Rank 2 Types for Term Graph Rewriting

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Abstract

We define a notion of type assignment with polymorphic intersection types of rank 2 for a term graph rewriting language that expresses sharing and cycles. We show that type assignment is decidable through defining, using the extended notion of unification from [5], a notion of principal pair which generalizes ML's principal type property.

Introduction

This paper presents a decidable notion of type assignment systems for a term-graph rewriting language that uses polymorphic types of rank 2, so allows for more than just the standard *shallow* polymorphism. In order to obtain principal typings, intersection types of rank 2 are added to the system.

In the past, many notions of type assignment have been studied for (functional) programming languages, all based on (extensions of) the Hindley-Milner type assignment system [27, 37]. Moreover, almost all notions of type assignment as proposed for use in functional programming, in reality are developed on (enriched) lambda calculi, and little work is available that discusses and studies types directly *on the level of the programming language*. However, to be able to study the role of types in practice, it is arguably important that type assignment is formally defined as close to the actual language as possible.

Furthermore, many aspects of those languages are not easily dealt with in the Lambda Calculus (LC) [8], or not expressible at all, like patterns, sharing, and cyclic structures. This motivated the investigation of type assignment for Term Rewriting Systems (TRS) [35] and Term Graph Rewriting Systems (TGRS) [10] presented in various papers [7, 6, 4, 13, 5], and the system presented in this paper is developed in much the same way as the systems studied there. As an example, take the problem of I/O in the context of functional programming: only when representing terms as graphs to express the *sharing* that is heavily used at run-time does it become possible to represent the number of different references to an object accurately; only when the reference is *unique* (see [13] for a discussion of *uniqueness types*; note that we do not consider a notion of uniquess typing here) is it possible to do a *destructive update*.

Although functional programming languages are normally viewed as enriched lambda calculi, studying properties on that level ignores features of the language, like patterns. The subject reduction property in particular, is lost in general in the context of patterns, a problem successfully dealt with in [7], where a notion of type assignment for TRS was developed. This system was later extended into type assignment systems for different calculi [4, 13, 6, 5]. The papers [7, 6, 4] studied type assignment for Curryfied TRS ($\mathcal{C}$TRS), a notion of first order TRS extended with application. A difference between $\mathcal{C}$TRS and @TGRS as considered (and defined) here, is that, for reasons of simplicity of definitions, we will consider (higher order) applicative systems only, with higher order variables and function symbols, rather than the first-order $\mathcal{C}$TRS.

The extension from LC to TRS is done via combinator systems; then term rewrite rules are written very much like the definitions of combinators, with the difference that a formal parameter can have structure and be a *pattern*: it need not be a term-variable. TGRS are obtained from TRS by lifting terms to graphs. This lifting consists of writing terms as trees and of sharing variables that occur more than

once in the term that is lifted. Term graph rewrite rules are obtained from term rewrite rules in very much the same way: the left- and right hand side terms of every term rewrite rule are lifted to term graphs, and the nodes that represent variables occurring in both terms are shared. If a variable appears in both the left and right hand side, this operation will generate a connected graph. Of course it is also possible to define term graphs and term graph rewrite rules directly, without first taking a term or term rewrite rule and lifting it; see [10, 11, 12, 24, 30, 42] for discussions of term graph rewriting and its suitability for programming.

The main point of focus for [6, 5] was *normalisation*, which motivated the choice to use intersections types [9]. This implied, however, that type assignment for those systems is undecidable. It is by now well-known that there are decidable restrictions of the intersection type assignment system [21, 34, 28, 4, 29, 22, 20, 31, 32], making the definition of notions of type assignment using those types feasible. In particular, in [4] a notion of type assignment for TRS was presented that uses intersection types of rank 2.

Another direction in the area of types is that of *quantified* or *polymorphic* types. This field originated in the context of LC with System F [26, 39], which provides a general notion of polymorphism, but lacks principal typings. Moreover, type inference in System F is undecidable in general [44], although it is decidable for some sub-systems, in particular if we consider types of rank 2 [33]. The type system of ML [19] uses (shallow) polymorphic types and has principal types. Since its polymorphism is limited, some programs that arise naturally cannot be typed, and it does *not* have principal *typings* [29], a property that is important for separate compilation, incremental type inference, and accurate type error messages.

Intersection type systems are somewhere in the middle with respect to polymorphism, and have principal typings.

The system of [4] was in [5] extended to a system for a combination of LC and ($\mathcal{U}$TRS) by adding '$\forall$' as an extra type-constructor (i.e. explicit polymorphism). Although the Rank 2 intersection system and the Rank 2 polymorphic system for LC type exactly the same set of terms [45], their combination results in a system with more expressive power: the set of assignable types increases, and types can better express the behaviour of terms [15]. Also, polymorphism can be expressed directly (using the universal quantifier) and, moreover, every typeable expression in [5] has a principal typing. This principal typing property does not hold in a system without intersection.

The decidability of a notion of unification on polymorphic intersection types of rank 2 as shown in [5] could be used in many different contexts. Since intersection types are the natural tool to type nodes that are shared in a notion of type assignment on graphs, in this paper, we adapt the notion of type assignment of [5] to one for (a kind of) TGRS. (Intersection types also provide a good formalism to express overloading.) We will show that the notion of type assignment as presented here has the principal typing property.

We will study type assignment on a class of graphs that can be defined via an abstract syntax definition, which makes an inductive approach to type assingment possible. Graphs will be written as terms, and type assignment will be treated on the level of terms. A first treatment of types for graph rewriting systems that uses this approach can be found in [13], which itself is based on the approach of [7] as far as the definition of type assignment is concerned. A draw-back of that system is that it uses the standard Curry types to type graphs, so that the types assignable to a graph are fewer than those assignable to the corresponding tree (obtained by unraveling the graph), since there a node shared in the graph would appear as two separate nodes, that can be typed with different types. Using intersection types, the concept of sharing in graphs causes no difficulties, since a shared node can now be typed with more than one type.

The only problem arises when the graph is allowed to have a cyclic structure, which causes the unraveling to generate an infinite tree. Then it is possible that the (infinite number of) copies of a node are all typed with different types, thus creating an intersection over an infinite number of types for the type assignment to the term graph. The solution for this problem used in this paper is to type a cyclic

node with *one* Curry type only, similar to the standard way of dealing with recursion.

In our Rank 2 system each typeable term has a principal typing; this is the case also in the Rank 2 *intersection* system of [4], but not in the Rank 2 *polymorphic* system of [33]. For the latter, a type inference algorithm of the same complexity of that of ML was given in [34], where the problems that occur due to the lack of principal types are discussed in detail. Our Rank 2 system (without the share and the cycle) generalizes also Jim's system P_2 [29], which is a combination of ML-types and Rank 2 intersection types. Having Rank 2 quantified types in the system allows us to type, for instance, the constant `runST` used in [36], which cannot be typed in P_2. Our system also generalises the system of [20] that combines rank 2 intersection types and shallow polymorphism, so does not have polymorphic types of rank 2.

The Rank 2 system as used in this paper can be seen as a combination of the systems of [4] and [33]. In [5] an incomplete notion of polymorphic intersection type assignment was presented for a language that is a combination of LC and CTRS; it contains a definition of a Rank 2 system for that combined calculus, and it claimed to show that type assignment in that system is decidable and has principal types; since there were some major flaws to definitions and proofs in that paper, a new correct presentation is necessary. This paper corrects those definitions and extends those result to a calculus with sharing and cycles, by defining a notion of Rank 2 type assignment on @TGRS, inspired by the system that was studied in [5].

This paper is organised as follows: in Section 1 we define term graph rewriting extended with application (@TGRS), and in Section 2 polymorphic intersection types of rank 2, and a notion of Rank 2 type assignment for @TGRS in Section 4. In Section 3, we will define four operations on types, needed in the notion of type assignment, that we will show to be sound in Section 5. We will conclude by presenting the notion of unification from [5] in Section 6, which will be used in Section 7 to show that all typeable terms have a principal typing. Finally, Section 8 contains some concluding remarks regarding implementation aspects and overloading.

We assume the reader to be familiar with LC [8], refer to [35, 23] for rewrite systems, and to [12, 10, 11, 30, 38, 42] for definitions of TGRS. The system defined here is aimed to be similar to those, although their relation is not studied here.

We will use a vector notation $\overrightarrow{g}$ as an abbreviation for $g_1, \ldots, g_n$, so such that $\overrightarrow{\langle x_i = t_i \rangle}$ stands for $\langle x_1 = t_1 \rangle, \ldots, \langle x_n = t_n \rangle$, and $\overrightarrow{x_i \mapsto r_i}$ for $x_1 \mapsto r_1, \ldots, x_n \mapsto r_n$, etc.

1 Applicative Term Graph Rewriting Systems

In this section, we will present a notion of *Applicative* Term Graph Rewriting (@TGRS) based on an inductive definition of graphs, following essentially a similar system presented in [13]. Term Graph Rewriting distinguishes itself from Term Rewriting in that the objects considered are no longer trees, but allow sharing and cycles; it is different from Generalised Graph Rewriting in that only those rewrites are allowed that can, essentially, be formulated through a term rewrite rule.

Definition 1.1 (TERMS) *i*) An *alphabet* or *signature* Σ consists of a countable, infinite set $\mathcal{X}$ of variables $x, y, z, \ldots$, a non-empty set $\mathcal{F}$ of *function symbols* $\mathsf{F}, \mathsf{G}, \ldots$, each with a fixed arity *arity*(F), and a special binary operator, called *application* (@, written in in-fix notation).

ii) The set $T(\mathcal{F}, \mathcal{X})$ of *terms*, ranged over by t, is defined by:

$$t ::= x \mid \mathsf{F} \mid (t_1 \mathbin{@} t_2) \mid (\mathsf{share}\, t_1\, \mathsf{via}\, x\, \mathsf{in}\, t_2) \mid (\mathsf{cycle}\, \langle \overrightarrow{x_i = t_i} \rangle\, \mathsf{in}\, t)$$

Rather than writing $(t_1 \mathbin{@} t_2)$, we will write $(t_1\, t_2)$; as usual, obsolete brackets will be omitted.

A thing to observe is that function symbols come with an arity, which is relevant when defining rewrite rules (Def. 1.5), and comes into play when translating a 'program' into a graph rewriting system; for details of such a translation, see [13] and below (Def. 1.5 (*ii*)).

Mainly for readability of proofs, the language of terms we study here differs from the one defined in [13], where *expressions* were defined by:

$$\begin{aligned} E &::= x \mid (\mathbf{F}\,(E_1,\ldots,E_n)) \mid (\mathsf{let}\,x = E_1\,\mathsf{in}\,E_2) \mid \\ &\qquad (\mathsf{letrec}\,\overrightarrow{x = E_1}\,\mathsf{in}\,E_2) \mid (\mathsf{case}\,E\,\mathsf{of}\,\overrightarrow{P}|\overrightarrow{E}) \\ P &::= \mathbf{C}\,(x_1,\ldots,x_n) \end{aligned}$$

Notice that, in Def. 1.1, we do not distinguish between function and constructor symbols, so we do not require a separate treatment of patterns; also, we deal with an *applicative* language; in [13], the language is first order, and application is considered in Section 4 of that paper only to allow higher order functions by partial application of function symbols, essentially by extending the set of rewrite rules via

$$(\mathbf{F}\,(t_1,\ldots,t_n)@t_{n+1}) \;=\; \mathbf{F}\,(t_1,\ldots,t_n,t_{n+1})$$

The distinction between that syntax and the one used here is cosmetic in that all results obtained here could be reached in a first-order system as that of [13]; it is the presentation of the results that benefits from an applicative syntax by giving less involved and shorter proofs. Using the keywords 'share' and 'cycle' rather than 'let' and 'letrec' serves to highlight the change in syntax and system.

Notice that the language of types (presented below) differs significantly from that considered in [13], in that, as far as assignable types are concerned, the systems are incompatible.

We will now formally introduce term graphs, as done in [10]. Following [13], graphs are written in an *equational style* [10, 1], rather than using drawings or 4-tuples (as in [10]).

Definition 1.2 (GRAPHS) [13, 24] A *graph* (over $\mathcal{F}$) is a pair $g = \langle r \mid G\rangle$, where r is a variable and stands for the *root* of the graph, and G is a set of equations of the shape '$x = @\,(y,z)$' or '$x = \mathbf{F}$', that describe the edges in the graph, where the variables that appear on the left appear there in only *one* equation and should all appear on the right as well.

The *variable set* of graph $g = \langle r \mid G\rangle$, *Var*$(g)$, is the collection of all variable names appearing in r, G. The set of *free* variables of g, *fv*(g), contains those variables that do not appear as the left-hand side of an equation in G, and a variable in *Var*(g) is *bound* if it is not free; we will identify graphs that differ only in the names of their bound variables.

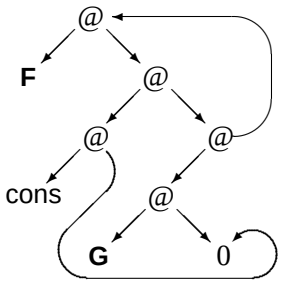
Definition 1.3 (GRAPH INTERPRETATION) (cf. [13]) For each term t, the *graph interpretation of* t, $[\![t]\!]$, is defined by ($[\overrightarrow{x_i \mapsto r_i}]$ stands for the simultaneous replacement of $\overrightarrow{r_i}$ for (the free occurrences of) $\overrightarrow{x_i}$, and different graphs are assumed to share no variable names):

$$\begin{aligned} [\![x]\!] &= \langle x \mid \emptyset\rangle \\ [\![\mathbf{F}]\!] &= \langle f \mid \{f = \mathbf{F}\}\rangle \\ [\![t_1\,t_2]\!] &= \langle r \mid \{r = @\,(r_1,r_2)\} \cup G_1 \cup G_2\rangle, \\ &\quad \text{where } [\![t_i]\!] = \langle r_i \mid G_i\rangle, i = 1,2, \text{ and } r \text{ is fresh} \\ [\![\mathsf{share}\,t_1\,\mathsf{via}\,x\,\mathsf{in}\,t_2]\!] &= \langle r_2 \mid G_1 \cup G_2\rangle\,[x \mapsto r_1], \\ &\quad \text{where } [\![t_i]\!] = \langle r_i \mid G_i\rangle, i = 1,2 \\ [\![\mathsf{cycle}\,\langle\,\overrightarrow{x_i = t_i}\,\rangle\,\mathsf{in}\,t']\!] &= \langle r' \mid G_1 \cup \cdots \cup G_n \cup G'\rangle\,[\overrightarrow{x_i \mapsto r_i}], \\ &\quad \text{where } [\![t_i]\!] = \langle r_i \mid G_i\rangle,\ (1 \le i \le n) \\ &\qquad\quad [\![t']\!] = \langle r' \mid G'\rangle, \end{aligned}$$

Via this interpretation, the notion of free and bound variables of a graph g induces a notion of free and bound variables on terms; as a result, in the term $(\mathsf{share}\,t_1\,\mathsf{via}\,x\,\mathsf{in}\,t_2)$, x does not occur free in t_1.

Example 1.4 (cf. [13]) The term $(\mathsf{share}\,0\,\mathsf{via}\,x\,\mathsf{in}\,(\mathsf{cycle}\,\langle\,z = \mathbf{F}\,(\mathsf{cons}\,x\,(\mathbf{G}\,x\,z))\,\rangle\,\mathsf{in}\,z))$ translates to the graph

$$\begin{aligned}\langle z \mid \{ z &= @(f,a),\\ f &= \mathsf{F},\\ a &= @(b,c),\\ b &= @(d,x),\\ c &= @(e,z),\\ d &= \mathsf{cons},\\ e &= @(g,x),\\ g &= \mathsf{G},\\ x &= 0\}\rangle\end{aligned}$$



Reduction on $T(\mathcal{F},\mathcal{X})$ is defined through rewrite rules.

Definition 1.5 (REWRITING RULES) *i*) A *rewrite rule* is a pair $(\mathit{left}, \mathit{right})$ of terms such that
- $\mathit{left} = \mathsf{F}\, t_1 \cdots t_n$, for some F with $n = \mathit{arity}(\mathsf{F})$, and terms $t_1, \ldots, t_n$, and
- $\mathit{fv}(\mathit{right}) \subseteq \mathit{fv}(\mathit{left})$.

