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The Relevance of Semantic Subtyping

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Abstract

We compare Meyer and Routley's minimal relevant logic $\mathbf{B}_+$ with the recent semantics-based approach to subtyping introduced by Frisch, Castagna and Benzaken in the definition of a type system with intersection and union. We show that – for the functional core of the system – such notion of subtyping, which is defined in purely set-theoretical terms, coincides with the relevant entailment of the logic $\mathbf{B}_+$.

1 Introduction

Thirty years ago Meyer and Routley introduced the logical system $\mathbf{B}_+$, a minimal negation-free relevant logic, along with a Kripke-style semantics [13]. As shown in 1999 in [9], such semantics basically describes a universe endowed with an application operator, where the Kripkian worlds are functions, and

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formulas holding in a world are types assignable to the corresponding function. In particular, as previously proved in [14], [8], the logic turns out to be equivalent to the type system for the λ-calculus defined and studied in [3], [4], in the sense that the valid formulas in $\mathbf{B}_+$ are exactly the types assignable in that system to the identity, which therefore plays the role, in the "possible worlds" semantics, of "real world". Whence it immediately follows that the relevant entailment coincides with the natural subtyping relation.

In [11] a type system is introduced where types are given a conceptually simple set-theoretic semantics, and all the mathematical complexity usually involved in other standard approaches is reduced to set-theoretic arguments. The novelty of the approach lies particularly in the fact that the subtyping relation, essential in this kind of systems, is defined semantically rather than syntactically, and the typing algorithms are directly derived from semantics.

With the present contribution we show that such type system, in the part that can be given a minimal logic interpretation, is also equivalent to the logic $\mathbf{B}_+$, in the sense that the semantically defined subtyping relation coincides with the relevant entailment of $\mathbf{B}_+$: $\sigma \leq \tau$ holds iff σ entails τ in $\mathbf{B}_+$, or – equivalently – iff $\sigma \to \tau$ is a theorem of $\mathbf{B}_+$. More generally, the types assignable to the identity are exactly the theorems of $\mathbf{B}_+$. We think that this sheds some light both on $\mathbf{B}_+$ and on the semantic subtyping system:

- it gives a further motivation for the definition of subtyping between arrow types in [11], expressed by an original formula derived from a natural semantic definition; this formula is here re-obtained syntactically from axioms and rules in $\mathbf{B}_+$, thus providing a better understanding of its meaning;
- this formula provides an explicit decomposition formula for the entailment relation in $\mathbf{B}_+$; in a sense, it can be seen as an algorithmic version for the rules defining $\mathbf{B}_+$.

The many other features that constitute the semantic subtyping system and characterize its new approach are not addressed, since the comparison with the minimal relevant logic seems to be meaningful, as the name itself indicates, only w.r.t. a minimalist version of the type system.

This paper is organized as follows. In Section 2 we give an informal and hopefully intuitive account of the whole argument, with some background. In Section 3 we start the technical part by giving two equivalent presentations of $\mathbf{B}_+$: as a deductive system à la Hilbert, and as a theory of an order relation $\preccurlyeq$ and a unary relation $\vdash_{\mathbf{B}+}$. In Section 4, we present the semantic approach of [11] for defining subtyping, and we extract the above mentioned essential formula to recast it in a setting similar to $\mathbf{B}_+$; formally, we define a theory $\mathbf{T}_{[\![\leq]\!]}$ of the order relation $\leq$ and the unary relation $\vdash_{\mathbf{T}}$. The main result of this paper is the equivalence between $\mathbf{B}_+$ and $\mathbf{T}_{[\![\leq]\!]}$, as stated by the following theorem:

Theorem 1.1 *A formula is valid in* $\mathbf{B}_+$ *if and only if it is a theorem in* $\mathbf{T}_{[\![\leq]\!]}$*,* *i.e.:*

$$\vdash_{\mathbf{B}+} \sigma \Leftrightarrow \vdash_{\mathbf{T}} \sigma$$

and the immediate corollary:

Corollary 1.2 *A formula σ entails a formula τ in $\mathbf{B}_+$ if and only if σ is a subtype of τ in $\mathbf{T}_{[\![\leq]\!]}$:*

$$\sigma \preccurlyeq \tau \Leftrightarrow \ \sigma \leq \tau$$

Sections 5 and 6 are dedicated to the proof of equivalence between $\mathbf{B}_+$ and $\mathbf{T}_{[\![\leq]\!]}$. In Section 7 we prove the reversibility of the distinguishing rule of $\mathbf{T}_{[\![\leq]\!]}$ (as could be predicted from [11] where the formula is indeed a reversible rule), using a syntactical tool in $\mathbf{B}_+$. In Section 8 we draw a short conclusion indicating possible directions for future work.

2 The intuition

In [11] a class of models for type systems with intersection, union and subtyping is defined, where types are interpreted as subsets of a universal set $\mathcal{D}$, and union, intersection and subtyping are interpreted as their obvious set-theoretic counterparts.

In addition, the interpretation of function types (i.e., intersections and unions of arrow types) is required to satisfy a constraint which can be roughly expressed as follows. Subtyping, i.e., the order defined on $\mathcal{D}$ by set inclusion, must be isomorphic to the one holding in a setting where:

- arrow types are interpreted as sets of (extensional) partial functions, given by the following definition:

$$[\![\sigma \to \tau]\!] = \{f \in \mathsf{F} \,|\, x \in [\![\sigma]\!] \Rightarrow f(x) \in [\![\tau]\!]\} \tag{1}$$

  where F is the set of partial functions from $\mathcal{D}$ to $\mathcal{D}$;
- intersections and unions of arrow types are also interpreted as the corresponding set operations.

Of course, for the usual cardinality reasons, the set of such interpretations of function types cannot coincide with $\mathcal{D}$ or be a subset of it; in [11] models (in the proper sense) satisfying the above condition are however proved to exist: in particular, a universal model is built by taking only finite approximations of functions.

If one abstracts from the nature of the semantic domain where functions and function types are interpreted (while still keeping to set-theoretical modelings, where types are interpreted as sets and the typing judgement as set membership) the definition may be written as:

$$[\![\sigma \to \tau]\!] = \{f \in \mathcal{F} \,|\, x \in [\![\sigma]\!] \Rightarrow f(x) \in [\![\tau]\!]\} \tag{2}$$

where $\mathcal{F}$ is the interpretation domain of function types, differently chosen in different semantic theories. Examples are the set-theoretic semantics of type systems for the λ-calculus, such as the F-semantics, the "simple" semantics or, more generally, the so-called inference semantics; they all assume this definition, each with a different specification of $\mathcal{F}$ [10], [12]. [4]

Observe that if one also abstracts from the semantics of the typing judgement $t\colon \sigma$ (which associates a type with a term), the above definition simply becomes the condition that

$$f\colon \sigma \to \tau \quad \text{iff} \quad x\colon \sigma \Rightarrow f(x)\colon \tau \tag{3}$$

which is the minimal condition that must be satisfied by whatever notion of function and of function type.

