{\nabla^{(n)}} \bigcap_{k \in K}(T_k \to U_k) \equiv \bigcap_{i \in I'} D_i.$$

Applying Lemma 5.2, we get $C' \leq_{\nabla} \bigcap_{k \in K}(T_k \to U_k) \equiv \bigcap_{i \in I'} D_i$. By (i) and ($\leq_{\nabla}$), we get

(d) $\vdash^{\nabla} R_p : \bigcap_{i \in I'} D_i$.

On the other hand, since $C' \leq_{\nabla} (\bigcap_{i \in I'} D_i) \to (\bigcap_{i \in I'} E_i)$, by ($\leq_{\nabla}$) we have

(e) $\vdash^{\nabla} R_p : (\bigcap_{i \in I'} D_i) \to (\bigcap_{i \in I'} E_i)$. Therefore, applying ($\to$ E) to (d) and (e), we get $\vdash^{\nabla} R_{p+1} : \bigcap_{i \in I'} E_i$. We are done, defining C as $\bigcap_{i \in I'} E_i$ and taking into account of (ii).

($\Leftarrow$) follows by standard computations, using Lemma 5.4(i). $\square$

Simple easiness of R_n is now an immediate consequence of previous Theorem.

Theorem 5.6
For any n, R_n is simple easy.

Proof. Take $p = n$ in the statement of the previous theorem and remember $\rho^{(n)}(\tilde{\sigma}, Z) \equiv Z$. □

Corollary 5.7
For any n, R_n is easy.

Acknowledgement

The authors wish to thanks Mariangiola Dezani Ciancaglini for enlightening discussions on the subject.

References

[1] S. Abramsky. Domain theory in logical form. *Ann. Pure Appl. Logic*, 51(1-2):1–77, 1991.

[2] S. Abramsky and C.-H. L. Ong. Full abstraction in the lazy lambda calculus. *Inform. and Comput.*, 105(2):159–267, 1993.

[3] F. Alessi. *Strutture di tipi, teoria dei domini e modelli del lambda calcolo*. PhD thesis, Torino University, 1991.

[4] F. Alessi, M. Dezani-Ciancaglini, and F. Honsell. Filter models and easy terms. In A. Restivo, S. Ronchi della Rocca, and L. Roversi, editors, *Mathematical Foundations of Computer Science 2000*, volume 2202 of *Lecture Notes in Comput. Sci.*, pages 17–37. Springer, 2001.

[5] J. Baeten and B. Boerboom. ω can be anything it shouldn't be. *Indag.Math.*, 41:111–120, 1979.

[6] H. Barendregt, M. Coppo, and M. Dezani-Ciancaglini. A filter lambda model and the completeness of type assignment. *J. Symbolic Logic*, 48(4):931–940, 1983.

[7] A. Berarducci. Infinite λ-calculus and non-sensible models. In M. D. Inc., editor, *Logic and Algebra*, number 180 in Lecture Notes in Pure and Applied Mathematics, pages 339–378. Springer-Verlag, 1996.

[8] A. Berarducci and B. Intrigila. Some new results on easy lambda-terms. *Theoret. Comput. Sci.*, 121:71–88, 1993.

[9] A. Berarducci and B. Intrigila. Church-Rosser λ-theories, infinite λ-calculus and consistency problems. In W. Hodges and M. Hyland et al., editors, *Logic: From Foundations to applications*, pages 33–58. Oxford Sci. Publ., New York, 1996.

[10] M. Coppo and M. Dezani-Ciancaglini. An extension of the basic functionality theory for the λ-calculus. *Notre Dame J. Formal Logic*, 21(4):685–693, 1980.

[11] M. Coppo, M. Dezani-Ciancaglini, F. Honsell, and G. Longo. Extended type structures and filter lambda models. In *Logic colloquium '82*, pages 241–262. North-Holland, Amsterdam, 1984.

[12] M. Coppo, M. Dezani-Ciancaglini, and B. Venneri. Principal type schemes and λ-calculus semantics. In R.Hindley and J.Seldin, editors, *To H. B. Curry: essays on combinatory logic, lambda calculus and formalism*, pages 535–560. Academic Press, London, 1980.

[13] M. Coppo, M. Dezani-Ciancaglini, and M. Zacchi. Type theories, normal forms, and D_∞-lambda-models. *Inform. and Comput.*, 72(2):85–116, 1987.

[14] M. Dezani-Ciancaglini, F. Honsell, and Y. Motohama. Compositional characterization of λ-terms using intersection types. In M. Nielsen and B. Rovan, editors, *Mathematical Foundations of Computer Science 2000*, volume 1893 of *Lecture Notes in Comput. Sci.*, pages 304–313. Springer, 2000.

[15] P. Di Gianantonio and F. Honsell. An abstract notion of application. In M.Bezem and J.F.Groote, editors, *Typed lambda calculi and applications*, volume 664 of *Lecture Notes in Comput. Sci.*, pages 124–138. Springer, Berlin, 1993.

[16] L. Egidi, F. Honsell, and S. Ronchi Della Rocca. Operational, denotational and logical descriptions: a case study. *Fund. Inform.*, 16(2):149–169, 1992.

[17] R. Hindley and G. Longo. Lambda-calculus models and extensionality. *Z. Math. Logik Grundlag. Math.*, 26(4):289–310, 1980.

[18] F. Honsell and M. Lenisa. Semantical analysis of perpetual strategies in λ-calculus. *Theoret. Comput. Sci.*, 212(1-2):183–209, 1999.

[19] F. Honsell and S. Ronchi Della Rocca. A filter model for Ω. Technical report, Torino University, 1984.

[20] F. Honsell and S. Ronchi Della Rocca. An approximation theorem for topological lambda models and the topological incompleteness of lambda calculus. *J. Comput. System Sci.*, 45(1):49–75, 1992.

[21] B. Intrigila. A problem on easy terms in λ-calculus. *Fund. Inform.*, 15.1:99–106, 1991.

[22] G. Jacopini. A condition for identifying two elements of whatever model of combinatory logic. In C.Böhm, editor, *λ-calculus and computer science theory*, volume 37 of *Lecture Notes in Comput. Sci.*, pages 213–219. Springer, Berlin, 1975.

[23] G. Jacopini and M. Venturini Zilli. Easy terms in the lambda calculus. *Fund. Inform.*, 80:225–233, 1985.

[24] J. Kuper. On the Jacopini technique. *Inform. and Comput.*, 138:101–123, 1997.

[25] G. D. Plotkin. Set-theoretical and other elementary models of the λ-calculus. *Theoret. Comput. Sci.*, 121(1-2):351–409, 1993.

[26] G. Pottinger. A type assignment for the strongly normalizable λ-terms. In R.Hindley and J.Seldin, editors, *To H. B. Curry: essays on combinatory logic, lambda calculus and formalism*, pages 561–577. Academic Press, London, 1980.

[27] A. Pravato, S. Ronchi, and L. Roversi. The call-by-value lambda calculus: a semantic investigation. *Math. Struct. in Comput. Sci.*, 9(5):617–650, 1999.

[28] D. Scott. Continuous lattices. In F. Lawvere, editor, *Toposes, algebraic geometry and logic*, volume 274 of *Lecture Notes in Math.*, pages 97–136. Springer, Berlin, 1972.

[29] S. Vickers. *Topology via logic.* Cambridge University Press, Cambridge, 1989.