Often, a rewrite rule will get a name, e.g. **r**, and we write $\mathit{left} \to_{\mathbf{r}} \mathit{right}$.

ii) The translation into graphs of Def. 1.3 is extended to rewrite rules through: Let $\mathit{left} \to \mathit{right}$ be a (recursive) rewrite rule with defined symbol F, then:

$$\begin{aligned}[\![\mathit{left} \to \mathit{right}]\!] &= \langle r_l \mid G_{\mathit{left}} \cup G_{\mathit{right}} \rangle\, [\overrightarrow{x_i \mapsto y_i}],\\ \text{where}\quad [\![\mathsf{F}]\!] &= \langle g \mid \{g = \mathsf{F}\}\rangle\\ [\![\mathit{left}]\!] &= \langle r_l \mid G_{\mathit{left}}\rangle\\ [\![\mathit{right}]\!] &= \langle r_r \mid G_{\mathit{right}}\rangle\\ \{x_1, \ldots, x_n\} &= \mathit{fv}(\mathit{left})\end{aligned}$$

and all $y_1, \ldots, y_n$ and g are unused variables.

We take the view that in a rewrite rule a certain symbol is defined.

Definition 1.6 (DEFINED SYMBOLS AND CONSTRUCTORS) In a rewrite rule $\mathsf{F}\, t_1 \cdots t_n \to_{\mathbf{r}} r$, F is called *the defined symbol* of **r**, and **r** is said to *define* F. F is *a defined symbol*, if there is a rewrite rule that defines F, and $\mathsf{Q} \in \mathcal{F}$ is called a *constructor* if Q is not a defined symbol.

Notice that, by the first condition of Def. 1.5, '@' cannot be a defined symbol.

We call a defined symbol F *recursive* if F occurs on a cycle in the dependency-graph, and call every rewrite rule that defines F *recursive*. All function symbols that occur on one cycle in the dependency-graph depend on each other and are, therefore, defined *simultaneously* and are called *mutually recursive*. Since it is always possible to introduce tuples into the language and solve the problem of mutual recursion using only recursive rules, we will assume that rules are *not* mutually recursive.

Definition 1.7 (TERM GRAPH REWRITING) The principle of term graph rewriting, presented formally in [10], can be summarised as follows:
- a graphs g contains a *redex* if a left-hand side *left* of a rewrite rule $\mathit{left} \to \mathit{right}$ can be mapped onto a graph, i.e. if there exists a homomorphism from *left* to the graph (MATCH), which respects the structure of graphs and maps free variables to graphs.
- Reduction (rewriting) of the redex then consists of adding an *instance* of *right* to the graph by adding the right hand side (graph) of the rewrite rule (BUILD), but by replacing an edge going into a free variable to one going into the image of the variable under the aforementioned homomorphism (LINK).
- All edges going into the image of the root of *left* are re-directed into the root of the added instance of *right* (RE-DIRECT).

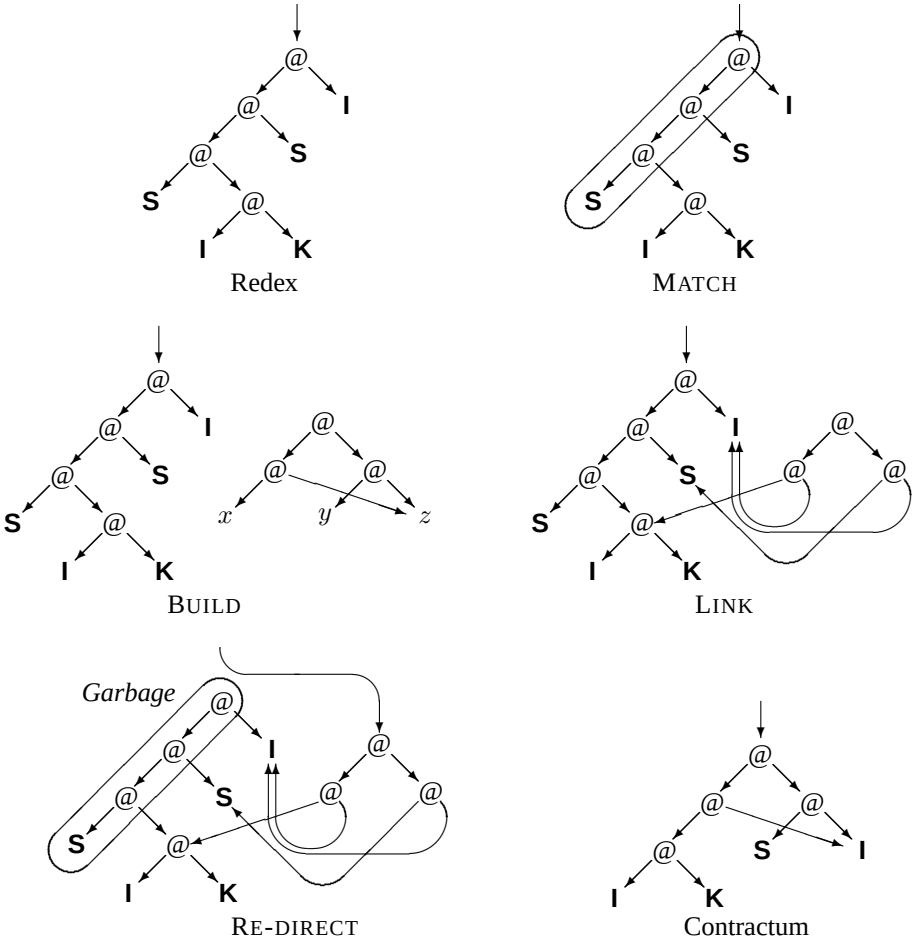


Figure 1: An example of term graph rewriting

- Now part of the graph has become *garbage*, in that it is no longer accessible from the root of g; this can be removed.

Example 1.8 As an example of term graph rewriting within the context of this paper, consider Fig. 1.

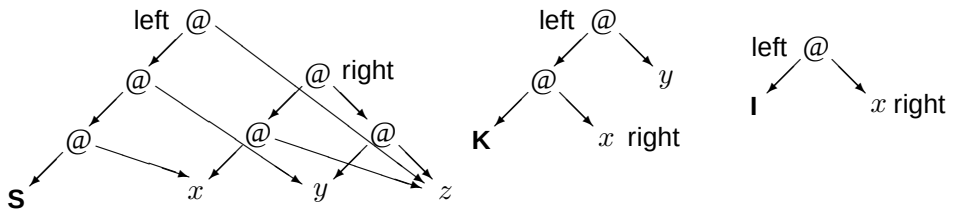
Definition 1.9 (REWRITING ON TERMS) We define a rewrite relation on terms by: $t_1 \to t_2$ if and only if there are graphs g_1 and g_2 such that $[\![t_1]\!] = g_1$, $[\![t_2]\!] = g_2$, and $g_1 \to g_2$.

Definition 1.10 An *Applicative Term Graph Rewriting System* (@TGRS) is defined as a pair $(\Sigma, \mathbf{R})$ of an alphabet Σ and a set $\mathbf{R}$ of rewrite rules.

Example 1.11 The rewrite rules that define Combinatory Logic are expressed as a @TGRS by:

$$\begin{array}{lll} \mathsf{S}\,x\,y\,z & \to & x\,z\,(y\,z) \\ \mathsf{K}\,x\,y & \to & x \\ \mathsf{I}\,x & \to & x \end{array}$$

Translated to term graph rewrite rules, these rules look like (using left and right rather than r_l and r_r):



Notice that, if we would have used '$\mathbf{S}\,x\,y\,z \to \mathsf{share}\,z\,\mathsf{via}\,v\,\mathsf{in}\,(x\,v)\,(y\,v)$' instead of the first rule, so would have expressed explicitly that we want the third parameter to be shared, the resulting graph rewrite rule would have been exactly the same.

Notice that the construct $(\mathsf{share}\,t_1\,\mathsf{via}\,x\,\mathsf{in}\,t_2)$ differs from the construct $(\mathsf{let}\,x = t_1\,\mathsf{in}\,t_2)$ as used in the ML language in that it is not a redex: in the rewriting relation in ML, $(\mathsf{let}\,x = t_1\,\mathsf{in}\,t_2) \to t_2[t_1/x]$, whereas here,

$$[\![\mathsf{share}\,t_1\,\mathsf{via}\,x\,\mathsf{in}\,t_2]\!] \;=\; \langle r_2 \mid G_1 \cup G_2\rangle\,[x \mapsto r_1] \;=\; [\![t_2[t_1/x]]\!],$$

where $[\![t_i]\!] = \langle r_i \mid G_i\rangle, i = 1, 2$. Of course the let-construct is normally implemented using sharing exactly as suggested by the share-construct, but that implies that then the language itself has changed from an extended lambda calculus to a language with lambda graphs, and the calculus no longer is the one presented in [37].

Since (free) variables in @TGRS may be substituted by function symbols, we obtain the usual functional programming paradigm, extended with definitions of operators and data structures. Notice, however, that we obtain more: in functional programs, the set $\mathcal{F}$ (Def. 1.1) is divided into *function symbols* and *(data-type) constructors*, and, in rewrite rules, function symbols are not allowed to appear in 'constructor position' and vice-versa. This does not hold for @TGRS.

Example 1.12 This example is an extension of one presented in [23]. It deals with the very well known definition of stacks of natural numbers and contains next to the operations *Top*, and *Pop*, the operation *Alternate* that combines two stacks, and shows the advantage of relaxing on the separation of function symbols and constructors.

The syntax of these representations is given by:

$$\begin{array}{lll} n & \in & \mathit{Nat} \\ n & ::= & 0 \mid \mathit{Succ}\,n \end{array} \qquad\qquad \begin{array}{lll} s & \in & \mathit{Stack} \\ s & ::= & \varepsilon \mid \mathit{Push}\,n\,s \end{array}$$

'*Push*' and 'ε' can be seen as stack constructors, and '0' and '*Succ*' can be seen as number constructors.

Semantics of the functions is given by the following rules:

$$\begin{array}{ll} \mathit{Top}\,(\mathit{Push}\,x\,y) & \to x \\ \mathit{Pop}\,(\mathit{Push}\,x\,y) & \to y \\ \mathit{Push}\,(\mathit{Top}\,x)\,(\mathit{Pop}\,x) & \to x \\ \mathit{Alternate}\,\varepsilon\,z & \to z \\ \mathit{Alternate}\,(\mathit{Push}\,x\,y)\,z & \to \mathit{Push}\,x\,(\mathit{Alternate}\,z\,y) \end{array}$$

With these rules, it can be shown, for example, that

$$\begin{array}{l} \mathit{Alternate}\;\;(\mathit{Push}\;(\mathit{Top}\,(\mathit{Push}\,0\,\varepsilon))\;(\mathit{Pop}\,(\mathit{Push}\,0\,\varepsilon))\;)\;(\mathit{Pop}\,(\mathit{Push}\,(\mathit{Succ}\,0)\,\varepsilon)) \\ \quad \to \mathit{Alternate}\;(\mathit{Push}\,0\,\varepsilon)\;(\mathit{Pop}\,(\mathit{Push}\,(\mathit{Succ}\,0)\,\varepsilon)) \\ \quad \to \mathit{Push}\,0\,(\mathit{Alternate}\,\varepsilon\,(\mathit{Pop}\,(\mathit{Push}\,(\mathit{Succ}\,0)\,\varepsilon))) \\ \quad \to \mathit{Push}\,0\,(\mathit{Pop}\,(\mathit{Push}\,(\mathit{Succ}\,0)\,\varepsilon)) \\ \quad \to \mathit{Push}\,0\,\varepsilon. \end{array}$$

2 Rank 2 types

In Section 4, we will present a decidable notion of type assignment on @TGRS, using polymorphic intersection types of rank 2. The system presented here is an extension, mainly by the '$\forall$' type constructor, of the Rank 2 system with intersection types as defined in [4].

We use strict intersection types over a set $V = \Phi \uplus \mathcal{A}$ of *free and bound type-variables* respectively, and a set S of *sorts* or *type constants*. For various reasons (definition of operations on types, definition of unification), we will distinguish syntactically between (names of) *free* type-variables (which belong to Φ) and (names of) *bound* type-variables (in $\mathcal{A}$).

Definition 2.1 (POLYMORPHIC INTERSECTION TYPES OF RANK 2) [5] We define types in layers: $\mathcal{T}_{\mathrm{C}}$ are Curry types, built out of type variables in Φ (ranged over by φ), sorts (type constants, ranged over by s) and '$\rightarrow$', $\mathcal{T}_{\mathrm{C}}^{\forall}$ are quantified Curry types, $\mathcal{T}_1$, the types of rank 1, are intersections of quantified Curry types, and $\mathcal{T}_2$ are types of Rank 2:

$$\begin{array}{llll} \mathcal{T}_{\mathrm{C}} & ::= \varphi \mid s \mid (\mathcal{T}_{\mathrm{C}} \rightarrow \mathcal{T}_{\mathrm{C}}) & \mathcal{T}_{\mathrm{C}}^{\forall} & ::= \mathcal{T}_{\mathrm{C}} \mid (\forall\alpha.\mathcal{T}_{\mathrm{C}}^{\forall}[\alpha/\varphi]) \\ \mathcal{T}_1 & ::= (\mathcal{T}_{\mathrm{C}}^{\forall} \cap \cdots \cap \mathcal{T}_{\mathrm{C}}^{\forall}) & \mathcal{T}_2 & ::= \varphi \mid s \mid (\mathcal{T}_1 \rightarrow \mathcal{T}_2) \end{array}$$

We use $\mathcal{T}_{\mathrm{R}}$ for the union of these sets, and use σ, τ for arbitrary elements of $\mathcal{T}_{\mathrm{R}}$. Notice that $\mathcal{T}_{\mathrm{C}} \subseteq \mathcal{T}_{\mathrm{C}}^{\forall} \subseteq \mathcal{T}_1$ and $\mathcal{T}_{\mathrm{C}} \subseteq \mathcal{T}_2$, but that $\mathcal{T}_{\mathrm{C}}^{\forall} \not\subseteq \mathcal{T}_2$.

In the notation of types, '$\rightarrow$' associates to the right, '$\cap$' binds stronger than '$\rightarrow$', which binds stronger than '$\forall$'; so $\rho\cap\mu\rightarrow(\forall\alpha.\gamma\rightarrow\delta)\rightarrow\sigma$ stands for $((\rho\cap\mu)\rightarrow((\forall\alpha.(\gamma\rightarrow\delta))\rightarrow\sigma))$. Also, $\forall\vec{\alpha}.\sigma$ is used for $\forall\alpha_1.\forall\alpha_2\ldots\forall\alpha_n.\sigma$, and we assume that each variable is bound at most once in a type (renaming if necessary). In the meta-language, we denote by $\sigma[\tau/\varphi]$ (resp. $\sigma[\tau/\alpha]$) the substitution of the type-variable φ (resp. α) by τ in σ.

Definition 2.2 (FREE AND BOUND TYPE-VARIABLES) $\mathit{fv}(\sigma)$, the set of *free variables* of a type σ is defined as usual (note that by construction, $\mathit{fv}(\sigma) \subseteq \Phi$). A type is called *closed* if it contains no free variables, and *ground* if it contains no variables at all. A type-variable will be *bound* if it is not free, and will then be an element of $\mathcal{A}$.

Below, we will define a unification procedure that will recursively go through types. However, using the sets defined above, not every sub-type of a type in $\mathcal{T}_2$ is a type in that same set. For example, $\alpha\rightarrow\varphi$ is not a type in any of the sets defined above; however, $\forall\alpha.\alpha\rightarrow\varphi \in \mathcal{T}_{\mathrm{C}}^{\forall}$, and therefore it can be that, when going through types in $\mathcal{T}_2$ recursively, $\alpha\rightarrow\varphi$ has to be dealt with. The distinction between free and bound variables is essential; for that reason we have introduced a different notation of both classes. We could therefore, for every set $\mathcal{T}_i$ defined above, also have defined the set $\mathcal{T}_i'$ of types, that contains also free occurrences of αs. We will not always use the '$'$' when speaking of these sets, however; it will be clear from the context which set is intended.