From the definition (2), the following property can be shown to hold:

$$\begin{gathered}
(\llbracket\sigma\rrbracket \subseteq \llbracket\sigma_1\rrbracket \cup \llbracket\sigma_2\rrbracket)\ \mathit{and}\ (\llbracket\tau_1\rrbracket \cap \llbracket\tau_2\rrbracket \subseteq \llbracket\tau\rrbracket)\\
\mathit{and}\ (\llbracket\sigma\rrbracket \subseteq \llbracket\sigma_1\rrbracket\ \mathit{or}\ \llbracket\tau_2\rrbracket \subseteq \llbracket\tau\rrbracket)\ \mathit{and}\ (\llbracket\sigma\rrbracket \subseteq \llbracket\sigma_2\rrbracket\ \mathit{or}\ \llbracket\tau_1\rrbracket \subseteq \llbracket\tau\rrbracket)\\
\mathit{implies}\\
\llbracket\sigma_1 \to \tau_1\rrbracket \cap \llbracket\sigma_2 \to \tau_2\rrbracket \subseteq \llbracket\sigma \to \tau\rrbracket
\end{gathered}$$

for, if we write the consequent through the set membership relation as:

$$\begin{gathered}
\Big(\big(\forall x\,.\, x \in \llbracket\sigma_1\rrbracket \Rightarrow f(x) \in \llbracket\tau_1\rrbracket\big)\ \mathit{and}\ \big(\forall y\,.\, y \in \llbracket\sigma_2\rrbracket \Rightarrow f(y) \in \llbracket\tau_2\rrbracket\big)\Big)\\
\Rightarrow (x \in \llbracket\sigma\rrbracket \Rightarrow f(x) \in \llbracket\tau\rrbracket)
\end{gathered}$$

then by assuming the antecedent we have:

$$x \in \llbracket\sigma\rrbracket \ \Rightarrow\ x \in \llbracket\sigma_1\rrbracket \cup \llbracket\sigma_2\rrbracket \ \Rightarrow\ x \in \llbracket\sigma_1\rrbracket\ \mathit{or}\ x \in \llbracket\sigma_2\rrbracket \ \Rightarrow\ f(x) \in \llbracket\tau\rrbracket$$

where the final implication is easily proved by cases:

(i) $x \in \llbracket\sigma_1\rrbracket$ *and* $x \in \llbracket\sigma_2\rrbracket$
then $f(x) \in \llbracket\tau_1\rrbracket$ *and* $f(x) \in \llbracket\tau_2\rrbracket$, i.e., $f(x) \in \llbracket\tau_1\rrbracket \cap \llbracket\tau_2\rrbracket \subseteq \llbracket\tau\rrbracket$;

(ii) $x \in \llbracket\sigma_1\rrbracket$ *and* $x \notin \llbracket\sigma_2\rrbracket$
then $\llbracket\sigma\rrbracket \not\subseteq \llbracket\sigma_2\rrbracket$, whence $\llbracket\tau_1\rrbracket \subseteq \llbracket\tau\rrbracket$; therefore $f(x) \in \llbracket\tau_1\rrbracket \subseteq \llbracket\tau\rrbracket$, i.e., $f(x) \in \llbracket\tau\rrbracket$;

(iii) $x \notin \llbracket\sigma_1\rrbracket$ *and* $x \in \llbracket\sigma_2\rrbracket$
symmetrical

[4] In the inference semantics $\mathcal{F}$ may depend on σ and τ: in the following we assume $\mathcal{F}$ constant.

The syntactic translation of such property, i.e.,

$$\left.\begin{array}{c}(\sigma \leq \sigma_1 \vee \sigma_2)\ \mathit{and}\ (\tau_1 \wedge \tau_2 \leq \tau) \\ \mathit{and}\ (\sigma \leq \sigma_1\ \mathit{or}\ \tau_2 \leq \tau)\ \mathit{and}\ (\sigma \leq \sigma_2\ \mathit{or}\ \tau_1 \leq \tau)\end{array}\right\} \\ \Rightarrow\ (\sigma_1 \to \tau_1) \wedge (\sigma_2 \to \tau_2) \leq \sigma \to \tau \tag{4}$$

or, more precisely, its extension to finite sets of intersections and unions, is in [11] proved to hold in the semantic subtyping system (actually, it is proved – in conjunction with the reverse implication – to be equivalent to the above sketched set-theoretic semantic characterization by means of extensional functions).

On the other hand, if we denote by id the identity function in $\mathcal{D} \to \mathcal{D}$, i.e., if id is the function such that $\forall x \in \mathcal{D} \,.\, \mathsf{id}(x) = x$, we have, like for any element of a set:

$$[\![\sigma]\!] \subseteq [\![\tau]\!]\ \mathit{and}\ \mathsf{id} \in [\![\sigma]\!]\ \Rightarrow\ \mathsf{id} \in [\![\tau]\!]$$

$$\mathsf{id} \in [\![\sigma]\!]\ \mathit{and}\ \mathsf{id} \in [\![\tau]\!]\ \Rightarrow\ \mathsf{id} \in [\![\sigma]\!] \cap [\![\tau]\!]$$

Moreover, the following property holds:

$$[\![\sigma]\!] \subseteq [\![\tau]\!] \Longleftrightarrow \mathsf{id} \in [\![\sigma \to \tau]\!]$$

as can be seen from:

$$[\![\sigma]\!] \subseteq [\![\tau]\!]\ \Longleftrightarrow\ (x \in [\![\sigma]\!] \Rightarrow x \in [\![\tau]\!])$$

$$\Longleftrightarrow\ (x \in [\![\sigma]\!] \Rightarrow \mathsf{id}(x) \in [\![\tau]\!])\ \Longleftrightarrow\ \mathsf{id} \in [\![\sigma \to \tau]\!]$$

The syntactic counterparts of these three properties will therefore hold in any simple type assignment system based on the constructs of semantic subtyping for languages where the identity function is expressible (by means of some expression id): [5]

(i) $id : \sigma,\ id : \tau\ \Rightarrow\ id : \sigma \wedge \tau$

(ii) $\sigma \leq \tau,\ id : \sigma\ \Rightarrow\ id : \tau$

(iii) $\sigma \leq \tau \Longleftrightarrow id : \sigma \to \tau$

If – following the well-known analogy – we interpret logic formulae as types, and in addition the subtyping relation as the logical entailment, the first two properties exactly translate into two inference rules of $\mathbf{B}_+$, while the third becomes what one might call the "relevant deduction theorem", i.e., a statement that, given an independent semantic definition of entailment, may be proved to hold in $\mathbf{B}_+$; alternatively, it may be merely assumed as the definition of the $\mathbf{B}_+$ (syntactic) entailment, as we have chosen to do in this paper, for clarity and self-containment.

[5] Notice that id is the denotation of id.

In $\mathbf{B}_+$ the set of classes of equivalent formulae is a distributive lattice w.r.t. the entailment relation, like in semantic subtyping the set of type interpretations w.r.t. set inclusion: then the equivalence between the minimal relevant logic and the functional core of semantic subtyping relies on the equivalence, in distributive lattices, between the remaining axioms and rules of $\mathbf{B}_+$ and the (extended version of) property (4). This is what will be shown in the rest of the paper for a formal setting intended to correspond to the intuitive content we have just described.

3 The positive minimal relevant logic $\mathbf{B}_+$

Definition 3.1 (The language of $\mathbf{B}_+$) The language $\mathcal{L}$ of $\mathbf{B}_+$ is defined by the following syntax:

$$\sigma, \tau, \rho ::= \alpha \mid \sigma \to \tau \mid \sigma \wedge \tau \mid \sigma \vee \tau$$

where α, β, γ denote atomic formulae, i.e., propositional variables.

In writing formulas we convene that $\wedge$ and $\vee$ take precedence over $\to$.

The logic $\mathbf{B}_+$ is usually presented by means of a deductive system à la Hilbert, consisting of axioms (axiom schemes) and rules. In the present paper we will only use the equivalent Definition 3.4.