Definition 2.3 (RELATIONS ON TYPES) [5] On $\mathcal{T}_{\mathrm{R}}$, the pre-order (i.e. reflexive and transitive relation) '$\leq$' is defined by:

$$\begin{array}{rcll} \sigma_1\cap\cdots\cap\sigma_n & \leq & \sigma_i, & (1 \leq i \leq n) \\ \forall\alpha.(\sigma[\alpha/\varphi]) & \leq & \sigma[\tau/\varphi], & (\tau \in \mathcal{T}_{\mathrm{C}}) \\ \forall 1 \leq i \leq n\ [\sigma \leq \sigma_i] & \Rightarrow & \sigma \leq \sigma_1\cap\cdots\cap\sigma_n & (n \geq 1) \\ \rho \leq \sigma, \tau \leq \mu & \Rightarrow & \sigma\rightarrow\tau \leq \rho\rightarrow\mu, & (\tau, \mu \in \mathcal{T}_2) \\ \sigma \leq \tau & \Rightarrow & \forall\alpha.\sigma[\alpha/\varphi] \leq \forall\alpha.\tau[\alpha/\varphi]. & \end{array}$$

The equivalence relation '$\sim$' is defined by:

$$\sigma \sim \tau \iff \sigma \leq \tau \leq \sigma.$$

For $\leq$, the following properties hold:

Lemma 2.4 *i) If $\sigma \in \mathcal{T}_1$, $\sigma \leq \tau \in \mathcal{T}_2$, and σ does not contain '$\forall$', then neither does τ.*
ii) If $\sigma \leq \tau_1 \cap \cdots \cap \tau_n$, then, for all $1 \leq i \leq n$, $\sigma \leq \tau_i$.

PROOF. Easy.

Definition 2.5 (STATEMENTS AND BASES) *i*) A *statement* is a term of the form $t{:}\sigma$, with $\sigma \in \mathcal{T}_{\mathrm{R}}$ and $t \in T(\mathcal{F}, \mathcal{X})$. t is the *subject* and σ the *predicate* of $t{:}\sigma$.

ii) A *basis* B is a partial mapping from $\mathcal{X}$ to $\mathcal{T}_1$, represented as set of statements with only distinct variables as subjects. By abuse of notation, we write $x \in B$ if there exists a τ such that $x{:}\tau \in B$, $\varphi \in B$ if there is a type in B in which φ occurs, and write $B \backslash x$ for the basis obtained from B by removing the statement that has x as subject.

iii) For bases B_1, B_2, the basis $B_1 \cap B_2$ is defined by:

$$\begin{array}{rcl} B_1 \cap B_2 & = & \{x{:}\tau \mid x{:}\tau \in B_1 \ \& \ x \notin B_2\} \cup \{x{:}\tau \mid x{:}\tau \in B_2 \ \& \ x \notin B_1\} \cup \\ & & \{x{:}\tau_1 \cap \tau_2 \mid x{:}\tau_1 \in B_1 \ \& \ x{:}\tau_2 \in B_2\} \\ B, x{:}\tau & = & B \setminus x \cup \{x{:}\tau\} \end{array}$$

iv) The relation' $\leq$ ' and ' $\sim$ ' are extended to bases by:

$$\begin{array}{rcl} B \leq B' & \iff & \forall x{:}\sigma' \in B' \ \exists x{:}\sigma \in B \ [\sigma \leq \sigma'] \\ B \sim B' & \iff & B \leq B' \leq B \end{array}$$

Notice that if $n = 0$, then $B_1 \cap \ldots \cap B_n = \emptyset$.

3 Operations on types

The Rank 2 versions for the various operations as presented below are defined in much the same way as in [4], with the exception of the operation of closure and lifting, that were not used there, and are taken from [5].

Substitution

We will define substitution as usual in first-order logic, but avoid to go out of the set of polymorphic intersection types of Rank 2. For example, the substitution of φ by $\tau_1 \cap \tau_2$ would transform $\sigma \to \varphi$ into $\sigma \to \tau_1 \cap \tau_2$, which is not in $\mathcal{T}_{\mathrm{R}}$. However, since $\mathcal{T}_{\mathrm{C}} \subseteq \mathcal{T}_2$, and $\mathcal{T}_{\mathrm{C}}$ is closed for (Curry-)substitution, also $\mathcal{T}_2$ is closed for that kind of substitution.

The following definition takes this fact into account.

Definition 3.1 (SUBSTITUTION) *i*) The *substitution* $(\varphi \mapsto \rho) : \mathcal{T}_2 \to \mathcal{T}_2$, where φ is a type-variable in Φ and $\rho \in \mathcal{T}_{\mathrm{C}}$, is defined by:

$$\begin{array}{ll} (\varphi \mapsto \rho)(\varphi) & = \rho \\ (\varphi \mapsto \rho)(\varphi') & = \varphi', \text{if } \varphi' \neq \varphi \\ (\varphi \mapsto \rho)(s) & = s \\ (\varphi \mapsto \rho)(\alpha) & = \alpha \\ (\varphi \mapsto \rho)(\sigma \to \tau) & = (\varphi \mapsto \rho)(\sigma) \to (\varphi \mapsto \rho)(\tau) \\ (\varphi \mapsto \rho)(\sigma_1 \cap \cdots \cap \sigma_n) & = (\varphi \mapsto \rho)(\sigma_1) \cap \cdots \cap (\varphi \mapsto \rho)(\sigma_n) \\ (\varphi \mapsto \rho)(\forall \alpha.\sigma) & = \forall \alpha.(\varphi \mapsto \rho)(\sigma) \end{array}$$

ii) We use Id_S for the substitution that replaces all type-variables by themselves, write $\mathcal{S}$ for the set of all substitutions, and use S to denote a generic substitution. Substitutions extend to bases in the natural way: $S(B) = \{x{:}S(\rho) \mid x{:}\rho \in B\}$, and the set of substitutions is closed under composition '$\circ$'.

iii) The set of substitutions is closed under composition '$\circ$': for substitutions S_1, S_2, the substitution $S_2 \circ S_1$ is defined as

$$S_2 \circ S_1(\sigma) = S_2(S_1(\sigma)).$$

We have the following property:

Lemma 3.2 If $\sigma \leq \tau$, then $S(\sigma) \leq S(\tau)$, for all S.

PROOF. Easy.

Lifting

The operation of *lifting* replaces basis and type by a smaller basis and a larger type, in the sense of '$\leq$'. This operation allows us to eliminate intersections and universal quantifiers, using the '$\leq$' relation.

Definition 3.3 (LIFTING) An operation of *lifting* is determined by a pair $L = <\langle B_1, \tau_1\rangle, \langle B_2, \tau_2\rangle>$ such that $\tau_1 \leq \tau_2$ and $B_2 \leq B_1$, and is defined by $L(\langle B, \sigma\rangle) = \langle B', \sigma'\rangle$ where

$$\begin{array}{ll} \sigma' = \tau_2, \text{ if } \sigma = \tau_1, & B' = B_2, \text{ if } B = B_1 \\ \sigma' = \sigma, \text{ otherwise} & B' = B, \text{ otherwise} \end{array}$$

A lifting on types is determined by a pair $L = \langle \tau_1, \tau_2\rangle$ such that $\tau_1 \leq \tau_2$ and is defined by

$$\begin{array}{rl} L(\sigma) = \tau_2, & \text{if } \sigma = \tau_1 \\ \sigma, & \text{otherwise} \end{array}$$

Closure

The operation of *closure* introduces quantifiers, taking into account the basis where a type might occur.

Definition 3.4 (CLOSURE) A *closure* is characterised by a pair of types $\langle \sigma, \varphi\rangle$ with $\sigma \in \mathcal{T}_C^{\forall}$, and is defined by:

$$\langle \sigma, \varphi\rangle(\langle B, \tau_1 \cap \cdots \cap \tau_n\rangle) = \langle B, \tau_1' \cap \cdots \cap \tau_n'\rangle$$

where, for all $1 \leq i \leq n$,

$$\begin{array}{ll} \tau_i' = \forall\alpha.\sigma[\alpha/\varphi], & \text{if } \tau_i = \sigma, \text{ and } \varphi \text{ does not appear in } B \ (\alpha \text{ is a fresh variable}), \\ \tau_i' = \tau_i, & \text{otherwise.} \end{array}$$

Closure is extended to types by: $\langle\varphi\rangle(\sigma) = (\tau)$, if $\langle\varphi, \sigma\rangle(\langle\emptyset, \sigma\rangle) = \langle\emptyset, \tau\rangle$.

Expansion

The variant of expansion used in the Rank 2 system is quite different from that normally used [2, 3, 41]. The reason for this is that expansion, normally, increases the rank of a type:

$$\langle\varphi_1, 2\rangle(\langle\{x{:}\varphi_1 \to \varphi_2\}, \varphi_1\rangle)(\varphi_1 \to \varphi_2) = (\varphi_1^1 \cap \varphi_1^2) \to \varphi_2,$$

a feature that is of course not allowed within a system that limits the rank of types. Since here expansion is only used in very precise situations (within the procedure *unify*$_2^{\forall}$, and in the proof of Thm. 7.7), the solution is relatively easy: in the context of Rank 2 types, expansion is only called on types in $\mathcal{T}_C^{\forall}$, so it is defined to work well there, by replacing *all* types by an intersection; in particular, intersections are not created at the right of an arrow.

Definition 3.5 (EXPANSION) Let B be a basis, $\sigma \in \mathcal{T}_{\mathrm{R}}$, and $n \geq 1$. The n-fold *expansion* with respect to the pair $\langle B, \sigma\rangle$, $n_{\langle B,\sigma\rangle} : \mathcal{T}_2 \to \mathcal{T}_2$ is constructed as follows: Suppose $F = \{\varphi_1, \ldots, \varphi_m\}$ is the set of all (free) variables occurring in $\langle B, \sigma\rangle$. Choose $m \times n$ different variables $\varphi_1^1, \ldots, \varphi_1^n, \ldots, \varphi_m^1, \ldots, \varphi_m^n$, such that each φ_j^i ($1 \leq i \leq n$, $1 \leq j \leq m$) does not occur in F. Let S_i be the substitution that replaces every φ_j by φ_j^i. Then expansion is defined on types, bases, and pairs, respectively, by:

$$\begin{array}{ll} n_{\langle B,\sigma\rangle}(\tau) & = S_1(\tau) \cap \cdots \cap S_n(\tau), \\ n_{\langle B,\sigma\rangle}(B') & = \{x{:}n_{\langle B,\sigma\rangle}(\rho) \mid x{:}\rho \in B\}, \\ n_{\langle B,\sigma\rangle}(\langle B', \sigma'\rangle) & = \langle n_{\langle B,\sigma\rangle}(B'), n_{\langle B,\sigma\rangle}(\sigma')\rangle. \end{array}$$

Notice that, if $\tau \in \mathcal{T}_2$, it can be that $S_1(\tau) \cap \cdots \cap S_n(\tau)$ is not a legal type. However, for the sake of clarity, and since each $S_i(\tau) \in \mathcal{T}_2$, we will not treat this case separately (see also Lem. 5.4).

Operations will be grouped in chains.

Definition 3.6 *i*) A *chain* is an object $[O_1, \ldots, O_n]$, where each O_i is an operation of substitution, expansion, lifting, or closure, and $[O_1, \ldots, O_n](\sigma) = O_n(\cdots(O_1(\sigma))\cdots)$.

ii) On chains the operation of concatenation is denoted by $*$, and: $[O_1, \ldots, O_i] * [O_{i+1}, \ldots, O_n] = [O_1, \ldots, O_n]$.

iii) We say that $Ch_1 = Ch_2$, if for all σ, $Ch_1(\sigma) = Ch_2(\sigma)$.

In [5], the following property is shown to hold for chains:

Lemma 3.7 [5] *Let Ch be a chain.*

i) If $\sigma \in \mathcal{T}_2$, and $Ch(\sigma) \in \mathcal{T}_{\mathrm{C}}^{\forall}$, then there are a substitution S, and closures $Cl_1, \ldots, Cl_n$, such that

$$Ch(\sigma) = [S, Cl_1, \ldots, Cl_n](\sigma).$$

ii) If $\sigma \in \mathcal{T}_{\mathrm{C}}$, and $Ch(\sigma) \in \mathcal{T}_1$, then there exists a lifting-free chain Ch' such that $Ch(\sigma) = Ch'(\sigma)$.

iii) If $\sigma \in \mathcal{T}_1$, and $Ch(\sigma) \in \mathcal{T}_{\mathrm{C}}$, then there is a substitution S such that $Ch(\sigma) = S(\sigma)$.

iv) If $\sigma \in \mathcal{T}_2$, and $Ch(\sigma) \in \mathcal{T}_{\mathrm{C}}$, then there is a substitution S such that $Ch(\sigma) = S(\sigma)$.

v) If $\sigma \in \mathcal{T}_2$, and $Ch(\sigma) \in \mathcal{T}_2$, then there are substitution S, and lifting L such that $Ch(\sigma) = [S, L](\sigma)$.

PROOF. For part one, expansion and closure are not needed, and by Lem. 2.4, neither is lifting. The other parts are just generalisations of the first.

4 Rank 2 Type Assignment

We now come to the definition of Rank 2 type assignment.

Definition 4.1 *i*) A *Rank 2 environment* $\mathcal{E}$ is a mapping from $\mathcal{F}$ to $\mathcal{T}_2$.

ii) *Rank 2 type assignment on terms* is defined by the following natural deduction system:

$$(Ax): \frac{}{B \vdash_{\mathcal{E}} x:\tau}\ (x{:}\sigma \in B \ \&\ \sigma \leq \tau \ \&\ \sigma \in \mathcal{T}_1 \ \&\ \tau \in \mathcal{T}_2)$$

$$(\cap I): \frac{B \vdash_{\mathcal{E}} t:\sigma_1 \quad \cdots \quad B \vdash_{\mathcal{E}} t:\sigma_n}{B \vdash_{\mathcal{E}} t:\sigma_1 \cap \cdots \cap \sigma_n}\ (n \geq 1 \ \&\ \forall 1 \leq i \leq n\,[\sigma_i \in \mathcal{T}_{\mathrm{C}}^{\forall}])$$

$$(\rightarrow E): \frac{B \vdash_{\mathcal{E}} t_1:\sigma \rightarrow \tau \quad B \vdash_{\mathcal{E}} t_2:\sigma}{B \vdash_{\mathcal{E}} t_1\, t_2:\tau}$$

$$(\forall I): \frac{B \vdash_{\mathcal{E}} t:\sigma}{B \vdash_{\mathcal{E}} t:\forall\alpha.\sigma[\alpha/\varphi]}\ (\varphi \notin B \ \&\ \sigma \in \mathcal{T}_{\mathrm{C}}^{\forall})$$

$$(\mathsf{share}): \frac{B, x{:}\sigma \vdash_{\mathcal{E}} t_2:\tau \quad B \vdash_{\mathcal{E}} t_1:\sigma}{B \vdash_{\mathcal{E}} (\mathsf{share}\, t_1\, \mathsf{via}\, x\, \mathsf{in}\, t_2):\tau}$$

$$(\mathcal{F}): \frac{}{B \vdash_{\mathcal{E}} \mathbf{F}:\sigma}\ (\exists Ch\,[Ch(\mathcal{E}(\mathbf{F})) = \sigma])$$

$$(\mathsf{cycle}): \frac{B, \overrightarrow{x_i{:}\sigma_i} \vdash_{\mathcal{E}} t_i:\sigma_1 \quad \ldots \quad B, \overrightarrow{x_i{:}\sigma_i} \vdash_{\mathcal{E}} t_i:\sigma_n \quad B, \overrightarrow{x_i{:}\sigma_i} \vdash_{\mathcal{E}} t:\tau}{B \vdash_{\mathcal{E}} \mathsf{cycle}\,\langle\, \overrightarrow{x_i = t_i}\,\rangle\, \mathsf{in}\, t:\tau}\ (\forall 1 \leq i \leq n\,[\sigma_i \in \mathcal{T}_{\mathrm{C}}])$$

We write $B \vdash_{\mathcal{E}} t:\sigma$ if this is derivable using the rules above.

Notice the use of an environment and chain in rule $(\mathcal{F})$; because of this rule, the notion of type assignment defined here is in fact a *partially typed* system: all function symbols are assumed to have a type to begin with, that is 'instantiated' by this rule.

Also, rule $(\mathcal{F})$ formalises the practice of functional languages in that it introduces a notion of polymorphism for function symbols, which is an extension (with intersection types and general quantification) of the ML-style of polymorphism. The environment returns the 'principal type' for a function symbol; this symbol can be used with types that are 'instances' of its principal type, obtained by applying chains of operations.

Although these rules express how to type terms, it is straightforward to extend this definition to one that expresses how to type graphs, such that $B \vdash_{\mathcal{E}} t:\sigma$ if and only if $B \vdash_{\mathcal{E}} [\![t]\!]:\sigma$.

Example 4.2 If we extend the definition of types with the alternative for list types and booleans

$$\mathcal{T}_{\mathrm{C}} ::= \varphi \mid s \mid (\mathcal{T}_{\mathrm{C}} \rightarrow \mathcal{T}_{\mathrm{C}}) \mid [\mathcal{T}_{\mathrm{C}}] \mid \mathsf{Bool}$$

then, using Rank 2 types, we can now express the function 'IsNil', that tests if a list is empty, defined by

$$\mathsf{IsNil}\,[\,] \ \rightarrow\ \mathrm{TT}$$

is typeable using the environment

$$\begin{array}{lcl} \mathcal{E}(\mathrm{TT}) & = & \mathsf{Bool} \\ \mathcal{E}([\,]) & = & [\varphi] \\ \mathcal{E}(\mathsf{Cons}) & = & \varphi \rightarrow [\varphi] \rightarrow [\varphi] \\ \mathcal{E}(\mathsf{IsNil}) & = & (\forall\alpha.[\alpha]) \rightarrow \mathsf{Bool} \end{array}$$

$$\dfrac{\dfrac{}{\emptyset \vdash_{\mathcal{E}} \mathsf{IsNil}:(\forall\alpha.[\alpha]) \rightarrow \mathsf{Bool}} \quad \dfrac{\dfrac{}{\emptyset \vdash_{\mathcal{E}} [\,]:[\varphi]}}{\emptyset \vdash_{\mathcal{E}} [\,]:\forall\alpha.[\alpha]}}{\emptyset \vdash_{\mathcal{E}} \mathsf{IsNil}\,[\,]:\mathsf{Bool}}$$

Notice that the type for this function 'IsNil' in the environment prohibits its use against 'concrete' lists that are not empty, since any list with an element is that is of type s is no longer polymorphic. Also, this is not a derivable result in any of the other systems mentioned in the introduction.

Notice that rule $(\mathcal{F})$ models a kind of polymorphism into our system, other than the kind obtained by having quantified types to our disposition. Quantification allows only the replacement of type-variables by Curry types, whereas rule $(\mathcal{F})$ allows any operation to be applied. It allows function symbols to appear in context that require a type that is more specific than the one provided by the environment; the soundness result we show below for the various operations justify the application of chains to the types provided by the environment.

Also, since quantification elimination is implicit in rule (Ax), when restricting the use of the quantifier to the left of arrows only, there is no longer need for a general $(\forall E)$ rule; as with a possible rule $(\cap E)$, its use is in a strict system limited to variables, and there its actions are already performed by (Ax). In fact, this restriction is justified by Lem. 5.2.

For this system to be of use in practice, a minimal requirement would be a *subject reduction* result, which expresses that types are preserved by reduction. To achieve this, we define a notion of type assignment on rewrite rules using the notion of principal pair (also called principal typing), that will be developed in Section 7 (see Def. 7.1), and culminates in Thm. 7.7, which states:

If $B \vdash_{\mathcal{E}} t:\sigma$, then there are a basis P and type π such that $pp_{\mathcal{E}}(t) = \langle P, \pi\rangle$,
and there is a chain Ch such that $Ch(\langle P, \pi\rangle) = \langle B, \sigma\rangle$.

This property, together with the result that all operations are sound, is used to prove the subject reduction result. (The same method was used in [7, 6, 5].)

Definition 4.3 (TYPE ASSIGNMENT ON REWRITE RULES) *i*) We say that *left* $\to$ *right* $\in \mathbf{R}$ with defined symbol F *is typeable with respect to* $\mathcal{E}$, if there are P, and $\pi \in \mathcal{T}_2$ such that:
a) $\langle P, \pi\rangle$ is a principal pair (Def. 7.1) for *left* with respect to $\mathcal{E}$.
b) In $P \vdash_{\mathcal{E}}$ *left*$:\pi$ and $P \vdash_{\mathcal{E}}$ *right*$:\pi$ each occurrenc of F is typed with $\mathcal{E}(\mathsf{F})$.
ii) We say that $(\Sigma, \mathbf{R})$ *is typeable with respect to* $\mathcal{E}$, if all rules in $\mathbf{R}$ are.

As an aside to part (*i.b*), remark that, by rule $(\mathcal{E})$, we know that each occurrence of F has a type *generated from* $\mathcal{E}(\mathsf{F})$ by applying a chain of operations. Part (*i.b*) states that, for the derivations involved here, these chains are all empty, i.e. are the identity operation. Since we forced the type of a function symbol F to be exactly $\mathcal{E}(\mathsf{F})$ in the rules that define F, the typeability of rules ensures consistency with respect to the environment.

Notice that, because in the translation of terms to graphs, the defined node is shared by all occurrences in the rule, when typing the graph rewrite rule the condition 'all occurrences of F are typed with $\mathcal{E}(\mathsf{F})$' becomes 'the occurrence of F is typed with $\mathcal{E}(\mathsf{F})$'.

Before we come to a subject reduction result, first we need to show that all operations defined are sound, which we will show in the next section. The main result there is Lem. 5.6, which states:

If $\sigma \in \mathcal{T}_1$, $B \vdash_{\mathcal{E}} t:\sigma$, and Ch is a chain of operations on types
such that $Ch(\langle B, \sigma\rangle) = \langle B', \sigma'\rangle$, then $B' \vdash_{\mathcal{E}} t:\sigma'$.

We will now take a short-cut, and show that reductions preserve types in our system, using the notion of principal pair and the soundness of operations on types.

The proof of Subject Reduction depends also on the following lemma:

Lemma 4.4 (REPLACEMENT) *Let $\mathcal{E}$ be an environment, t a term, and f a mapping from free variables to terms (which extends naturaly to a mapping from terms to terms).*

i) If $B \vdash_{\mathcal{E}} t:\sigma$ and B' is such that $B' \vdash_{\mathcal{E}} f(x):\rho$ for every statement $x:\rho \in B$, then $B' \vdash_{\mathcal{E}} f(t):\sigma$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{B_1 \vdash_{\mathcal{E}} \mathbf{S} : \mathcal{E}(\mathbf{S})} \quad \dfrac{}{B_1 \vdash_{\mathcal{E}} x : \sigma\to\tau\to\rho}}{B_1 \vdash_{\mathcal{E}} \mathbf{S}\, x : (\mu\to\tau)\to\sigma\cap\mu\to\rho} \quad \dfrac{}{B_1 \vdash_{\mathcal{E}} y : \mu\to\tau}}{B_1 \vdash_{\mathcal{E}} \mathbf{S}\, x\, y : \sigma\cap\mu\to\rho} \quad \dfrac{\dfrac{}{B_1 \vdash_{\mathcal{E}} z : \sigma} \quad \dfrac{}{B_1 \vdash_{\mathcal{E}} z : \mu}}{B_1 \vdash_{\mathcal{E}} z : \sigma\cap\mu}}{B_1 \vdash_{\mathcal{E}} \mathbf{S}\, x\, y\, z : \rho}$$

$$\dfrac{\dfrac{\dfrac{}{B_1 \vdash_{\mathcal{E}} x : \sigma\to\tau\to\rho} \quad \dfrac{}{B_1 \vdash_{\mathcal{E}} z : \sigma}}{B_1 \vdash_{\mathcal{E}} x\, z : \tau\to\rho} \quad \dfrac{\dfrac{}{B_1 \vdash_{\mathcal{E}} y : \mu\to\tau} \quad \dfrac{}{B_1 \vdash_{\mathcal{E}} z : \mu}}{B_1 \vdash_{\mathcal{E}} y\, z : \tau}}{B_1 \vdash_{\mathcal{E}} x\, z\, (y\, z) : \rho}$$

$$\dfrac{\dfrac{\dfrac{}{B_2 \vdash_{\mathcal{E}} \mathbf{K} : \mathcal{E}(\mathbf{K})} \quad \dfrac{}{B_2 \vdash_{\mathcal{E}} x : \nu}}{B_2 \vdash_{\mathcal{E}} \mathbf{K}\, x : \gamma\to\nu} \quad \dfrac{}{B_2 \vdash_{\mathcal{E}} y : \gamma}}{B_2 \vdash_{\mathcal{E}} \mathbf{K}\, x\, y : \nu} \qquad \dfrac{}{B_2 \vdash_{\mathcal{E}} x : \nu} \qquad \dfrac{\dfrac{\dfrac{}{B_3 \vdash_{\mathcal{E}} \mathbf{I} : \mathcal{E}(\mathbf{I})} \quad \dfrac{}{B_3 \vdash_{\mathcal{E}} x : \delta}}{B_3 \vdash_{\mathcal{E}} \mathbf{I}\, x : \delta} \qquad \dfrac{}{B_3 \vdash_{\mathcal{E}} x : \delta}$$

Figure 2: Type derivations for Ex. 4.6 (where $B_1 = \{x{:}\sigma\to\tau\to\rho, y{:}\mu\to\tau, z{:}\sigma\cap\mu\}$, $B_2 = \{x{:}\nu, y{:}\gamma\}$ and $B_3 = \{x{:}\delta\}$).

ii) If there are B and σ such that $B \vdash_{\mathcal{E}} f(t) : \sigma$, then for every x occurring in t there is a type ρ_x such that $\{x{:}\rho_x \mid x \in fv(t)\} \vdash_{\mathcal{E}} t : \sigma$, and $B \vdash_{\mathcal{E}} f(x) : \rho_x$.

PROOF. By induction on the structure of derivations.

Using this lemma, the following result follows easily.

Theorem 4.5 (SUBJECT REDUCTION) *If $B \vdash_{\mathcal{E}} t : \sigma$ and $t \to t'$, then $B \vdash_{\mathcal{E}} t' : \sigma$.*

PROOF. We consider only the case of a rewrite step. Let *left* $\to$ *right* be the (typeable) rewrite rule applied in the rewrite step $t \to t'$. We will prove that for every term-substitution R and type μ, if $B \vdash_{\mathcal{E}} f(\textit{left}) : \mu$, then $B \vdash_{\mathcal{E}} f(\textit{right}) : \mu$, which proves the theorem.

Since **r** is typeable, there are P, π such that $\langle P, \pi\rangle$ is a principal pair for *left* with respect to $\mathcal{E}$, and $P \vdash_{\mathcal{E}} \textit{right} : \pi$. Suppose R is a term-substitution such that $B \vdash_{\mathcal{E}} f(\textit{left}) : \mu$. By Lem. 4.4(*ii*) there is a B' such that for every $x{:}\rho \in B'$, $B \vdash_{\mathcal{E}} f(x) : \rho$, and $B' \vdash_{\mathcal{E}} \textit{left} : \mu$. Since $\langle P, \pi\rangle$ is a principal typing for *left* with respect to $\mathcal{E}$, by Thm. 7.7 there is a chain *Ch* such that $\textit{Ch}(\langle P, \pi\rangle) = \langle B', \mu\rangle$. Since $P \vdash_{\mathcal{E}} \textit{right} : \pi$, by Lem. 5.6 also $B' \vdash_{\mathcal{E}} \textit{right} : \mu$. Then by Lem. 4.4(*i*) $B \vdash_{\mathcal{E}} f(\textit{right}) : \mu$.

Example 4.6 Let $\sigma, \tau, \rho, \mu, \nu, \gamma$, and δ be (arbitrary) types. Take the rewrite rules that define Combinatory Logic of Ex. 1.11, and the environment $\mathcal{E}$:

$$\begin{aligned} \mathcal{E}(\mathbf{S}) &= (\sigma\to\tau\to\rho)\to(\mu\to\tau)\to\sigma\cap\mu\to\rho \\ \mathcal{E}(\mathbf{K}) &= \nu\to\gamma\to\nu \\ \mathcal{E}(\mathbf{I}) &= \delta\to\delta \end{aligned}$$

Then these rules are typeable with respect to $\mathcal{E}$; we show the derivations in Fig. 2.

Example 4.7 Take the rewrite rule

$$\mathbf{D}\, y \to (\mathsf{share}\, y\, \mathsf{via}\, v\, \mathsf{in}\, v\, v)$$

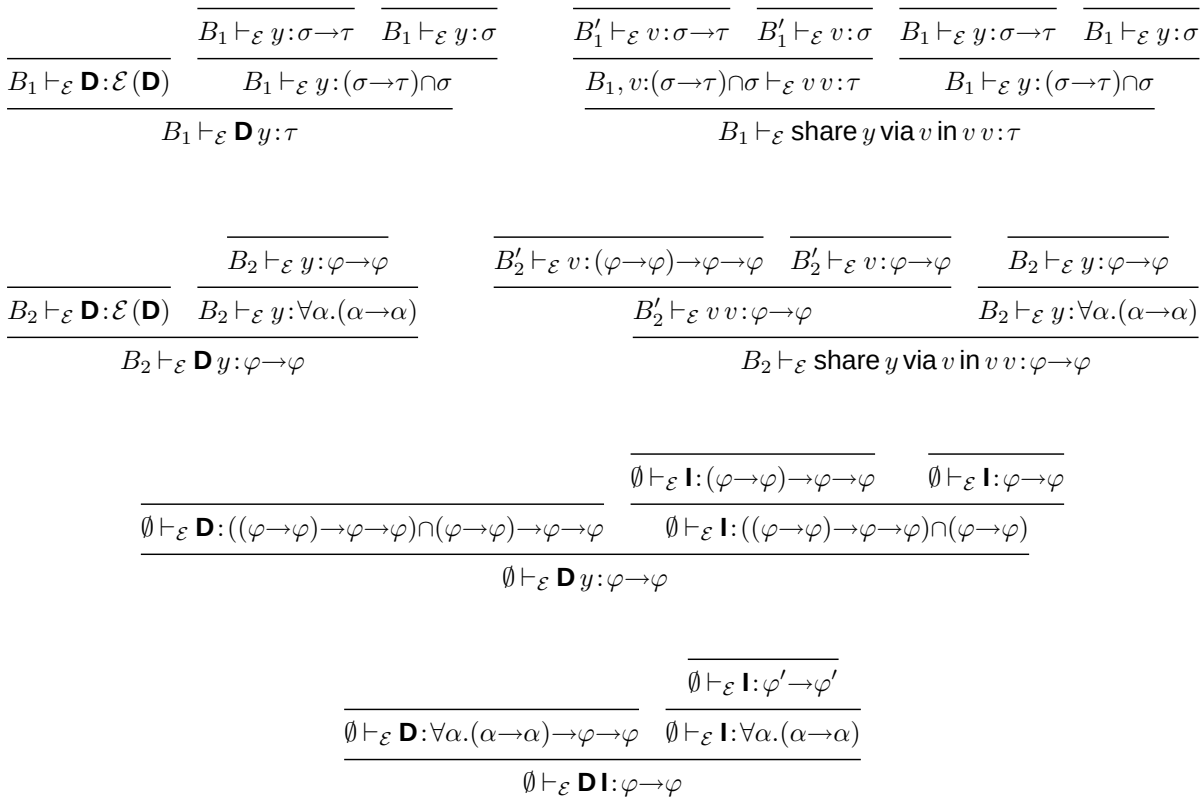


Figure 3: Type derivations for Example 4.7

(notice that the rule corresponds to the lambda term $\lambda y.yy$, but expresses that the variable y is shared; as far as our system is concerned, this is already the case for the rule $\mathbf{D}\,y \to y\,y$). This rule can be typed, as shown in Figure 3, using $\mathcal{E}(\mathbf{D}) = (\sigma{\to}\tau){\cap}\sigma{\to}\tau$, as well as using $\mathcal{E}(\mathbf{D}) = \forall\alpha.(\alpha{\to}\alpha){\to}\varphi{\to}\varphi$, where $B_1 = \{y{:}(\sigma{\to}\tau){\cap}\sigma\}$, and $B_1' = B_1, v{:}(\sigma{\to}\tau){\cap}\sigma$ for the first environment, and $B_2 = \{y{:}\forall\alpha.(\alpha{\to}\alpha)\}$ and $B_2' = B_2, v{:}\forall\alpha.(\alpha{\to}\alpha)$ for the second.