Definition 3.2 (The positive minimal relevant logic $\mathbf{B}_+$)

Refl.	$\sigma \to \sigma$
$\wedge$**E.**	$\sigma \wedge \tau \to \sigma,\ \sigma \wedge \tau \to \tau$
$\to \wedge$**I.**	$(\rho \to \sigma) \wedge (\rho \to \tau) \to \rho \to \sigma \wedge \tau$
$\to \vee$**E.**	$(\sigma \to \rho) \wedge (\tau \to \rho) \to \sigma \vee \tau \to \rho$
$\vee$**I.**	$\sigma \to \sigma \vee \tau,\ \tau \to \sigma \vee \tau$
Dist$\wedge\vee$.	$\rho \wedge (\sigma \vee \tau) \to (\rho \wedge \sigma) \vee (\rho \wedge \tau)$
$\to$**E.**	$\sigma \to \tau,\ \sigma \quad \Rightarrow \quad \tau$
$\wedge$**I.**	$\sigma, \tau \quad \Rightarrow \quad \sigma \wedge \tau$
Pre.	$\sigma \to \tau \quad \Rightarrow \quad (\rho \to \sigma) \to \rho \to \tau$
Suf.	$\rho \to \sigma \quad \Rightarrow \quad (\sigma \to \tau) \to \rho \to \tau$

Of course, the $\Rightarrow$ symbol in the inference rules is a (classical!) meta-implication, and a formula is a $\mathbf{B}_+$-*theorem* (or is valid) iff it is the final formula of a sequence where each element is either an instance of an axiom, or an instance of the consequent of a rule whose correspondingly instantiated antecedents occur previously in the sequence. We will write $\vdash_{\mathbf{B}+} \sigma$ for the statement that σ is a $\mathbf{B}_+$-theorem.

In such presentation there is no interesting notion of "deduction from premises", since the meta-implication, being classical, cannot be internalized in the (relevant) arrow of the logic, and therefore there is no deduction theorem connecting $\Rightarrow$ and $\rightarrow$. A notion of entailment is semantically defined with reference to à la Kripke interpretations [6]; we will denote it by the symbol $\preccurlyeq$ instead of the more usual $\models$ or $\vdash$, somewhat in the spirit of [9], as a hint to the fact that the entailment is, as usual, a pre-order relation. The syntactical correspondent of the entailment is the principal arrow of a formula: soundness and completeness hold for $\mathbf{B}_+$ in the sense that a formula σ entails a formula τ iff $\sigma \rightarrow \tau$ is a theorem:

$$\sigma \preccurlyeq \tau \quad \text{iff} \quad \vdash_{\mathbf{B}+} \sigma \rightarrow \tau \tag{5}$$

Here we can take this equivalence as a definition for the entailment relation $\preccurlyeq$. We will use the symbol $\sim$ to denote $\mathbf{B}_+$-logical equivalence, i.e., $\sigma \sim \tau$ iff $\sigma \preccurlyeq \tau$ and $\tau \preccurlyeq \sigma$.

Of course, the entailment is an order relation on the set of classes of logically equivalent formulae, which is then a distributive lattice, as can be seen from the axioms and rules. The logic $\mathbf{B}_+$ can then be viewed as the following theory (with equality):

Definition 3.4 The theory $\mathbf{B}_+$ of the order relation $\preccurlyeq$ and of the unary relation $\vdash_{\mathbf{B}+}$, consists of:

- (i) the axioms of distributive lattices, i.e.:
  - (a) $\preccurlyeq$ is an order relation ($\sigma \preccurlyeq \tau, \tau \preccurlyeq \sigma \Rightarrow \sigma \sim \tau$, etc.);
  - (b) intersection and union respectively are the meet and join operations ($\sigma \wedge \tau \preccurlyeq \sigma$, $\sigma \wedge \tau \preccurlyeq \tau$, etc.);
  - (c) the distributivity law: $\rho \wedge (\sigma \vee \tau) \preccurlyeq (\rho \wedge \sigma) \vee (\rho \wedge \tau)$

[6] For reader convenience we report here the definition of Kripke semantics.

Definition 3.3 (KRIPKE-STYLE POSSIBLE WORLDS SEMANTICS OF $\mathbf{B}_+$) We define a *model structure* to be a structure $\mathbf{K} =< \mathcal{K}, \mathsf{R} >$, where $\mathcal{K}$ is a set (of worlds) and R is a ternary relation on K. A valuation v on the model structure $\mathbf{K}$ is a function from the set of variables to the set $\mathcal{P}(\mathcal{K})$ of all subsets of $\mathcal{K}$. A valuation on $\mathbf{K}$ is extended to an interpretation $\mathcal{I}(-)$ from the set of all formulas to $\mathcal{P}(\mathcal{K})$, as follows (for $w \in \mathcal{K}$):

- (i) $w \in \mathcal{I}(\alpha) \iff w \in \mathsf{v}(\alpha)$;
- (ii) $w \in \mathcal{I}(\sigma \wedge \tau) \iff w \in \mathcal{I}(\sigma)$ and $w \in \mathcal{I}(\tau)$;
- (iii) $w \in \mathcal{I}(\sigma \vee \tau) \iff w \in \mathcal{I}(\sigma)$ or $w \in \mathcal{I}(\tau)$;
- (iv) $w \in \mathcal{I}(\sigma \rightarrow \tau) \iff \forall x \forall y\ \mathsf{R}wxy \Rightarrow x \in \mathcal{I}(\sigma) \Rightarrow y \in \mathcal{I}(\tau)$.

We say that σ *entails* τ iff $\mathcal{I}(\sigma) \subseteq \mathcal{I}(\tau)$ for all valuations and all model structures.

(ii) the additional $\mathbf{B}_+$ axioms and rules:

$$\rightarrow\!\wedge\mathbf{I}_{\preccurlyeq}.\ (\rho \to \sigma) \wedge (\rho \to \tau) \preccurlyeq \rho \to \sigma \wedge \tau$$

$$\rightarrow\!\vee\mathbf{E}_{\preccurlyeq}.\ (\sigma \to \rho) \wedge (\tau \to \rho) \preccurlyeq \sigma \vee \tau \to \rho$$

$$\mathbf{MP}_{\preccurlyeq}.\ \sigma \preccurlyeq \tau,\ \vdash_{\mathbf{B}+} \sigma\ \Rightarrow \vdash_{\mathbf{B}+} \tau$$

$$\wedge\mathbf{I}_{\preccurlyeq}.\ \vdash_{\mathbf{B}+} \sigma, \vdash_{\mathbf{B}+} \tau\ \Rightarrow \vdash_{\mathbf{B}+} \sigma \wedge \tau$$

$$\mathbf{Pre}_{\preccurlyeq}.\ \sigma \preccurlyeq \tau\ \Rightarrow\ \rho \to \sigma\ \preccurlyeq\ \rho \to \tau$$

$$\mathbf{Suf}_{\preccurlyeq}.\ \rho \preccurlyeq \sigma\ \Rightarrow\ \sigma \to \tau\ \preccurlyeq\ \rho \to \tau$$

(iii) the entailment rule relating the predicate $\vdash_{\mathbf{B}+}$ and the relation $\preccurlyeq$:

$$\mathbf{Ent}.\quad \sigma \preccurlyeq \tau\ \iff\ \vdash_{\mathbf{B}+} \sigma \to \tau$$

Remark that, in agreement with the statement (5), the new forms of the axioms are obtained from the original ones by replacing the principal arrow with the entailment.

4 Semantic subtyping

In [11] a generic type language is considered, quite independently from the programming languages to which it may be applied; besides basic, function and product types, it includes a universal and an empty type, intersection, union and complement types, and recursive types.