We can, using these environments, also derive $\emptyset \vdash_{\mathcal{E}} \mathbf{D}\,\mathbf{I} : \varphi{\to}\varphi$.

5 Soundness of operations

We will now show that the operations defined in Section 3 are sound. First, we show this for substitution.

Lemma 5.1 (SOUNDNESS OF SUBSTITUTION) *If $B \vdash_{\mathcal{E}} t{:}\sigma$, then, for every substitution S, $S(B) \vdash_{\mathcal{E}} t{:}S(\sigma)$.*

PROOF. By induction on the structure of derivations.

(Ax) : Then $t \equiv x, \sigma \in \mathcal{T}_2$, and there is $\rho \in \mathcal{T}_1$ such that $x{:}\rho \in B$ and $\rho \leq \sigma$. Then, by Lemma 3.2, $S(\rho) \leq S(\sigma)$, and since $x{:}S(\rho) \in S(B)$, also $S(B) \vdash_{\mathcal{E}} x{:}S(\sigma)$.

$(\cap I)$: Then $\sigma = \sigma_1{\cap}\cdots{\cap}\sigma_n$, and, for $1 \leq i \leq n$, $B \vdash_{\mathcal{E}} t{:}\sigma_i$. Then, by induction, for all $1 \leq j \leq m$, $S(B) \vdash_{\mathcal{E}} t{:}S(\tau_j)$, so, by rule $(\cap I)$, also $S(B) \vdash_{\mathcal{E}} t{:}S(\tau_1{\cap}\cdots{\cap}\tau_m)$.

$(\mathcal{F})$: Then $t \equiv \mathbf{F}$, and $Ch(\mathcal{E}(\mathbf{F})) = \sigma$ for some chain Ch. Since $[S] * Ch(\mathcal{E}(\mathbf{F})) = S(\sigma)$, by rule $(\mathcal{F})$, $S(B) \vdash_{\mathcal{E}} \mathbf{F}{:}S(\sigma)$.

($\rightarrow E$) : Then $t \equiv (t_1\, t_2)$, and $B \vdash_{\mathcal{E}} t_1 : \rho \rightarrow \sigma$, and $B \vdash_{\mathcal{E}} t_2 : \rho$ for some $\rho \in \mathcal{T}_1$. By induction, $S(B) \vdash_{\mathcal{E}} t_1 : S(\rho \rightarrow \sigma)$, and $S(B) \vdash_{\mathcal{E}} t_2 : S(\rho)$, so $S(B) \vdash_{\mathcal{E}} t_1\, t_2 : S(\sigma)$, by rule ($\rightarrow E$).

($\forall I$) : Then $\sigma = \forall\alpha.\rho[\alpha/\varphi]$, $B \vdash_{\mathcal{E}} t : \rho$, and $S(B) \vdash_{\mathcal{E}} t : S(\rho)$ by induction. We can assume, without loss of generality, that φ is not affected by S, so, φ occurs in ρ if and only if it occurs in $S(\rho)$. Therefore also $S(B) \vdash_{\mathcal{E}} t : S(\rho)[\alpha/\varphi]$ by rule ($\forall I$), so $S(B) \vdash_{\mathcal{E}} t : S(\rho[\alpha/\varphi])$.

(share) : Then $t \equiv (\mathsf{share}\, t_1\, \mathsf{via}\, x\, \mathsf{in}\, t_2)$, and there exists a τ such that $B, x{:}\tau \vdash_{\mathcal{E}} t_1 : \sigma$ and $B \vdash_{\mathcal{E}} t_2 : \tau$. By induction, we get $S(B, x{:}\tau) \vdash_{\mathcal{E}} t_1 : S(\sigma)$ and $S(B) \vdash_{\mathcal{E}} t_2 : S(\tau)$. But then, since $S(B, x{:}\tau) = S(B), x{:}S(\tau)$, by rule (share) we get $S(B) \vdash_{\mathcal{E}} \mathsf{share}\, t_1\, \mathsf{via}\, x\, \mathsf{in}\, t_2 : S(\sigma)$.

(cycle) : Then $t \equiv (\mathsf{cycle}\, \langle\, \overrightarrow{x_i = t_i}\, \rangle\, \mathsf{in}\, t')$. Then $B, \overrightarrow{x_i{:}\sigma_i} \vdash_{\mathcal{E}} t_i : \sigma_i$ and $B, \overrightarrow{x_i{:}\sigma_i} \vdash_{\mathcal{E}} t' : \sigma$ for some $\sigma_1, \ldots, \sigma_n$. Then we have $S(B, \overrightarrow{x_i{:}\sigma_i}) \vdash_{\mathcal{E}} t_i : S(\sigma_i)$ and $S(B, \overrightarrow{x_i{:}\sigma_i}) \vdash_{\mathcal{E}} t' : S(\sigma)$, by induction. Since $S(B, \overrightarrow{x_i{:}\sigma_i}) = S(B), \overrightarrow{x_i{:}S(\sigma_i)}$, we get $S(B) \vdash_{\mathcal{E}} \mathsf{cycle}\, \langle\, \overrightarrow{x_i = t_i}\, \rangle\, \mathsf{in}\, t : S(\sigma)$ by rule (cycle).

The next lemma essentially states that lifting is a sound operation.

Lemma 5.2 (SOUNDNESS OF LIFTING) *If $B \vdash_{\mathcal{E}} t : \sigma$, and let B', τ be such that $B' \leq B$, and $\sigma \leq \tau$, then $B' \vdash_{\mathcal{E}} t : \tau$.*

PROOF. By induction on the structure of derivations. First we deal with the case that τ is not an intersection.

(*Ax*) : Then $t \equiv x, \sigma \in \mathcal{T}_{\mathrm{C}}$, and there is $\rho \in \mathcal{T}_1$ such that $x{:}\rho \in B$ and $\rho \leq \sigma$. Since $B' \leq B$, there is $\rho' \in \mathcal{T}_1$ such that $x{:}\rho' \in B'$ and $\rho' \leq \rho \leq \sigma$. Since $\sigma \leq \tau$, also $\rho' \leq \tau$ and $B' \vdash_{\mathcal{E}} x : \tau$.

($\cap I$) : Then $\sigma = \sigma_1 \cap \cdots \cap \sigma_n$, and, for $1 \leq i \leq n$, $B \vdash_{\mathcal{E}} t : \sigma_i$. Then there is an $1 \leq i \leq n$, such that $\sigma_i \leq \tau$. Then, by induction, $B' \vdash_{\mathcal{E}} t : \tau$.

($\mathcal{F}$) : Then $t \equiv \mathbf{F}$, and $Ch(\mathcal{E}(\mathbf{F})) = \sigma$ for some chain Ch. Since $\sigma \leq \tau$, $L = <\langle\emptyset, \sigma\rangle, \langle\emptyset, \tau\rangle>$ is a lifting. Notice that $[L] * Ch(\mathcal{E}(\mathbf{F})) = \tau$, and therefore $B' \vdash_{\mathcal{E}} \mathbf{F} : \tau$ by rule ($\mathcal{F}$).

($\rightarrow E$) : Then $t \equiv (t_1\, t_2)$, and $B \vdash_{\mathcal{E}} t_1 : \rho \rightarrow \sigma$, and $B \vdash_{\mathcal{E}} t_2 : \rho$, for some $\rho \in \mathcal{T}_1$. Since $\sigma \leq \tau$, also $\rho \rightarrow \sigma \leq \rho \rightarrow \tau$, so by induction, $B' \vdash_{\mathcal{E}} t_1 : \rho \rightarrow \tau$ and $B' \vdash_{\mathcal{E}} t_1\, t_2 : \tau$ by rule ($\rightarrow E$).

($\forall I$) : Then $\sigma = \forall\alpha.\rho[\alpha/\varphi]$, and $B \vdash_{\mathcal{E}} t : \rho$, and, by definition of '$\leq$', either:

($\tau = \rho[\mu/\varphi]\, (\mu \in \mathcal{T}_{\mathrm{C}})$) : By induction, $B' \vdash_{\mathcal{E}} t : \rho$, and, by Lem. 5.1, $B' \vdash_{\mathcal{E}} t : \rho[\mu/\varphi]$ (notice that φ occurs in ρ only).

($\tau = \forall\alpha.\mu[\alpha/\varphi]\, (\rho \leq \mu)$) : Then $B' \vdash_{\mathcal{E}} t : \mu$ by induction, and $B' \vdash_{\mathcal{E}} t : \forall\alpha.\mu[\alpha/\varphi]$ by rule ($\forall I$).

(share) : Then $t \equiv (\mathsf{share}\, t_1\, \mathsf{via}\, x\, \mathsf{in}\, t_2)$, and there exists a ρ such that $B, x{:}\rho \vdash_{\mathcal{E}} t_1 : \sigma$ and $B \vdash_{\mathcal{E}} t_2 : \rho$. Since we have $B', x{:}\rho \leq B, x{:}\rho$, by induction $B', x{:}\rho \vdash_{\mathcal{E}} t_1 : \tau$, and $B' \vdash_{\mathcal{E}} t_2 : \rho$, so, by rule (share), also $B' \vdash_{\mathcal{E}} (\mathsf{share}\, t_1\, \mathsf{via}\, x\, \mathsf{in}\, t_2) : \tau$.

(cycle) : Then $t \equiv (\mathsf{cycle}\, \langle\, \overrightarrow{x_i = t_i}\, \rangle\, \mathsf{in}\, t')$. Then, for some $\sigma_1, \ldots \sigma_n \in \mathcal{T}_{\mathrm{C}}$, $B, \overrightarrow{x_i{:}\sigma_i} \vdash_{\mathcal{E}} t_i : \sigma_i$ and $B, \overrightarrow{x_i{:}\sigma_i} \vdash_{\mathcal{E}} t' : \sigma$. Since $B', \overrightarrow{x_i{:}\sigma_i} \leq B, \overrightarrow{x_i{:}\sigma_i}$, by induction, $B', \overrightarrow{x_i{:}\sigma_i} \vdash_{\mathcal{E}} t_i : \sigma_i$, for $1 \leq i \leq n$, and $B', \overrightarrow{x_i{:}\sigma_i} \vdash_{\mathcal{E}} t' : \tau$. Then, by rule (cycle), $B' \vdash_{\mathcal{E}} (\mathsf{cycle}\, \langle\, \overrightarrow{x_i = t_i}\, \rangle\, \mathsf{in}\, t') : \tau$.

If $\tau = \tau_1 \cap \cdots \cap \tau_n$, then, by Lem. 2.4(*ii*), for all $1 \leq i \leq n$, $\sigma \leq \tau_i$. The result then follows from the above, and rule ($\cap I$).

The next lemma states that closure is a sound operation.

Lemma 5.3 (SOUNDNESS OF CLOSURE) *If $B \vdash_{\mathcal{E}} t : \tau$ and $Cl = \langle\sigma, \varphi\rangle$ is a closure such that $Cl(\langle B, \tau\rangle) = \langle B', \rho\rangle$, then $B' \vdash_{\mathcal{E}} t : \rho$.*

PROOF. Let $\tau = \tau_1 \cap \cdots \cap \tau_n$ ($n \geq 1$), then

$$\langle\sigma, \varphi\rangle\, (\langle B, \tau_1 \cap \cdots \cap \tau_n\rangle) = \langle B, \tau_1' \cap \cdots \cap \tau_n'\rangle$$

so $B' = B$ and, for all $1 \leq i \leq n$,

- φ occurs in B, and $\tau_i = \sigma$, and the result is trivial, or
- φ does not occur in B, $\tau_i = \sigma$, and $\tau_i' = \forall\alpha.\sigma[\alpha/\varphi]$, and the result follows from rule $(\forall I)$.

Since expansion just creates an intersection of types, it could be that the type created is not in $\mathcal{T}_2$, but would be an intersection of types from that set. Therefore, we cannot show a general soundness result. However, we can show the following:

Lemma 5.4 (SOUNDNESS OF EXPANSION) *Let Ex be an expansion, and* $Ex(\sigma) = \sigma_1 \cap \cdots \cap \sigma_n$. *If* $B \vdash_{\mathcal{E}} t{:}\sigma$, *then, for every* $1 \leq i \leq n$, *there is a* B' *such that* $B' \vdash_{\mathcal{E}} t{:}\sigma_i$.

PROOF. By Def. 3.5, there are substitutions $S_1, \ldots, S_n$ such that $Ex(\sigma) = S_1(\sigma) \cap \cdots \cap S_n(\sigma)$. The result then follows from Lem. 5.1 (notice that $B' = S_i(B)$).

In case expansion gets applied to a type in $\mathcal{T}_1$, the result is stronger.

Lemma 5.5 *Let Ex be an expansion. If* $\sigma \in \mathcal{T}_1$ *and* $B \vdash_{\mathcal{E}} t{:}\sigma$, *then* $Ex(B) \vdash_{\mathcal{E}} t{:}Ex(\sigma)$.

PROOF. By the previous lemma, if $Ex(\sigma) = \sigma_1 \cap \cdots \cap \sigma_n$, then, for every $1 \leq i \leq n$, there is a B' such that $B' \vdash_{\mathcal{E}} t{:}\sigma_i$. Since $\sigma \in \mathcal{T}_1$, also each $\sigma_i \in \mathcal{T}_1$. Notice that $Ex(B) \leq S_i(B)$, for $1 \leq i \leq n$, so, by Lem. 5.2 and rule $(\cap I)$, we get the result.

These soundness results are combined in the following:

Lemma 5.6 (SOUNDNESS OF CHAINS) *If* $\sigma \in \mathcal{T}_1$, $B \vdash_{\mathcal{E}} t{:}\sigma$, *and Ch is a chain such that* $Ch(\langle B, \sigma\rangle) = \langle B', \sigma'\rangle$, *then* $B' \vdash_{\mathcal{E}} t{:}\sigma'$.

PROOF. By lemmas 5.1 to 5.4.

The following properties of chains will be used in the proof of Thm. 7.7 below.

Lemma 5.7
i) If there exists a chain Ch such that $Ch(\langle P \cup \{x{:}\nu\}, \pi\rangle) = \langle B \cup \{x{:}\rho\}, \mu\rangle$, *where* $\pi, \mu \in \mathcal{T}_2$, *then there exists a chain Ch′ such that* $Ch'(\langle P, \nu{\to}\pi\rangle) = \langle B, \rho{\to}\mu\rangle$.
ii) If there exists a chain Ch such that $Ch(\langle P, \pi\rangle) = \langle B, \mu\rangle$, *where* $\pi, \mu \in \mathcal{T}_2$, *then there exists a chain Ch′ such that* $Ch'(\langle P, \varphi{\to}\pi\rangle) = \langle B, \varphi{\to}\mu\rangle$, *where* φ *is a fresh type variable.*

PROOF. Straightforward.

6 Unification of Rank 2 Types

In the context of types, unification is a procedure normally used to find a common instance for demanded and provided type for applications, i.e: if t_1 has type $\sigma{\to}\tau$, and t_2 has type ρ, then unification looks for a common instance of the types σ and ρ such that $(t_1\, t_2)$ can be typed properly. The unification algorithm $unify_2^{\forall}$ presented in the next definition (a corrected version of the algorithm presented in [5]) deals with just that problem. This means that it is not a full unification algorithm for types of Rank 2, but only an algorithm that finds the most general unifying chain for demanded and provided type. It is defined as a natural extension of Robinson's well-known unification algorithm *unify* [40], and can be seen as an extension of the notion of unification as presented in [4], in that it deals with quantification as well.

Definition 6.1 (UNIFICATION) Unification of Curry types (extended with bound variables and type

constants) is defined by:

$$
\begin{array}{lll}
\multicolumn{3}{l}{\mathit{unify} : \mathcal{T}'_{\mathrm{C}} \times \mathcal{T}'_{\mathrm{C}} \to \mathcal{S}} \\
\mathit{unify}(\varphi, \varphi') & = & (\varphi \mapsto \varphi'), \\
\mathit{unify}(\varphi, \tau) & = & (\varphi \mapsto \tau), \text{ if } \varphi \text{ not in } \tau, \\
\mathit{unify}(\alpha, \alpha) & = & \mathit{Id}_S, \\
\mathit{unify}(s, s) & = & \mathit{Id}_S, \\
\mathit{unify}(\sigma, \varphi) & = & \mathit{unify}(\varphi, \sigma), \\
\mathit{unify}(\sigma{\to}\tau, \rho{\to}\mu) & = & S_2 \circ S_1, \\
 & & \quad \text{where } S_1 = \mathit{unify}(\sigma, \rho), \\
 & & \qquad\quad S_2 = \mathit{unify}(S_1(\tau), S_1(\mu)).
\end{array}
$$

(All non-specified cases, like $\mathit{unify}(\alpha_1, \alpha_2)$ with $\alpha_1 \neq \alpha_2$, fail.)

It is worthwhile to notice that the operation on types returned by *unify* is not really a substitution, since it allows, e.g., $(\varphi \mapsto \alpha)$, without keeping track of the binder for α. This potentially will create wrong results, since unification can now substitute bound variables in unbound places. Therefore, special care has to be taken before applying a substitution, to guarantee its application to the argument acts as a 'real' substitution.

The following property is well-known, and formulates that *unify* returns the most general unifier for two Curry types, if it exists.

Property 6.1 ([40]) *If two types have an instance in common, they have a* highest common instance *which is returned by unify: for all $\sigma, \tau \in \mathcal{T}_{\mathrm{C}}$, substitutions S_1, S_2: if $S_1(\sigma) = S_2(\tau)$, then there are substitutions S_u and S' such that*

$$S_u = \mathit{unify}(\sigma, \tau), \text{ and } S_1(\sigma) = S' \circ S_u(\sigma) = S' \circ S_u(\tau) = S_2(\tau).$$

The unification algorithm $\mathit{unify}_2^{\forall}$ as defined below gets, typically, called during the computation of the principal pair for an application $t_1\, t_2$. Suppose the algorithm has derived $P_1 \vdash_{\mathcal{E}} t_1 : \pi_1$ and $P_2 \vdash_{\mathcal{E}} t_2 : \pi_2$ as principal pairs for t_1 and t_2, respectively, and that $\pi_1 = \sigma{\to}\tau$. Thus the demanded type σ is in $\mathcal{T}_1$ and the provided type π_2 is in $\mathcal{T}_2$. In order to be consistent, the result of the unification of σ and π_2 – a chain *Ch* – should always be such that $\mathit{Ch}(\pi_2) \in \mathcal{T}_1$. However, if $\pi_2 \notin \mathcal{T}_{\mathrm{C}}$, then in general $\mathit{Ch}(\pi_2) \notin \mathcal{T}_1$. To overcome this difficulty, an algorithm $\mathit{to}\mathcal{T}_C$ will be inserted that, when applied to the type ρ, returns a chain of operations that removes, if possible, intersections in ρ. This can be understood by the observation that, for example, $((\sigma{\to}\sigma){\to}\sigma{\to}\sigma){\to}\sigma$ is a substitution instance of $((\varphi_1{\to}\varphi_1){\to}\varphi_2) \cap (\varphi_3{\to}\varphi_4{\to}\varphi_4){\to}\varphi_5$. Note that if quantifiers appear in ρ, $\mathit{to}\mathcal{T}_C(\rho)$ should fail, since quantifiers that appear before an arrow cannot be removed by any of the operations on types defined above. Finally,

$$\mathit{unify}_2^{\forall}(\sigma, S_2(\pi_2), S_2(P_2))$$

is called (with $S_2 = \mathit{to}\mathcal{T}_C(\pi_2)$). The basis $S_2(P_2)$ is needed to calculate the expansion of $S_2(\pi_2)$ in case σ is an intersection type.

Definition 6.2 The function $\mathit{to}\mathcal{T}_C : \mathcal{T}_2 \to \mathcal{S}$ is defined by:

$$
\begin{array}{lll}
\mathit{to}\mathcal{T}_C(\sigma) & = & [\mathit{Id}_S], \quad \text{if } \sigma \in \mathcal{T}_{\mathrm{C}} \\
\mathit{to}\mathcal{T}_C((\sigma_1 \cap \cdots \cap \sigma_n){\to}\mu) & = & S' \circ S_n, \text{ otherwise,}
\end{array}
$$

where $S_i = \mathit{unify}(S_{i-1}(\sigma_1), S_{i-1}(\sigma_{i+1})) \circ S_{i-1}, (1 \leq i \leq n{-}1, \text{ with } S_0 = \mathit{Id}_S)$
$S' = \mathit{to}\mathcal{T}_C(S_n(\mu))$

(Again, notice that $to\mathcal{T}_C(\sigma)$ fails if σ contains '$\forall$'.)

The algorithm $unify_2^{\forall}$ is called with the types σ and ρ', the latter being ρ in which the intersections are removed (so $\rho' = to\mathcal{T}_C(\rho)(\rho)$; notice that $to\mathcal{T}_C(\rho)$ is an operation on types that removes all intersections in ρ, and needs to be applied to ρ). Since none of the derivation rules, nor one of the operations, allows for the removal of a quantifier that occurs *inside* a type, if $\sigma = \forall\vec{\alpha}.\sigma'$, the unification of σ with ρ' will not remove the '$\forall\vec{\alpha}$' part.

The following definition presents the main unification algorithm, $unify_2^{\forall}$. It gets, typically, called as

$$unify_2^{\forall}(\sigma, S_2(\pi_2), S_2(P_2))$$

during the calculation of the principal pair for an application; after deriving $\langle P_1, \pi_1\rangle$ and $\langle P_2, \pi_2\rangle$ as principal pairs for t_1 and t_2, respectively, with $\pi_1 = \sigma\rightarrow\tau$ and $S_2 = to\mathcal{T}_C(\pi_2)$. The basis is needed to calculate the expansion in case σ is an intersection type, as mentioned above.

Definition 6.3 Let $\mathcal{B}$ be the set of all bases, and $\mathcal{C}h$ the set of all chains. The function $unify_2^{\forall}$ is defined by:

$$\begin{array}{lll} unify_2^{\forall}(\varphi, \tau, B) & = & [(\varphi \mapsto \tau)], \\ unify_2^{\forall}((\forall\vec{\alpha_1}.\sigma_1)\cap\ldots\cap(\forall\vec{\alpha_n}.\sigma_n), \tau, B) & = & [Ex, S_n], \quad \text{otherwise} \end{array}$$

where
$$\begin{array}{rcl} Ex & = & n_{\langle B,\tau\rangle}, \\ \tau_1\cap\cdots\cap\tau_n & = & Ex(\tau), \text{ and} \\ \text{for every } 1\leq i\leq n,\ S_i & = & unify(S_{i-1}(\sigma_i), \tau_i)\circ S_{i-1} \text{ (with } S_0 = Id_S). \end{array}$$

It is worthwhile to notice that *unify*, $to\mathcal{T}_C$, and $unify_2^{\forall}$ all return chains without liftings or closures. Moreover, both *unify* and $to\mathcal{T}_C$ return a substitution, and the chain returned by $unify_2^{\forall}(\sigma, \tau)$ acts on σ as a substitution: the expansion in the chain is defined for the sake of τ only. Notice also that $unify_2^{\forall}$ does not really return a unifying chain for its first two arguments; to achieve this, also closures would have to be inserted. They are not needed for the present purpose.

The procedure $unify_2^{\forall}$ fails when *unify* fails, and $to\mathcal{T}_C$ fails when either *unify* fails or when the argument contains '$\forall$'. Because of this relation between $unify_2^{\forall}$ and $to\mathcal{T}_C$ on one side, and *unify* on the other, the procedures defined here are terminating and type assignment in the system defined in this paper is decidable.

Using Property 6.1, the following lemma is shown:

Lemma 6.4 Let Ch be a chain.

i) If $\sigma \in \mathcal{T}_2$, and $Ch(\sigma) = \tau \in \mathcal{T}_C$, then there is a S such that $S\circ to\mathcal{T}_C(\sigma)(\sigma) = \tau$.

ii) If $\sigma \in \mathcal{T}_2$, and $Ch(\sigma) = \tau \in \mathcal{T}_1$, then there is a chain Ch' such that $[to\mathcal{T}_C(\sigma)] * Ch'(\sigma) = \tau$.

PROOF. Easy, using Lem. 3.7 (*ii*) and (*iv*).

7 Principal pairs for terms

In this section, the principal pair for a term t with respect to the environment $\mathcal{E}$ – $pp_{\mathcal{E}}(t)$ – is defined, consisting of basis P and type π. In Thm. 7.7 it will be shown that, for every term, this is indeed the principal one.

Definition 7.1 Let t be a term in $T(\mathcal{F}, \mathcal{X})$. $pp_{\mathcal{E}}(t) = \langle P, \pi\rangle$, with $\pi \in \mathcal{T}_2$, is defined, using $unify_2^{\forall}$, by induction to the structure of terms through:

(x) : Then $pp_{\mathcal{E}}(x) = \langle\{x{:}\varphi\}, \varphi\rangle$.

(F) : $pp_{\mathcal{E}}(\mathsf{F}) = \langle\emptyset, \mathcal{E}(\mathsf{F})\rangle$.

$(t_1\,t_2)$: Let $pp_{\mathcal{E}}(t_1) = \langle P_1, \pi_1\rangle$, $pp_{\mathcal{E}}(t_2) = \langle P_2, \pi_2\rangle$ (choose, if necessary, trivial variants such that these pairs are disjoint), and $S_2 = to\mathcal{T}_C(\pi_2)$, then

$(\pi_1 = \varphi)$: $pp_{\mathcal{E}}(t_1\,t_2) = \langle P, \pi\rangle$, where

$$\begin{aligned}\langle P, \pi\rangle &= \langle S_1(P_1 \cap S_2(P_2)), \varphi'\rangle,\\ S_1 &= (\varphi \mapsto S_2(\pi_2) \to \varphi'),\ \text{and}\\ \varphi' &\ \text{is a fresh variable.}\end{aligned}$$

$(\pi_1 = \sigma \to \tau)$: $pp_{\mathcal{E}}(t_1\,t_2) = \langle P, \pi\rangle$, provided P and π contain no unbound occurrences of αs, where

$$\begin{aligned}\langle P, \pi\rangle &= \langle S(P_1 \cap Ex(S_2(P_2))), S(\tau)\rangle,\\ [Ex, S] &= unify_2^{\forall}(\sigma, S_2(\pi_2), S_2(P_2)).\end{aligned}$$

$(\mathsf{share}\,t_1\,\mathsf{via}\,x\,\mathsf{in}\,t_2)$: Let $pp_{\mathcal{E}}(t_i) = \langle P_i, \pi_i\rangle$, for $i = 1, 2$. Then either:

– (x occurs in t_1). Then there exists $P', \sigma \in \mathcal{T}_1$ such that $P_1 = P', x{:}\sigma$. Let $S_2 = to\mathcal{T}_C(\pi_2)$. Then

$$pp_{\mathcal{E}}(\mathsf{share}\,t_1\,\mathsf{via}\,x\,\mathsf{in}\,t_2) = \langle P, \pi\rangle,$$

provided P and π contain no unbound occurrences of αs, where

$$\begin{aligned}\langle P, \pi\rangle &= \langle S(P' \cap Ex(S_2(P_2))), S(\pi_1)\rangle\\ [Ex, S] &= unify_2^{\forall}(\sigma, S_2(\pi_2), S_2(P_2)).\end{aligned}$$

– (x does not occur in t_1). Then

$$pp_{\mathcal{E}}(\mathsf{share}\,t_1\,\mathsf{via}\,x\,\mathsf{in}\,t_2) = \langle P_1, \pi_1\rangle.$$

$(\mathsf{cycle}\,\langle \overrightarrow{x_i = t_i}\rangle\,\mathsf{in}\,t')$: Let, for $1 \le i \le n$, $pp_{\mathcal{E}}(t_i) = \langle P_i, \pi_i\rangle$, and $pp_{\mathcal{E}}(t') = \langle P', \pi'\rangle$, and assume, without loss of generality, that these pairs share no type variables. Let

$$P_i = P^i, x_1{:}\rho_1^i, \ldots, x_n{:}\rho_n^i$$

Let S be such that $S(\pi_i) = \tau_i \in \mathcal{T}_{\mathrm{C}}$, and $S(\rho_j^i) = \mu_j^i \in \mathcal{T}_{\mathrm{C}}$, for all $1 \le i, j \le n$, and let

$$S_i = unify(S_{i-1}(\mu_i^i), S_{i-1}(\tau_i)) \circ S_{i-1}$$

(with $S_0 = Id_S$). Then

$$pp_{\mathcal{E}}(\mathsf{cycle}\,\langle \overrightarrow{x_i = t_i}\rangle\,\mathsf{in}\,t') = S_n \circ S(\langle P' \cap P_1 \cap \ldots \cap P_n, \pi'\rangle).$$

(Notice that S can be built out of $to\mathcal{T}_C(\pi_i)$, $to\mathcal{T}_C(\rho_j^i)$, and unification.)

Since *unify* or $unify_2^{\forall}$ may fail, not every term has a principal pair.

Notice that closures are not needed when calculating the new basis and type.

Notice that, if $pp_{\mathcal{E}}(t) = \langle P, \pi\rangle$, then $\pi \in \mathcal{T}_2$. For example, the principal pair for **I** with rewrite rule $\mathbf{I}\,x \to x$ is $\langle \emptyset, \varphi \to \varphi\rangle$, so, in particular, it is not $\langle \emptyset, \forall\alpha.\alpha \to \alpha\rangle$. Although one could argue that the latter type is more 'principal' in the sense that it expresses the generic character the principal type is supposed to have, we have chosen to use the former instead. This is mainly for technical reasons: because unification is used in the definition below, using the latter type, we would often be forced to remove the external quantifiers. Both types can be seen as 'principal' though, since $\forall\alpha.\alpha \to \alpha$ can be obtained from $\varphi \to \varphi$ by closure, and $\varphi \to \varphi$ from $\forall\alpha.\alpha \to \alpha$ by lifting.

Example 7.2 Take the rewrite rules and environment

$$\begin{aligned} \mathsf{I}\, x &\to x \\ \mathsf{F}\, z &\to z \\ \mathsf{F}\, z &\to z\, z \end{aligned}$$

$$\begin{aligned} \mathcal{E}(\mathsf{I}) &= \varphi_1 \to \varphi_1 \\ \mathcal{E}(\mathsf{F}) &= (\forall\alpha.\alpha\to\alpha)\to\varphi_1\to\varphi_1 \end{aligned}$$

and using Def. 7.1, the following can easily be checked:

$$\emptyset \vdash_{\mathcal{E}} \mathsf{F}\,\mathsf{I} : \varphi_2 \to \varphi_2, \text{ and } \emptyset \vdash_{\mathcal{E}} \mathsf{I}\,\mathsf{I} : \varphi_2 \to \varphi_2.$$

These types are the principal types for these terms.