As anticipated in Section 2, in order to define a meaningful subtyping relation on the type algebra, a set-theoretic semantics is defined, where types are interpreted as subsets of a universal set: the type operations intersection, union, complement, etc. are interpreted as the homonymous set operations, and subtyping simply corresponds to set inclusion. Nothing is explicitly said about the nature of the elements of such sets (except for cartesian products, which of course have to be isomorphic to actual cartesian products). In particular, nothing is explicitly said about the nature of function types; they, however, must be appropriate for describing sets of functions, for example in allowing the subject reduction for a reasonable language to hold.

The semantic definition of subtyping must therefore be required to satisfy some constraints that implicitly restrict the interpretation of arrow types to actual function sets. In [11] this is obtained through a particular condition, which asserts that the subtyping must behave *as if* arrow types were interpreted extensionaly, as sets of binary relations (graphs of possibly non-deterministic and non-terminating functions that may raise a type error Ω); formally, if $\mathcal{D}$ denotes the structure where types are interpreted, the extensional interpretation of an arrow type $\sigma \to \tau$ is the set of graphs $f \subset \mathcal{D} \times (\mathcal{D} \cup \{\Omega\})$ such that $(x, y) \in f$ and $x \in [\![\sigma]\!]$ imply $y \in [\![\tau]\!]$. Set-theoretic manipulations allow to derive a syntactic characterization of the subtyping relation between an intersection and a union of arrow types; for any finite sets $\{\sigma_i \to \tau_i\}_{i\in I}$ and $\{\sigma'_j \to \tau'_j\}_{j\in J}$ of arrow types, the following equivalence holds:

$$\mathfrak{A} \Longleftrightarrow \mathfrak{F},$$

where

$$\begin{cases} \mathfrak{A} \triangleq \bigwedge_{i\in I}(\sigma_i \to \tau_i) \leq \bigvee_{j\in J}(\sigma'_j \to \tau'_j) \\ \mathfrak{F} \triangleq \exists j \in J\,.\,(\sigma'_j \leq \bigvee_{i\in I}\sigma_i)\ \mathit{and}\, \forall I' \subsetneq I\,.\,(\sigma'_j \leq \bigvee_{i\in I'}\sigma_i)\ \mathit{or}\ (\bigwedge_{i\in I\setminus I'}\tau_i \leq \tau'_j) \end{cases}$$

This very condition, in the context of the standard set properties, turns out to be equivalent to the axioms and rules of $\mathbf{B}_+$.

The type system in [11] has some features that are absent in $\mathbf{B}_+$ and that cannot be easily dropped from the framework (empty and universal types); also, it has no type variables (atomic types in [11] are quite different from type variables, for instance they do not intersect any arrow type), nor any natural notion of theorem (we would like to speak of the "types of the identity", but we have no programming language to refer to), so we cannot directly relate the two systems. In order to isolate the essential connection between semantic subtyping and the minimal relevant logic, we will just pick the rule $\mathfrak{F} \Rightarrow \mathfrak{A}$ from [11], and show that it is equivalent to $\mathbf{B}_+$ axioms $\to\wedge\mathbf{I}_\preccurlyeq$, $\to\vee\mathbf{E}_\preccurlyeq$ and rules $\mathbf{Pre}_\preccurlyeq$, $\mathbf{Suf}_\preccurlyeq$ in presence of the other axioms and rules of $\mathbf{B}_+$.

Because there is no universal or empty type in the type language $\mathcal{L}$, the finite sets I and J in formulae $\mathfrak{F}$ and $\mathfrak{A}$ must be non-empty, and when I' is empty in $\mathfrak{F}$, the rightmost disjunction should be read as $(\bigwedge_{i\in I}\tau_i \leq \tau'_j)$.

The minimal core of the semantic subtyping system we wish to study is then the theory (with equality) defined as follows.

Definition 4.1 The theory $\mathbf{T}_{[\![\leq]\!]}$ of the order relation $\leq$ and of the unary relation $\vdash_{\mathbf{T}}$ consists of:

(i) the axioms for distributive lattices (see Definition 3.4);

(ii) the three rules:

MP$_{\mathbf{T}}$. $\sigma \leq \tau,\ \vdash_{\mathbf{T}} \sigma\ \Rightarrow \vdash_{\mathbf{T}} \tau$

$\wedge\mathbf{I}_{\mathbf{T}}$. $\vdash_{\mathbf{T}} \sigma,\ \vdash_{\mathbf{T}} \tau\ \Rightarrow \vdash_{\mathbf{T}} \sigma \wedge \tau$

M-rule. $\mathfrak{F} \Rightarrow \mathfrak{A}$, where $\mathfrak{F}$ and $\mathfrak{A}$ are the two formulae defined above.

(iii) the entailment rule:

$$\mathbf{Ent}.\quad \sigma \leq \tau \Longleftrightarrow\ \vdash_{\mathbf{T}} \sigma \to \tau$$

The axioms and rules concerning $\vdash_{\mathbf{T}}$ may be interpreted as a miniature type assignment system for a programming language with a single term (the identity): $\mathbf{MP_T}$ and $\wedge\mathbf{I_T}$ are the classical subsumption and intersection rules. We call $\mathbf{T}_{[\![\leq]\!]}$-*theorems* the theorems for the relation $\vdash_{\mathbf{T}}$.

Section 7 will prove the reversibility of the **M-rule**, which means that we could also consider the full implication $\mathfrak{F} \iff \mathfrak{A}$ instead of the **M-rule**, and get an equivalent theory.

5 $\mathbf{B}_+$-theorems are $\mathbf{T}_{[\![\leq]\!]}$-theorems

Theorem 5.1 *If a formula of the language $\mathcal{L}$ is valid in $\mathbf{B}_+$, then it is a $\mathbf{T}_{[\![\leq]\!]}$-theorem, i.e.:*

$$\vdash_{\mathbf{B}+} \sigma \Rightarrow \vdash_{\mathbf{T}} \sigma$$

Proof. We have to show that $\rightarrow\!\wedge\mathbf{I}_{\preccurlyeq}$, $\rightarrow\!\vee\mathbf{E}_{\preccurlyeq}$, $\mathbf{Pre}_{\preccurlyeq}$, $\mathbf{Suf}_{\preccurlyeq}$ are derivable from $\mathfrak{F} \Rightarrow \mathfrak{A}$; but they are easily seen to be mere instances of it.

In particular, the axiom $\rightarrow\!\wedge\mathbf{I}_{\preccurlyeq}$ is obtained by taking $I=\{1,2\}$, $J=\{1\}$, $\sigma_1 = \sigma_2 = \sigma'_1 = \sigma$, $\tau'_1 = \tau_1 \wedge \tau_2$. The proposition $\mathfrak{F}$ becomes:

$$(\sigma \leq \sigma \vee \sigma) \textit{ and } (\sigma \leq \sigma \textit{ or } \tau_1 \leq \tau_1 \wedge \tau_2) \textit{ and}$$
$$(\sigma \leq \sigma \textit{ or } \tau_2 \leq \tau_1 \wedge \tau_2) \textit{ and } (\tau_1 \wedge \tau_2 \leq \tau_1 \wedge \tau_2)$$

which is trivially true, and so is the consequence $\mathfrak{A}$:

$$(\sigma \rightarrow \tau_1) \wedge (\sigma \rightarrow \tau_2) \leq \sigma \rightarrow \tau_1 \wedge \tau_2$$

The axiom $\rightarrow\!\vee\mathbf{E}_{\preccurlyeq}$ is analogous.