Example 7.3 Take the rewrite rules

$$\begin{aligned} \mathsf{G}\, x &\to x\, \mathsf{D} \\ \mathsf{D}\, y &\to y\, y \end{aligned}$$

For these rules to be typeable, we are forced to use:

$$\begin{aligned} \mathcal{E}(\mathsf{G}) &= (((\varphi_2\to\varphi_1)\cap\varphi_2\to\varphi_1)\to\varphi_3)\to\varphi_3 \\ \mathcal{E}(\mathsf{D}) &= (\varphi_2\to\varphi_1)\cap\varphi_2\to\varphi_1 \\ &\quad \text{or } (\forall\alpha.\alpha\to\alpha)\to\varphi_1\to\varphi_1 \end{aligned}$$

The principal derivation for $(x\,\mathsf{D})$ is:

$$\frac{\dfrac{}{\{x{:}\mu\to\varphi_3\} \vdash_{\mathcal{E}} x : \mu\to\varphi_3} \qquad \dfrac{}{\{x{:}\mu\to\varphi_3\} \vdash_{\mathcal{E}} \mathsf{D} : \mu}}{\{x{:}\mu\to\varphi_3\} \vdash_{\mathcal{E}} x\,\mathsf{D} : \varphi_3}$$

(where $\mu = (\varphi_1\to\varphi_2)\cap\varphi_1\to\varphi_2$), so, when typing the left-hand term $(\mathsf{G}\,x)$, we are forced to use

$$\frac{\dfrac{}{\{x{:}\mu\to\varphi_3\} \vdash_{\mathcal{E}} \mathsf{G} : (\mu\to\varphi_3)\to\varphi_3} \quad \dfrac{}{\{x{:}\mu\to\varphi_3\} \vdash_{\mathcal{E}} x : \mu\to\varphi_3}}{\{x{:}\mu\to\varphi_3\} \vdash_{\mathcal{E}} \mathsf{G}\,x : \rho}$$

An attempt to type these rewrite rules using Rank 2 types would fail: since $\mathit{to}\mathcal{T}_C((\sigma\to\tau)\cap\sigma\to\tau)$ will fail on $\mathit{to}\mathcal{T}_C((\sigma\to\tau)\cap\sigma)$, D can only be typed with a Rank 2 type, so the type of x has to be of rank 3, and the type for G has to be of rank 4.

The following lemma is needed in the proof of Thm. 7.7. It states that if a chain maps the principal pairs of terms t_1, t_2 in an application $t_1\,t_2$ to pairs that allow the application itself to be typed, then these pairs can also be obtained by first performing a unification.

Lemma 7.4 [5] *Let $\sigma \in \mathcal{T}_2$, and $pp_{\mathcal{E}}(t_i) = \langle P_i, \pi_i\rangle$, for $i = 1, 2$, such that these pairs are disjoint. Let Ch_1, Ch_2 be chains such that $Ch_1(pp_{\mathcal{E}}(t_1)) = \langle B, \sigma\to\tau\rangle$ and $Ch_2(pp_{\mathcal{E}}(t_2)) = \langle B, \sigma\rangle$. Then there are chains Ch_u and Ch_p, and type $\rho \in \mathcal{T}_2$ such that*

$$\begin{aligned} pp_{\mathcal{E}}(t_1\,t_2) &= Ch_u(\langle P_1\cap P_2, \rho\rangle), \text{ and} \\ Ch_p(pp_{\mathcal{E}}(t_1\,t_2)) &= \langle B, \tau\rangle. \end{aligned}$$

PROOF. Let $S_2 = \mathit{to}\mathcal{T}_C(\pi_2)$, and $\pi_2' = S_2(\pi_2)$, $P_2' = S_2(P_2)$. We distinguish the cases:

($\pi_1 \in \mathcal{T}_C$) : Since $Ch_1(\pi_1) = \sigma \to \tau \in \mathcal{T}_2$, we have that, by Lem. 3.7, $Ch_1(\pi_1) \in \mathcal{T}_C$ and there is a substitution S_1 such that $Ch_1(\pi_1) = S_1(\pi_1)$. Since Ch_1 contains no closure nor expansion, there is at most one lifting L_1 such that $Ch_1 = [S_1, L_1]$, and L_1 can at most be of the kind $<\langle B_1, \sigma \to \tau\rangle, \langle B, \sigma \to \tau\rangle>$, with $B \leq B_1$, and $S_1(P_1) = B_1$.

Since $Ch_2(\pi_2) = \sigma \in \mathcal{T}_C$, by Lem. 6.4(*i*) there is a substitution S_3 such that $Ch_2(\pi_2) = S3\pi_2'$. Again, since Ch_2 contains no closure nor expansion, there is at most one lifting L_2 such that $Ch_2 = [S_3 \circ S_2, L_2]$, and L_2 can at most be of the kind $<\langle B_2, \sigma\rangle, \langle B, \sigma\rangle>$, with $B \leq B_2$, and $S_1(P_2) = B_2$.

Let φ be a fresh type-variable, and assume, without loss of generality, that $S3\varphi = \tau$. Then we have $S_1(\pi_1) = \sigma \to \tau = S3\pi_2' \to \varphi$, so by Property 6.1, there exists substitutions S_u, S' such that $S_u = \mathit{unify}(\pi_1, \pi_2' \to \varphi)$, and $S_1 \circ S_3 = S' \circ S_u = S_3 \circ S_1$.

Notice that, since $B \leq B_1$ and $B \leq B_2$,

$$L = <\langle B_1 \cap B_2, \tau\rangle, \langle B, \tau\rangle>$$

is a lifting. Take $Ch_u = [S_u]$, and $Ch_p = [S_1 \circ S_3, L]$, and $\rho = \varphi$.

($\pi_1 \notin \mathcal{T}_C$) : Then, in particular, there are $\mu_1 \in \mathcal{T}_1, \mu_2 \in \mathcal{T}_2$ such that $\pi_1 = \mu_1 \to \mu_2$.

Notice that $Ch_1(\pi_1) = Ch_1(\mu_1 \to \mu_2) = \sigma \to \tau \in \mathcal{T}_2$, so, by Lem. 3.7(*v*), there are substitution S, and lifting L such that

$$Ch_1(\pi_1) = [S, L](\pi_1).$$

So there are $B' \geq B$, $\sigma \leq \rho_1$, $\rho_2 \leq \tau$ such that

$$\begin{aligned} S(\pi_1) = S(\mu_1 \to \mu_2) &= \rho_1 \to \rho_2, \text{ (So } S(\mu_1) = \rho_1\text{), and} \\ L &= <\langle B', \rho_1 \to \rho_2\rangle, \langle B, \sigma \to \tau\rangle>. \end{aligned}$$

In particular,

$$\begin{aligned} \pi_1 &= \mu_1 \to \mu_2, \text{ so} && (\mu_1 \in \mathcal{T}_1, \mu_2 \in \mathcal{T}_2) \\ \pi_1 &= (\forall\overrightarrow{\alpha_1}.\delta_1 \cap \cdots \cap \forall\overrightarrow{\alpha_m}.\delta_m) \to \mu_2, \text{ so} \\ & && (\delta_j \in \mathcal{T}_C\ (1 \leq j \leq m),\ \mu_2 \in \mathcal{T}_2) \\ S(\pi_1) &= (\forall\overrightarrow{\alpha_1}.S(\delta_1) \cap \cdots \cap \forall\overrightarrow{\alpha_m}.S(\delta_m)) \to S(\mu_2) \\ &= \rho_1 \to \rho_2. \end{aligned}$$

Notice that $<\langle B, \sigma\rangle, \langle B, \rho_1\rangle>$ is a lifting, and

$$Ch_2 * [<\langle B, \sigma\rangle, \langle B, \rho_1\rangle>](\pi_2) = \rho_1 \in \mathcal{T}_1.$$

Then, by Lem. 6.4(*ii*), there is a lifting-free chain Ch' such that $Ch'(\pi_2') = \rho_1$. Then

$$\begin{aligned} Ch' &= [Ex', S', \overrightarrow{Cl}], \\ \text{with } Ex'(\pi_2') &= S_1'(\pi_2') \cap \cdots \cap S_n'(\pi_2'), \quad (\forall \text{ free}) \\ [Ex', S'](\langle P_2', \pi_2'\rangle) &= \langle B_1, \nu\rangle, \\ \text{with } \nu &= S'(S_1'(\pi_2')) \cap \cdots \cap S'(S_n'(\pi_2')), \text{"} \\ \overrightarrow{Cl}(\langle B, \nu\rangle) &= \langle B, \rho_1\rangle, \\ \text{with } \rho_1 &= \forall\overrightarrow{\alpha_1}.\nu_2^1 \cap \cdots \cap \forall\overrightarrow{\alpha_m}.\nu_2^m \end{aligned}$$

So, for $1 \leq j \leq m$, $S(\delta_j) = \nu_2^j = S'(S_j'(\pi_2'))$, and, by Property 6.1, there exists substitutions S_u^j, S^j such that

$$\begin{aligned} S_u^j &= \mathit{unify}(\delta_j, \pi_2') \\ S(\delta_j) = S^j(S_u^j(\delta_j)) = \nu_2^j &= S^j(S_u^j(\pi_2')) = S'(S_j'(\pi_2')) \end{aligned}$$

Since the substitutions S_u^j agree on type-variables, without loss of generality, we can assume that exists substitutions $S_1, \ldots, S_m$ such that

$$\begin{aligned} S_i &= \mathit{unify}(S_{i-1}(\delta_i), \pi_2') \circ S_{i-1} \\ &\quad \text{(with } S_0 = Id_S\text{), and} \\ \text{for } 1 \leq j \leq m,\ S^j(S_n(\delta_j)) &= \nu_2^j = S^j(S_n(\pi_2')) \end{aligned}$$

so, by Def. 6.3

$$\mathit{Ch}_u' = \mathit{unify}_2^{\forall}((\forall\overrightarrow{\alpha_1}.\delta_1) \cap \ldots \cap (\forall\overrightarrow{\alpha_n}.\delta_n), \pi_2', P_2').$$

exists. Take

$$\begin{aligned} \mathit{Ch}_u &= [S_2] * \mathit{Ch}_u', \\ \mathit{Ch}_p &= [S^1 \circ \cdots \circ S^m, <\langle B, \rho_2\rangle, \langle B, \tau\rangle>], \text{ and} \\ \rho &= \mu_2. \end{aligned}$$

Notice that, since B can be assumed to not contain free occurrences of αs, the last chain is well-defined.

Similarly, we can show the following property

Lemma 7.5 Let $\sigma \in \mathcal{T}_2$, and $pp_{\mathcal{E}}(t_1) = \langle P_1 \cup \{x{:}\rho\}, \pi_1\rangle$, and $pp_{\mathcal{E}}(t_2) = \langle P_2, \pi_2\rangle$, such that these pairs are disjoint. Let Ch_1, Ch_2 be chains such that

$$\mathit{Ch}_1(pp_{\mathcal{E}}(t_1)) = \langle B \cap \{x{:}\sigma\}, \tau\rangle \ \&\ \mathit{Ch}_2(pp_{\mathcal{E}}(t_2)) = \langle B, \sigma\rangle.$$

Then there are chains Ch_u and Ch_p such that

$$\begin{aligned} pp_{\mathcal{E}}(\mathsf{share}\, x\, \mathsf{via}\, t_1\, \mathsf{in}\, t_2) &= \mathit{Ch}_u(\langle P_1 \cap P_2, \pi_1\rangle), \text{ and} \\ \mathit{Ch}_p(pp_{\mathcal{E}}(\mathsf{share}\, x\, \mathsf{via}\, t_1\, \mathsf{in}\, t_2)) &= \langle B_1 \cap B_2, \tau\rangle. \end{aligned}$$

The main result of this section then becomes the soundness and completeness result for $pp_{\mathcal{E}}$.

Theorem 7.6 (SOUNDNESS OF $pp_{\mathcal{E}}$) *If $pp_{\mathcal{E}}(t) = \langle P, \pi\rangle$, then $P \vdash_{\mathcal{E}} t{:}\pi$.*

PROOF. By induction on the structure of terms.

(x) : Since $pp_{\mathcal{E}}(x) = \langle\{x{:}\varphi\}, \varphi\rangle$, the result follows from rule (Ax).

(F) : Immediate.

$(t_1\, t_2)$: Then $pp_{\mathcal{E}}(t_1) = \langle P_1, \pi_1\rangle$, $pp_{\mathcal{E}}(t_2) = \langle P_2, \pi_2\rangle$, for some P_1, P_2, and $P_1 \vdash_{\mathcal{E}} t_1{:}\pi_1$ and $P_2 \vdash_{\mathcal{E}} t_2{:}\pi_2$ by induction. Then either:

$(\pi_1 \in \mathcal{T}_C)$: Then $pp_{\mathcal{E}}(t_1\, t_2) = \langle S(P_1 \cap P_2), S(\varphi)\rangle$, where

$$\begin{aligned} S &= S_1 \circ S_2, \\ S_1 &= \mathit{unify}(\pi_1, S_2(\pi_2) \rightarrow \varphi), \\ S_2 &= \mathit{to}\mathcal{T}_C(\pi_2), \text{ and} \\ \varphi &\text{ is a fresh variable} \end{aligned}$$

Then, by the soundness lemmas above,

$$S(P_1) \vdash_{\mathcal{E}} t_1{:}S(\pi_1) \text{ and } S(P_2) \vdash_{\mathcal{E}} t_2{:}S(\pi_2).$$

Since $S_1(\pi_1) = S_1(S_2(\pi_2) \rightarrow \varphi)$, and S_2 does not affect any variable in $\langle P_1, \pi_1\rangle$ or φ, we can also state that $S(\pi_1) = S(\pi_2 \rightarrow \varphi)$. So, by rule $(\rightarrow E)$,

$$S(P_1 \cap P_2) \vdash_{\mathcal{E}} t_1\, t_2{:}S(\varphi).$$

$(\pi_1 = \sigma \to \tau\ (\sigma \in \mathcal{T}_1, \tau \in \mathcal{T}_2))$: Similar to the previous part, using *unify*$_2^{\forall}$ rather than *unify*.

$(\mathsf{share}\, t_1\, \mathsf{via}\, x\, \mathsf{in}\, t_2)$: Let $pp_{\mathcal{E}}(t_i) = \langle P_i, \pi_i \rangle$, for $i = 1, 2$. Then either:

- (x occurs in t_1). Then there exists $P', \sigma \in \mathcal{T}_1$ such that $P = P', x{:}\sigma$. Let $S_2 = to\mathcal{T}_C(\pi_2)$. Then

$$pp_{\mathcal{E}}(\mathsf{share}\, t_1\, \mathsf{via}\, x\, \mathsf{in}\, t_2) = \langle P, \pi \rangle,$$

  provided P and π contain no unbound occurrences of αs, where

$$\begin{aligned} \langle P, \pi \rangle &= \langle S(P' \cap Ex(S_2(P_2))), S(\pi_1) \rangle \\ [Ex, S] &= unify_2^{\forall}(\sigma, S_2(\pi_2), S_2(P_2)). \end{aligned}$$

- (x does not occur in t_1). Then

$$pp_{\mathcal{E}}(\mathsf{share}\, t_1\, \mathsf{via}\, x\, \mathsf{in}\, t_2) = \langle P_1, \pi_1 \rangle.$$

$(\mathsf{cycle}\, \langle \overrightarrow{x_i = t_i} \rangle\, \mathsf{in}\, t')$: Let, for $1 \le i \le n$, $pp_{\mathcal{E}}(t_i) = \langle P_i, \pi_i \rangle$, and $pp_{\mathcal{E}}(t') = \langle P', \pi' \rangle$, and assume, without loss of generality, that these pairs share no type variables. Let

$$P_i = P^i, x_1{:}\rho_1^i, \ldots, x_n{:}\rho_n^i$$

Let S be such that $S(\pi_i) = \tau_i \in \mathcal{T}_C$, and $S(\rho_j^i) = \mu_j^i \in \mathcal{T}_C$, for all $1 \le i, j \le n$, and let

$$S_i = unify(S_{i-1}(\mu_i^i), S_{i-1}(\tau_i)) \circ S_{i-1}$$

(with $S_0 = Id_S$). Then

$$pp_{\mathcal{E}}(\mathsf{cycle}\, \langle \overrightarrow{x_i = t_i} \rangle\, \mathsf{in}\, t') = S_n \circ S(\langle P_1, \ldots, P_n \cap, \pi \rangle).$$

(Notice that S can be built out of $to\mathcal{T}_C(\pi_i)$, $to\mathcal{T}_C(\rho_j^i)$, and unification.)

Theorem 7.7 (COMPLETENESS OF $pp_{\mathcal{E}}$) *If $B \vdash_{\mathcal{E}} t{:}\sigma$, then there are a basis P and type π such that $pp_{\mathcal{E}}(t) = \langle P, \pi \rangle$, and there is a chain Ch such that $Ch(\langle P, \pi \rangle) = \langle B, \sigma \rangle$.*

PROOF. By induction on the structure of derivations.