The proof for the rules $\mathbf{Pre}_{\preccurlyeq}$ and $\mathbf{Suf}_{\preccurlyeq}$ is obtained by taking $I = \{1\}$, $J=\{1\}$; the propositions $\mathfrak{F}$ and $\mathfrak{A}$ respectively become:

$$(\sigma'_1 \leq \sigma_1) \textit{ and } (\tau_1 \leq \tau'_1)$$
$$\sigma_1 \rightarrow \tau_1 \leq \sigma'_1 \rightarrow \tau'_1$$

So the implication $\mathfrak{F} \Rightarrow \mathfrak{A}$ gives both $\mathbf{Pre}_{\preccurlyeq}$ and $\mathbf{Suf}_{\preccurlyeq}$. □

6 $\mathbf{T}_{[\![\leq]\!]}$-theorems are $\mathbf{B}_+$-theorems

To prove the converse of Theorem 5.1 it is sufficient to prove that the rule $\mathfrak{F} \Rightarrow \mathfrak{A}$ (with the translation of $\leq$ into $\preccurlyeq$) holds in $\mathbf{B}_+$. By fixing the j in $\mathfrak{F}$, we have to prove the statement:

$$(\mathcal{B}\ \ and\ \ \mathcal{C}) \Rightarrow\ \mathcal{A}, \quad \text{where} \begin{cases} \mathcal{A}\ \triangleq\ \bigwedge_{i\in I}(\sigma_i \to \tau_i) \preccurlyeq \sigma \to \tau \\ \mathcal{B}\ \triangleq\ \sigma \preccurlyeq \bigvee_{i\in I}\sigma_i \\ \mathcal{C}\ \triangleq\ \forall I' \subsetneq I\,.\,(\sigma \preccurlyeq \bigvee_{i\in I'}\sigma_i)\ \ or\ \ (\bigwedge_{j\in I\setminus I'}\tau_j \preccurlyeq \tau) \end{cases}$$

We introduce a preliminary definition of two sets of indices related to the proposition $\mathcal{C}$, and we prove a lemma that concerns them.

Definition 6.1 The sets H and K are sets of subsets of I defined as follows:

$$H\ =\ \{I' \subseteq I \mid \sigma \preccurlyeq \bigvee_{i\in I'}\sigma_i\} \qquad K\ =\ \{I' \subseteq I \mid I\setminus I' \notin H\}.$$

Some elementary properties of I, H and K, listed in the following lemma, are immediately derived.

Lemma 6.2

(i) $I \in H$ *and* $\emptyset \notin H$.
(ii) $I \in K$ *and* $\emptyset \notin K$.
(iii) $I' \in H, I' \subseteq I'' \Rightarrow I'' \in H$.
(iv) $I' \in K, I' \subseteq I'' \Rightarrow I'' \in K$.
(v) $\forall I' \in H\ \forall I'' \in K, I' \cap I'' \neq \emptyset$.
(vi) $\forall I' \in K,\ \bigwedge_{j\in I'}\tau_j \preccurlyeq \tau$.

Proof.
(i), (ii), (iii), (vi): trivial.
(iv): since $I\setminus I' \supseteq I\setminus I''$ and $I\setminus I' \notin H$, then $I\setminus I'' \notin H$, i.e. $I'' \in K$.
(v): Suppose $I' \cap I'' = \emptyset$. So, $I' \subseteq I\setminus I''$ because $I', I'' \subseteq I$. Since $I' \in H$, then $I\setminus I'' \in H$ by (iii) above, whence $I'' \notin K$, which is a contradiction. □

Remark 6.3 The usual distributivity law, given by the axiom **Dist**$\wedge\vee$ and also holding in classical logics, when applied to an arbitrary number of conjunctions and disjunctions may be expressed as follows. If $K = \{J_1, \ldots, J_k\}$ is a set of sets of indices, then:

$$\bigvee_{J\in K}\Big(\bigwedge_{j\in J}\sigma_j\Big) = \Big(\bigwedge_{j_1\in J_1}\sigma_{j_1}\Big) \vee \ldots \vee \Big(\bigwedge_{j_k\in J_k}\sigma_{j_k}\Big) \sim \bigwedge_{j_1\in J_1}\cdots\bigwedge_{j_k\in J_k}(\sigma_{j_1}\vee\ldots\vee\sigma_{j_k})$$

Example 6.4 Let $J_1 = \{1,2,3\}$, $J_2 = \{4,5\}$, then

$$(\sigma_1\wedge\sigma_2\wedge\sigma_3)\vee(\sigma_4\wedge\sigma_5) \sim (\sigma_1\vee\sigma_4)\wedge(\sigma_1\vee\sigma_5)\wedge(\sigma_2\vee\sigma_4)\wedge(\sigma_2\vee\sigma_5)\wedge(\sigma_3\vee\sigma_4)\wedge(\sigma_3\vee\sigma_5)$$

Lemma 6.5 *If H and K are the sets of indices defined in the Definition 6.1, then the following property holds:*

$$\bigwedge_{I'\in H}\Big(\bigvee_{i\in I'}\sigma_i\Big)\sim\bigvee_{J\in K}\Big(\bigwedge_{j\in J}\sigma_j\Big)$$

Proof.

(i) Proof of $\bigwedge_{I'\in H}\Big(\bigvee_{i\in I'}\sigma_i\Big)\preccurlyeq\bigvee_{J\in K}\Big(\bigwedge_{j\in J}\sigma_j\Big)$:
Following the above remark on distributivity, we can write:

$$\bigvee_{J\in K}\Big(\bigwedge_{j\in J}\sigma_j\Big)\sim\bigwedge_{j_1\in J_1}\cdots\bigwedge_{j_k\in J_k}(\sigma_{j_1}\vee\ldots\vee\sigma_{j_k})$$

So we have to prove $\bigwedge_{I'\in H}\Big(\bigvee_{i\in I'}\sigma_i\Big)\preccurlyeq\bigvee_{j\in R}\sigma_j$ for any set $R=\{j_1,\ldots,j_k\}$ where $j_1\in J_1,\ldots,j_k\in J_k$. Observe that for every $J\in K$ we have $R\cap J\neq\emptyset$, since R contains at least one j for each J in K. This allows us to easily prove that $R\in H$. For suppose $R\notin H$: then, by definition, $I\backslash R\in K$; hence by the above observation $R\cap(I\backslash R)\neq\emptyset$, which is obviously absurd.
Because $R\in H$, we can immediately conclude $\bigwedge_{I'\in H}\Big(\bigvee_{i\in I'}\sigma_i\Big)\preccurlyeq\bigvee_{j\in R}\sigma_j$.

(ii) Proof of $\bigvee_{J\in K}\Big(\bigwedge_{j\in J}\sigma_j\Big)\preccurlyeq\bigwedge_{I'\in H}\Big(\bigvee_{i\in I'}\sigma_i\Big)$:
For any two sets of indices J and I' such that $J\cap I'\neq\emptyset$, independently from the definitions of H and K, we have:

$$\bigwedge_{j\in J}\sigma_j\preccurlyeq\bigwedge_{j\in J\cap I'}\sigma_j\preccurlyeq\bigvee_{j\in J\cap I'}\sigma_j\preccurlyeq\bigvee_{j\in J}\sigma_j$$

Then, by respectively taking the union over all the J's and the intersection over all the I''s we conclude the proof.