(Ax) : Then $t \equiv x$, $\sigma \in \mathcal{T}_C$, and there is $\tau \in \mathcal{T}_1$ such that $x{:}\tau \in B$ and $\tau \le \sigma$. Also, $\pi = \varphi$ and $P = \{x{:}\varphi\}$. Since $\tau \le \sigma$, $B \le \{x{:}\sigma\}$, so $<\langle \{x{:}\sigma\}, \sigma \rangle, \langle B, \sigma \rangle>$ is a lifting. Take

$$Ch = [\varphi \mapsto \sigma, <\langle \{x{:}\sigma\}, \sigma \rangle, \langle B, \sigma \rangle>].$$

$(\cap I)$: Then $\sigma = \sigma_1 \cap \cdots \cap \sigma_n$, and, for $1 \le i \le n$, $B \vdash_{\mathcal{E}} t{:}\sigma_i$. Let $pp_{\mathcal{E}}(t) = \langle P, \pi \rangle$, and let $Ex = n_{\langle P, \pi \rangle}$, then $Ex(\langle P, \pi \rangle) = \langle P_1 \cap \ldots \cap P_n, \pi_1 \cap \cdots \cap \pi_n \rangle$, with each pair $\langle P_i, \pi_i \rangle$ a trivial variant of $\langle P, \pi \rangle$. So, without loss of generality, we can even say $pp_{\mathcal{E}}(t) = \langle P_i, \pi_i \rangle$, for all $1 \le i \le n$. By induction, there exist chains $Ch_1, \ldots, Ch_n$ such that for $1 \le i \le n$, $Ch_i(\langle P_i, \pi_i \rangle) = \langle B, \sigma_i \rangle$. By Lem. 3.7 (*i*), $Ch_i = [S_i, \overrightarrow{Cl}_i]$. Take $Ch = [Ex, S_1 \circ \cdots \circ S_n, \overrightarrow{Cl}_n, \ldots, \overrightarrow{Cl}_n]$.

$(\mathcal{F})$: Then $t \equiv \mathsf{F}$. There is a chain Ch such that $Ch(\mathcal{E}(\mathsf{F})) = \sigma$,and $pp_{\mathcal{E}}(\mathsf{F}) = \langle \emptyset, \mathcal{E}(\mathsf{F}) \rangle$. Take the lifting $L = <\langle \emptyset, \sigma \rangle, \langle B, \sigma \rangle>$, then $Ch(pp_{\mathcal{E}}(\mathsf{F})) = \langle B, \sigma \rangle$.

$(\to E)$: Then $t \equiv t_1\, t_2$, and $B \vdash_{\mathcal{E}} t_1{:}\tau \to \sigma$, and $B \vdash_{\mathcal{E}} t_2{:}\tau$ for some $\tau \in \mathcal{T}_2$. By induction, for $i = 1, 2$, there are P_i, π_i, and chain Ch_i such that

$$\begin{aligned} pp_{\mathcal{E}}(t_i) &= \langle P_i, \pi_i \rangle, \\ Ch_1(pp_{\mathcal{E}}(t_1)) &= \langle B, \tau \to \sigma \rangle, \text{ and} \\ Ch_2(pp_{\mathcal{E}}(t_2)) &= \langle B, \tau \rangle. \end{aligned}$$

Then, by Lem. 7.4, there is a chain Ch such that

$$Ch(pp_{\mathcal{E}}(t_1\, t_2)) = \langle B, \sigma \rangle.$$

($\forall I$) : Then $\sigma = \forall\alpha.\tau[\alpha/\varphi]$, and $B \vdash_{\mathcal{E}} t{:}\tau$. Let $pp_{\mathcal{E}}(t) = \langle P, \pi\rangle$, then by induction $Ch'(\langle P, \pi\rangle) = \langle B, \tau\rangle$, for some chain Ch'. Take $Cl = \langle \tau, \varphi\rangle$, and $Ch = Ch' * [Cl]$.

(share) : Then $t \equiv (\mathsf{share}\, t_1\, \mathsf{via}\, x\, \mathsf{in}\, t_2)$. Then there exists a τ such that $B, x{:}\tau \vdash_{\mathcal{E}} t_1{:}\sigma$ and $B \vdash_{\mathcal{E}} t_2{:}\tau$. Let $pp_{\mathcal{E}}(t_i) = \langle P_i, \pi_i\rangle$, for $i = 1, 2$. Then, by induction, there are chains Ch_1, Ch_2 such that

$$\begin{aligned} Ch_1(\langle P_1, \pi_1\rangle) &= \langle B \cup \{x{:}\tau\}, \sigma\rangle \text{ and} \\ Ch_2(\langle P_2, \pi_2\rangle) &= \langle B, \tau\rangle. \end{aligned}$$

Then either:

- (x occurs in t_1). Then $P_1 = P, x{:}\rho$, and, by Lem. 7.5, there is a chain Ch such that

$$Ch(pp_{\mathcal{E}}(\mathsf{share}\, x\, \mathsf{via}\, t_1\, \mathsf{in}\, t_2)) = \langle B_1 \cap B_2, \tau\rangle.$$

- (x does not occur in t_1). Then

$$pp_{\mathcal{E}}(\mathsf{share}\, t_1\, \mathsf{via}\, x\, \mathsf{in}\, t_2) = \langle P_1, \pi_1\rangle.$$

  Take $Ch = Ch_1$.

(cycle) : Then $t \equiv (\mathsf{cycle}\, \langle \overrightarrow{x_i = t_i} \rangle\, \mathsf{in}\, t')$. Then there are $\sigma_1, \ldots, \sigma_n \in \mathcal{T}_{\mathrm{C}}$ such that $B, \overrightarrow{x_i{:}\sigma_i} \vdash_{\mathcal{E}} t'{:}\tau$ and $B, \overrightarrow{x_i{:}\sigma_i} \vdash_{\mathcal{E}} t_i{:}\sigma_i$, for $1 \le i \le n$. Let $pp_{\mathcal{E}}(t') = \langle P', \pi'\rangle$ and $pp_{\mathcal{E}}(t_i) = \langle P_i, \pi_i\rangle$, for $1 \le i \le n$. Then, by induction, there are chains $Ch_1, \ldots, Ch_n, Ch'$ such that

$$\begin{aligned} Ch_i(pp_{\mathcal{E}}(t_i)) &= \langle B \cup \{\overrightarrow{x_i{:}\sigma_i}\}, \sigma_i\rangle \text{ for } 1 \le i \le n, \text{ and} \\ Ch' pp_{\mathcal{E}}(t') &= \langle B \cup \{\overrightarrow{x_i{:}\sigma_i}\}, \tau\rangle \end{aligned}$$

Let $P_i = P^i, x_1{:}\rho_1^i, \ldots, x_n{:}\rho_n^i$. Since $Ch_i(\rho_1^i) = \sigma_i \in \mathcal{T}_{\mathrm{C}}$, by Lem. 3.7(*iii*), Ch_i acts like a substitution on ρ_1^i, and we can say that there exist $Ch^1, \ldots, Ch^n$, $S^1, \ldots, S^n$ such that

$$Ch_i = Ch^i * [S^i]$$

Since $S^i(\rho_j^i) = \sigma_j$ for all $1 \le j \le n$, by Prop. 6.1, we can assume that all S^i are one and the same S. Then, using *to*$\mathcal{T}_C$ and *unify* we can define S_u such that

$$pp_{\mathcal{E}}(\mathsf{cycle}\, \langle \overrightarrow{x_i = t_i} \rangle\, \mathsf{in}\, t') = S_u(\langle P' \cap P_1 \cap \ldots \cap P_n, \pi'\rangle),$$

and there exists a S' such that $S = S' \circ S_u$. Take

$$Ch = Ch_1 * \cdots * Ch_n * S'.$$

8 Concluding remarks

On implementation

The results of this paper could be used to implement a type-check algorithm for @TGRS. It should be pointed out that the notion of type assignment as defined in this paper is really a *type-check* system, in the sense that it is not possible to create a type-inference algorithm, based on the approach of this paper. Take for example the rewrite rules

$$\begin{aligned} \mathsf{F}\, x &\to x \\ \mathsf{F}\, x &\to x\, x \end{aligned}$$

A type-inference algorithm could for example type both alternatives separately and try to combine the results found. For the first rule it would find $\mathcal{E}(\mathsf{F}) = \varphi_1 \to \varphi_1$, and for the second $\mathcal{E}(\mathsf{F}) = (\varphi_2 \to \varphi_3) \cap \varphi_2 \to \varphi_3$. It is possible to create the desired type,

$$(\varphi_5 \cap (\varphi_4 \to \varphi_5) \cap \varphi_4) \to \varphi_5,$$

from these other two applying the operations specified in this paper, but this is not part of, for example, the algorithm $pp_{\mathcal{E}}$.

So, it is allowed to give an environment for function symbols that is not a combination of possible environments for the various rules. This implies that, in particular, combining types found for one function symbol defined by several rules, applying the here defined operations, does not always lead to the right solution. It can be that the user 'sees' the right type for the rules, which the type-check algorithm is not capable of deducing, but will be capable of checking on its correctness. This can be seen as a disadvantage of the system, but, on the other hand, it can be considered to be good programming hygiene to explicitly state the types for function definitions.

Although type assignment (and type-checking) using the here defined notion of type assignment is decidable, the complexity of type-checking is bigger than for a system based on Curry-types. The biggest problem arises when checking the type provided for a function symbol. Suppose $l \to r$ is a rewrite rule. One way to implement type-checking for this rule would be to construct the principal pair $\langle P, \pi \rangle$ for the term l and to try to type r using this pair. Let $\sigma_1 \cap \cdots \cap \sigma_n$ be the type assigned to the term-variable x in P. Then, for every occurrence of x in r, some selection of the types in $\sigma_1 \cap \cdots \cap \sigma_n$ should be made. In the worst case the number of possibilities that must be tried is huge: 2^n. There are some more efficient ways to type-check a rule, but the complexity is still exponential. However, in every day programming life n will rarely be larger than 2.

Overloading

The concept of overloading in programming languages is normally used to express that different objects (typically procedures) can have the same identifier. (For another approach to overloading, see [17, 16].) At first sight this seems to be nothing but a tool to obtain programming convenience, but the implementation aspects of languages with overloading are not at all trivial. In functional programming languages, functions are *first-order citizens* which means that they can be handled as any object, like for example numbers. In particular, a function can be passed as argument to another one, or could be its result. Especially in the first case it can occur that at compile time it is not possible to decide which of the several bodies (or pieces of code) for an overloaded identifier should be linked into the object-code. If this decision cannot be made, the compiler should generate code that contains all possible functions and some kind of case-construct that makes it possible to select at run-time which is the code to use. For reasons of efficiency – and to avoid run-time checks on function types – it seems natural to allow for overloaded objects only if at compile time it can be decided which of the different function definitions is meant, since then, for every occurrence of an overloaded symbol, the compiler can decide which of the several function definitions should be linked into the object code.

The intersection type constructor is a good candidate to express overloading. It seems natural to say for example that the type for addition $+$ is $(\mathsf{int} \to \mathsf{int} \to \mathsf{int}) \cap (\mathsf{real} \to \mathsf{real} \to \mathsf{real})$. Bringing the notion of overloading into a formal system for type assignment as defined in this paper implies that the restriction on the types that can be provided by an environment should be dropped; in such a formalism, types provided by the environment should be an intersection type, not just an element of $\mathcal{T}_2$.

However, this extension itself creates strange effects. Let, for example,

$$\begin{aligned} \mathcal{E}(\mathsf{F}) &= (\mathsf{int} \to \mathsf{int} \to \mathsf{int}) \cap (\mathsf{real} \to \mathsf{real} \to \mathsf{real}) \to \alpha \\ \mathcal{E}(+) &= (\mathsf{int} \to \mathsf{int} \to \mathsf{int}) \cap (\mathsf{real} \to \mathsf{real} \to \mathsf{real}) \end{aligned}$$

Then, by the notion of type assignment as defined here, the term $\mathsf{F}+$ can be typed by α. In general, let G be a function symbol that has the type $\sigma \cap \tau \to \rho$, and let H be an overloaded function symbol with

$\mathcal{E}(\mathsf{H}) = \alpha \cap \beta$. Then finding the principal pair for the term $\mathsf{G}\,\mathsf{H}$ requires more than just the kind of unification defined in this paper. In general, there can be several cases, since all possible combinations have to be tried:

- *unify* (σ, α) and *unify* (τ, β) are both successful.
- *unify* (σ, β) and *unify* (τ, α) are both successful.
- *unify*$_2^{\forall}$ $(\sigma \cap \tau, \alpha)$ and *unify*$_2^{\forall}$ $(\sigma \cap \tau, \beta)$ are both successful.
- *unify*$_2^{\forall}$ $(\sigma \cap \tau, \beta)$ fails, *unify*$_2^{\forall}$ $(\sigma \cap \tau, \alpha)$ is successful.
- *unify*$_2^{\forall}$ $(\sigma \cap \tau, \alpha)$ fails, *unify*$_2^{\forall}$ $(\sigma \cap \tau, \beta)$ is successful.

It can even be that more than one of these cases is true at the same time, like for example the first and second. This in particular is troublesome, since it is not obvious at all what in this case the type of $\mathsf{G}\,\mathsf{H}$ should be. One solution for this problem would be to allow, like in [18], for more than one principal pair for a term (notice that this is not the same as saying that a principal type can be an intersection). Another would be to introduce – formally – an extra type constructor $\cap\!\!\!\!\cdot$ with the same meaning as $\cap$, and to define overloading using this notion. Then the unification of $\sigma \cap \tau$ and $\alpha \cap\!\!\!\!\cdot\, \beta$ can be defined as the combination of the results of unifying $\sigma \cap \tau$ and α, and unifying $\sigma \cap \tau$ and β.

A good solution to the aforementioned problem is to *force selection* of one of the function definitions for an overloaded identifier. This can be accomplished by defining, as in Definition 4.3, how a rewrite rule can be typed, but by adding that, for every $\sigma \in \mathcal{T}_2$ such that $\mathcal{E}(\mathsf{F}) \leq \sigma$, all the rewrite rules that define F should be typeable using the type σ, for every occurrence of F. (Another approach would be to introduce a new syntactic construct into the language that is used to separate the rules that define F in groups, and to ask that, for every $\sigma \in \mathcal{T}_2$ such that $\mathcal{E}(\mathsf{F}) \leq \sigma$, there is at least one group of rules that can be typed using σ.) Moreover, it is possible to define, as in rule $(\mathcal{F})$ how a type for a function symbol can be obtained form the one provided by the environment, in the following way:

$$(\mathcal{F}) : \frac{}{B \vdash_{\mathcal{E}} \mathsf{F} : \sigma}\ (\exists \tau \in \mathcal{T}_2, \mathit{Ch}\,[\mathcal{E}(\mathsf{F}) \leq \tau\ \&\ \mathit{Ch}(\tau) = \sigma])$$

Then the term $(\mathsf{F}+)$ mentioned above cannot be typed. This selection is then reflected in the way intersection types are unified. Since only *one* of the types in an 'overloaded' type can be used, the unification should try to unify the demanded type with *each individual type* occurring in the provided type.

Using this definition, the notion of 'principal pair' becomes slightly more complicated. This is best explained by discussing the implementation of the type-checker that is looking for such a pair. Take the well-known function foldr that is defined by

$$\begin{array}{ll} \mathsf{foldr}\ f\ i\ [\,] & = i \\ \mathsf{foldr}\ f\ i\ (a : b) & = f\ a\ (\mathsf{foldr}\ f\ i\ b) \end{array}$$

and can be typed by $(\varphi_1 \to \varphi_2 \to \varphi_2) \to \varphi_2 \to [\varphi_1] \to \varphi_2$. Take the term

$$\mathsf{foldr}\ +\ 1\ [2, 3, 4]$$

then it is clear that this term should be typeable by the type int. When constructing the type assignment for this term, the subterm $(\mathsf{foldr}+)$ is typed. For this term as such the type needed for $+$ cannot be uniquely determined: it is the second argument of foldr that forces the selection. Since there is a chance of success, the type-checker should postpone the decision to reject the term and consider both possibilities simultaneously. This means that formally the term $(\mathsf{foldr}+)$ has *two* principal types.

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