□

Lemma 6.6 ($\mathcal{B}$ *and* $\mathcal{C}\Rightarrow\mathcal{A}$)

Proof. Since $J\subseteq I$ for all $J\in K$,

$$\bigwedge_{i\in I}(\sigma_i\to\tau_i)\preccurlyeq\bigwedge_{j\in J}(\sigma_j\to\tau_j).$$

So,

$$\begin{aligned}
\bigwedge_{i\in I}(\sigma_i \to \tau_i) &\preccurlyeq \bigwedge_{J\in K}\bigwedge_{j\in J}(\sigma_j \to \tau_j)\\
&\preccurlyeq \bigwedge_{J\in K}\bigwedge_{j\in J}(\bigwedge_{j\in J}\sigma_j \to \tau_j) \quad \text{by rule } \mathbf{Suf}_{\preccurlyeq}\\
&\preccurlyeq \bigwedge_{J\in K}(\bigwedge_{j\in J}\sigma_j \to \bigwedge_{j\in J}\tau_j) \quad \text{by rule } {\to}{\wedge}\mathbf{I}_{\preccurlyeq}\\
&\preccurlyeq \bigwedge_{J\in K}(\bigwedge_{j\in J}\sigma_j \to \tau) \quad \text{by rule } \mathbf{Pre}_{\preccurlyeq} \text{ because } \forall J\in K,\ \bigwedge_{j\in J}\tau_j \preccurlyeq \tau\\
&\preccurlyeq \bigvee_{J\in K}(\bigwedge_{j\in J}\sigma_j) \to \tau \quad \text{by rule } {\to}{\vee}\mathbf{E}_{\preccurlyeq}\\
&\preccurlyeq \bigwedge_{I'\in H}(\bigvee_{i\in I'}\sigma_i) \to \tau \quad \text{by Lemma 6.5}\\
&\preccurlyeq \sigma \to \tau \quad \text{by rule } \mathbf{Suf}_{\preccurlyeq} \text{ because } \forall I'\in H,\ \sigma \preccurlyeq \bigvee_{i\in I'}\sigma_i
\end{aligned}$$

$\square$

Then we get immediately:

Theorem 6.7 *If a formula of the language $\mathcal{L}$ is a theorem in $\mathbf{T}_{[\![\leq]\!]}$, then it is a valid formula in $\mathbf{B}_+$, i.e.:*

$$\vdash_{\mathbf{T}} \sigma \;\Rightarrow \vdash_{\mathbf{B}+} \sigma$$

7 Reversibility of the M-rule

In this section, we will prove the reversibility of the **M-rule** in the theory $\mathbf{T}_{[\![\leq]\!]}$, i.e., the implication $\mathfrak{A} \Rightarrow \mathfrak{F}$. To do this, we will use the equivalence between $\mathbf{B}_+$ and $\mathbf{T}_{[\![\leq]\!]}$ and will reason in $\mathbf{B}_+$ using a classical stratification approach.

7.1 *Stratification: disjunctive and conjunctive normal forms in* $\mathbf{B}_+$

To be able to prove properties of the system $\mathbf{B}_+$, it is useful to introduce – as in classical logics – conjunctive and disjunctive normal forms, along with specialized inference rules for them; we give the main definitions and results, following [2].

Definition 7.1 (Stratified and normal forms) Stratified forms, among them conjunctive normal forms $\sigma^{\wedge\vee}$ and disjunctive normal forms $\sigma^{\vee\wedge}$, are $\mathbf{B}_+$ formulas specified by the following simple grammar, where α is – as before – an atomic formula:

$$\sigma^{\to}, \tau^{\to} ::= \alpha \mid \sigma^{\wedge} \to \tau^{\vee}$$

$$\sigma^{\wedge}, \tau^{\wedge} ::= \sigma^{\to} \mid \sigma^{\wedge} \wedge \tau^{\wedge} \qquad\qquad \sigma^{\wedge\vee}, \tau^{\wedge\vee} ::= \sigma^{\vee} \mid \sigma^{\wedge\vee} \wedge \tau^{\wedge\vee}$$

$$\sigma^{\vee}, \tau^{\vee} ::= \sigma^{\to} \mid \sigma^{\vee} \vee \tau^{\vee} \qquad\qquad \sigma^{\vee\wedge}, \tau^{\vee\wedge} ::= \sigma^{\wedge} \mid \sigma^{\vee\wedge} \vee \tau^{\vee\wedge}$$

We will also denote by $\mathcal{L}^\star$, with $\star = \to, \wedge, \vee, \wedge\vee, \vee\wedge$, the respective sets of stratified formulae $\sigma^\star$, but we will usually write a formula in $\mathcal{L}^\star$ simply as $\sigma^\star$.

Transformations from arbitrary formulae into their conjunctive or disjunctive normal forms are also defined as expected.

Definition 7.2 The maps $\mathbf{m}_{\wedge\vee} : \mathcal{L} \to \mathcal{L}^{\wedge\vee}$ and $\mathbf{m}_{\vee\wedge} : \mathcal{L} \to \mathcal{L}^{\vee\wedge}$ are defined by simultaneous induction on the structure of formulae:

(i) $\mathbf{m}_{\wedge\vee}(\alpha) = \mathbf{m}_{\vee\wedge}(\alpha) = \alpha$

(ii) $\left.\begin{array}{l}\mathbf{m}_{\wedge\vee}(\sigma \to \tau)\\ \mathbf{m}_{\vee\wedge}(\sigma \to \tau)\end{array}\right\} = \bigwedge_{i\in I} \bigwedge_{j\in J} (\sigma_i^\wedge \to \tau_j^\vee)$
if $\mathbf{m}_{\vee\wedge}(\sigma) = \bigvee_{i\in I} \sigma_i^\wedge$ and $\mathbf{m}_{\wedge\vee}(\tau) = \bigwedge_{j\in J} \tau_j^\vee$

(iii) $\mathbf{m}_{\wedge\vee}(\sigma \wedge \tau) = \mathbf{m}_{\wedge\vee}(\sigma) \wedge \mathbf{m}_{\wedge\vee}(\tau)$

(iv) $\mathbf{m}_{\vee\wedge}(\sigma \wedge \tau) = \bigvee_{i\in I} \bigvee_{j\in J} (\sigma_i^\wedge \wedge \tau_j^\wedge)$
if $\mathbf{m}_{\vee\wedge}(\sigma) = \bigvee_{i\in I} \sigma_i^\wedge$ and $\mathbf{m}_{\vee\wedge}(\tau) = \bigvee_{j\in J} \tau_j^\wedge$

(v) $\mathbf{m}_{\vee\wedge}(\sigma \vee \tau) = \mathbf{m}_{\vee\wedge}(\sigma) \vee \mathbf{m}_{\vee\wedge}(\tau)$

(vi) $\mathbf{m}_{\wedge\vee}(\sigma \vee \tau) = \bigwedge_{i\in I} \bigwedge_{j\in J} (\sigma_i^\vee \vee \tau_j^\vee)$
if $\mathbf{m}_{\wedge\vee}(\sigma) = \bigwedge_{i\in I} \sigma_i^\vee$ and $\mathbf{m}_{\wedge\vee}(\tau) = \bigwedge_{j\in J} \tau_j^\vee$

We then define specialized inference rules for the entailment between stratified formulae.

Definition 7.3 (Entailment on stratified forms)
The relations $\preccurlyeq_\star \subseteq \mathcal{L}^\star \times \mathcal{L}^\star$ ($\star = \to, \wedge, \vee, \wedge\vee, \vee\wedge$) are the least preorders such that:

$$\alpha \preccurlyeq_\to \alpha$$

$$\sigma^\wedge \to \sigma^\vee \preccurlyeq_\to \tau^\wedge \to \tau^\vee \quad \text{iff} \quad \tau^\wedge \preccurlyeq_\wedge \sigma^\wedge \text{ and } \sigma^\vee \preccurlyeq_\vee \tau^\vee$$

$$\textstyle\bigwedge_{i\in I} \sigma_i^\to \preccurlyeq_\wedge \bigwedge_{j\in J} \tau_j^\to \quad \text{iff} \quad \forall j \in J \,.\, \exists i \in I \,.\, \sigma_i^\to \preccurlyeq_\to \tau_j^\to$$

$$\textstyle\bigvee_{i\in I} \sigma_i^\to \preccurlyeq_\vee \bigvee_{j\in J} \tau_j^\to \quad \text{iff} \quad \forall i \in I \,.\, \exists j \in J \,.\, \sigma_i^\to \preccurlyeq_\to \tau_j^\to$$

$$\textstyle\bigwedge_{i\in I} \sigma_i^\vee \preccurlyeq_{\wedge\vee} \bigwedge_{j\in J} \tau_j^\vee \quad \text{iff} \quad \forall j \in J \,.\, \exists i \in I \,.\, \sigma_i^\vee \preccurlyeq_\vee \tau_j^\vee$$

$$\textstyle\bigvee_{i\in I} \sigma_i^\wedge \preccurlyeq_{\vee\wedge} \bigvee_{j\in J} \tau_j^\wedge \quad \text{iff} \quad \forall i \in I \,.\, \exists j \in J \,.\, \sigma_i^\wedge \preccurlyeq_\wedge \tau_j^\wedge$$

Lemma 7.4 $\preccurlyeq_\star$ ($\star = \to, \wedge, \vee, \wedge\vee, \vee\wedge$) *are reflexive and transitive.*

Proof. By induction on the definition of $\preccurlyeq_\star$. □

The following proposition states that conjunctive and disjunctive normal forms are logically equivalent to their counterimages under $\mathbf{m}_{\vee\wedge}()$ and $\mathbf{m}_{\wedge\vee}()$, and that the specialized relations $\preccurlyeq_\star$ are restrictions of $\preccurlyeq$ to the respective sets $\mathcal{L}^\star$.

Proposition 7.5 *For all $\sigma, \tau \in \mathcal{L}$:*

(i) $\sigma \sim \mathbf{m}_{\vee\wedge}(\sigma) \sim \mathbf{m}_{\wedge\vee}(\sigma)$.

(ii) $\sigma^\star \preccurlyeq_\star \tau^\star \Rightarrow \sigma^\star \preccurlyeq \tau^\star$ *for* $\star = \to, \wedge, \vee, \vee\wedge, \wedge\vee$.

(iii) $\sigma \preccurlyeq \tau \iff \mathbf{m}_{\wedge\vee}(\sigma) \preccurlyeq_{\wedge\vee} \mathbf{m}_{\wedge\vee}(\tau) \iff \mathbf{m}_{\vee\wedge}(\sigma) \preccurlyeq_{\vee\wedge} \mathbf{m}_{\vee\wedge}(\tau)$.

Proof.

(i) By induction on σ. E.g., if $\sigma = \tau \to \rho$ then, by induction hypothesis, we have $\tau \sim \mathbf{m}_{\vee\wedge}(\tau) = \bigvee_{i\in I} \tau_i^\wedge$ and $\rho \sim \mathbf{m}_{\wedge\vee}(\rho) = \bigwedge_{j\in J} \rho_j^\vee$, so that, by repeated use of the rules $\to\wedge\mathbf{I}_\preccurlyeq$, $\to\vee\mathbf{E}_\preccurlyeq$, $\mathbf{Pre}_\preccurlyeq$ and $\mathbf{Suf}_\preccurlyeq$, we conclude that

$$\tau \to \rho \sim \bigvee_{i\in I} \tau_i^\wedge \to \bigwedge_{j\in J} \rho_j^\vee \sim \bigwedge_{i\in I}\bigwedge_{j\in J}(\tau_i^\wedge \to \rho_j^\vee) \sim \mathbf{m}_{\wedge\vee}(\tau \to \rho) = \mathbf{m}_{\vee\wedge}(\tau \to \rho)$$

(ii) Straightforward, by induction on the definition of $\preccurlyeq_\star$.

(iii) Implications ($\Leftarrow$) are immediate consequences of (i) and (ii). To prove ($\Rightarrow$) we use induction over $\preccurlyeq$. All cases are simple computations.
For example, the case ($\to\vee\mathbf{E}_\preccurlyeq$): $(\sigma \to \rho) \wedge (\tau \to \rho) \preccurlyeq \sigma \vee \tau \to \rho$.
Let $\mathbf{m}_{\vee\wedge}(\sigma) = \bigvee_{i\in I} \sigma_i^\wedge$, $\mathbf{m}_{\vee\wedge}(\tau) = \bigvee_{j\in J} \tau_j^\wedge$ and $\mathbf{m}_{\wedge\vee}(\rho) = \bigwedge_{k\in K} \rho_k^\vee$. Therefore $\mathbf{m}_{\vee\wedge}(\sigma \vee \tau) = \mathbf{m}_{\vee\wedge}(\sigma) \vee \mathbf{m}_{\vee\wedge}(\tau) = (\bigvee_{i\in I} \sigma_i^\wedge) \vee (\bigvee_{j\in J} \tau_j^\wedge)$. Then we have

$$\begin{aligned}
\mathbf{m}_{\wedge\vee}((\sigma \to \rho) \wedge (\tau \to \rho)) &= \mathbf{m}_{\wedge\vee}(\sigma \to \rho) \wedge \mathbf{m}_{\wedge\vee}(\tau \to \rho) \\
&= \bigwedge_{i\in I}\bigwedge_{k\in K}(\sigma_i^\wedge \to \rho_k^\vee) \wedge \bigwedge_{j\in J}\bigwedge_{k\in K}(\tau_j^\wedge \to \rho_k^\vee) \\
\mathbf{m}_{\wedge\vee}(\sigma \wedge \tau \to \rho) &= \bigwedge_{i\in I}\bigwedge_{k\in K}(\sigma_i^\wedge \to \rho_k^\vee) \wedge \bigwedge_{j\in J}\bigwedge_{k\in K}(\tau_j^\wedge \to \rho_k^\vee).
\end{aligned}$$

□

Remark 7.6 The converse of Proposition 7.5 (ii) is false, an example is just the axiom $\to\vee\mathbf{E}_\preccurlyeq$.

7.2 Strong disjunction lemma for arrows

Our first step to prove the translation of $\mathfrak{A} \Rightarrow \mathfrak{F}$ in $\mathbf{B}_+$ is to get rid of the $(\exists j)$ in $\mathfrak{F}$, through the following lemma:

Lemma 7.7 *The entailment $\bigwedge_{i\in I}(\sigma_i \to \tau_i) \preccurlyeq \bigvee_{j\in J}(\sigma_j \to \tau_j)$ holds iff there exists a $j_0 \in J$ such that $\bigwedge_{i\in I}(\sigma_i \to \tau_i) \preccurlyeq \sigma_{j_0} \to \tau_{j_0}$ holds.*

Proof. The if-part is trivial, being $\vee$ the join in the lattice of the $\preccurlyeq$-order. For the only-if-part, first observe that the disjunctive-conjunctive normal form of an arrow type never is a disjunction. Let

$$\mathbf{m}_{\vee\wedge}(\bigwedge_{i\in I}(\sigma_i \to \tau_i)) = \bigwedge_{l\in L} \phi_l^{\to} \qquad \mathbf{m}_{\vee\wedge}(\bigvee_{j\in J}(\sigma_j \to \tau_j)) = \bigvee_{j\in J} \psi_j^{\wedge}$$

where, by Definition 7.2, $\psi_j^{\wedge} = \mathbf{m}_{\vee\wedge}(\sigma_j \to \tau_j)$; then one has:

$$\begin{aligned}
\bigwedge_{i\in I}(\sigma_i \to \tau_i) \preccurlyeq \bigvee_{j\in J}(\sigma_j \to \tau_j) &\Rightarrow (7.5\text{ iii}) \quad \bigwedge_{l\in L} \phi_l^{\to} \preccurlyeq_{\vee\wedge} \bigvee_{j\in J} \psi_j^{\wedge} \\
&\Rightarrow (\text{Def. } 7.3)\ \exists j_0 \in J \,.\, \bigwedge_{l\in L} \phi_l^{\to} \preccurlyeq_{\wedge} \psi_{j_0}^{\wedge} \\
&\Rightarrow \qquad \bigwedge_{i\in I}(\sigma_i \to \tau_i) \preccurlyeq (\sigma_{j_0} \to \tau_{j_0})
\end{aligned}$$

□

7.3 Proof core

Now, using the notation of Section 6, we have to prove that $\mathcal{A} \Rightarrow (\mathcal{B}\ \mathit{and}\ \mathcal{C})$.

Lemma 7.8 ($\mathcal{A} \Rightarrow (\mathcal{B}$ *and* $\mathcal{C})$) *In the logic* $\mathbf{B}_+$ *the following property holds:*

$$\bigwedge_{i\in I}(\sigma_i \to \tau_i) \preccurlyeq \sigma \to \tau \quad \Rightarrow \begin{cases} \sigma \preccurlyeq \bigvee_{i\in I} \sigma_i \quad \mathit{and} \\ \forall I' \subsetneq I \,.\, (\sigma \preccurlyeq \bigvee_{i\in I'} \sigma_i) \ \ \mathit{or}\ \ (\bigwedge_{j\in I\setminus I'} \tau_j \preccurlyeq \tau) \end{cases}$$

Proof. Let

(i) $\mathbf{m}_{\vee\wedge}(\sigma_i) = \bigvee_{l\in L_i} \phi_l^{(i)} \quad (\phi_l^{(i)} \in \mathcal{L}^{\wedge})$ (ii) $\mathbf{m}_{\wedge\vee}(\tau_i) = \bigwedge_{p\in P_i} \psi_p^{(i)} \quad (\psi_p^{(i)} \in \mathcal{L}^{\vee})$

(iii) $\mathbf{m}_{\vee\wedge}(\sigma) = \bigvee_{m\in M} \chi_m \quad (\chi_m \in \mathcal{L}^{\wedge})$ (iv) $\mathbf{m}_{\wedge\vee}(\tau) = \bigwedge_{q\in Q} \theta_q \quad (\theta_q \in \mathcal{L}^{\vee})$

Hence, by the definition of $\mathbf{m}_{\wedge\vee}$:

$$\begin{aligned}
\mathbf{m}_{\wedge\vee}(\bigwedge_{i\in I}(\sigma_i \to \tau_i)) &= \bigwedge_{i\in I} \mathbf{m}_{\wedge\vee}(\sigma_i \to \tau_i) = \bigwedge_{i\in I} \bigwedge_{l\in L_i} \bigwedge_{p\in P_i} (\phi_l^{(i)} \to \psi_p^{(i)}) \\
\mathbf{m}_{\wedge\vee}(\sigma \to \tau) &= \bigwedge_{m\in M} \bigwedge_{q\in Q} (\chi_m \to \theta_q)
\end{aligned}$$

The lemma assumption $\mathcal{A}$ then becomes:

$$\bigwedge_{i\in I} \bigwedge_{l\in L_i} \bigwedge_{p\in P_i} (\phi_l^{(i)} \to \psi_p^{(i)}) \preccurlyeq_{\wedge\vee} \bigwedge_{m\in M} \bigwedge_{q\in Q} (\chi_m \to \theta_q)$$

By the Definition 7.3 of entailment on stratified forms one derives:

$$\forall m \in M \,.\, \forall q \in Q \,.\, \exists i \in I \,.\, \exists l \in L_i \,.\, \exists p \in P_i \,.\, \chi_m \preccurlyeq_{\wedge} \phi_l^{(i)} \ \mathit{and}\ \psi_p^{(i)} \preccurlyeq_{\vee} \theta_q \quad (6)$$

From (6) we get

$$\bigvee_{m\in M} \chi_m \preccurlyeq \bigvee_{i\in I} \bigvee_{l\in L_i} \phi_l^{(i)}$$

i.e.,

$$\sigma \preccurlyeq \bigvee_{i\in I} \sigma_i$$

which is the condition $\mathcal{B}$. To prove the condition $\mathcal{C}$, we assume $I' \subsetneq I$ and $\sigma \not\preccurlyeq \bigvee_{i\in I'} \sigma_i$, and we show $\bigwedge_{j\in I\setminus I'} \tau_j \preccurlyeq \tau$. From (ii) and (i) above, we obtain: $\bigvee_{m\in M} \chi_m \not\preccurlyeq_{\vee\wedge} \bigvee_{i\in I'} \bigvee_{l\in L_i} \phi_l^{(i)}$. Then, by stratified entailment:

$$\begin{array}{rl} & \exists m \in M \,.\, \forall i \in I' \,.\, \forall l \in L_i \,.\, \chi_m \not\preccurlyeq_{\wedge} \phi_l^{(i)} \\ & \Rightarrow \exists m \in M \,.\, (i \in I' \; and \; l \in L_i \Rightarrow \chi_m \not\preccurlyeq_{\wedge} \phi_l^{(i)}) \\ \text{by contraposition} & \Rightarrow \exists m \in M \,.\, (l \in L_i \; and \; \chi_m \preccurlyeq_{\wedge} \phi_l^{(i)} \Rightarrow i \notin I') \\ & \Rightarrow \exists m \in M \,.\, (l \in L_i \; and \; \chi_m \preccurlyeq_{\wedge} \phi_l^{(i)} \Rightarrow i \in I\setminus I') \end{array}$$

Applying the last line to the statement (6) above, one obtains:

$$\forall q \in Q \,.\, \exists i \in I\setminus I' \,.\, \exists p \in P_i \,.\, \psi_p^{(i)} \preccurlyeq_{\vee} \theta_q$$

i.e., for every θ_q there is a $\psi_p^{(i)}$ that entails it; then, by definition of $\preccurlyeq_{\wedge\vee}$:

$$\bigwedge_{i\in I\setminus I'} \bigwedge_{p\in P_i} \psi_p^{(i)} \preccurlyeq_{\wedge\vee} \bigwedge_{q\in Q} \theta_q$$

whence finally, by (iii) and (iv) above, $\bigwedge_{j\in I\setminus I'} \tau_j \preccurlyeq \tau$. $\square$

8 Conclusion and future work

The equivalence between the minimal relevant logic and the minimal functional core of semantic subtyping is clearly connected with a common basic aspect of the two systems.

On the one hand relevant logic, as shown in [14], [8], [9] and recently further developed in [6], corresponds, in a Curry-Howard-like analogy, to the natural type system for λ-calculus with intersection, union and subtyping. This is reflected by the particular functional character of its Kripke semantics, where a notion of application between two worlds naturally arises, in addition to the usual notion of application between terms (i.e., proofs).

On the other hand, the set-theoretic characterization of functional types in [11] is basically the same as the analogous definition in λ-calculus type theories with intersection and arrow only, which has been proved by Coppo et al. in [7] to exactly characterize the set of continuous functions in filter models.

Natural directions for further work are therefore both the study of whether analogous characterizations w.r.t. filter models hold for the functional type system of semantic subtyping (i.e., in the presence of the union operation), and at the same time a further and more explicative analysis of the fundamental correspondence between relevant logic systems and set-based type systems.

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