A Filter Model for the $\lambda\mu$ Calculus

(Extended Abstract with proofs in appendix)

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Abstract

We introduce an intersection type assignment system for the pure $\lambda\mu$-calculus, which is invariant under subject reduction and expansion. The system is obtained by describing Streicher and Reus's denotational model of continuations in the category of ω-algebraic lattices via Abramsky's domain logic approach. This provides a tool for showing the completeness of the type assignment system with respect to the continuation models via a filter model construction. We also show that typed $\lambda\mu$-terms in Parigot's system have a non-trivial intersection typing in our system.

1 Introduction

The $\lambda\mu$-calculus is a pure calculus introduced by Parigot [27] to denote classical proofs and to compute with them. It is an extension of the proofs-as-programs paradigm where types can be understood as classical formulas and (closed) terms inhabiting a type as the respective proofs in a variant of Gentzen's natural deduction calculus for classical logic [16]. Since the early days, the study of the syntactic properties of the $\lambda\mu$-calculus has been challenging, motivating the introduction of variants of term syntax, of reduction rules, and of type assignment, as is the case of de Groote's variant of the $\lambda\mu$-calculus [18]. These changes have an impact on the deep nature of the calculus which emerges both in the typed and in the untyped setting [12, 29, 19].

Types are of great help in understanding the computational properties of terms in an abstract way. Although in [7] Barendregt treats the theory of the pure λ-calculus without a reference to types, most of the fundamental results of the theory can be exposed in a quite elegant way by using the Coppo-Dezani intersection type system [10]. This is used by Krivine [22], where the treatment of the pure λ-calculus relies on intersection type assignment systems called $\mathcal{D}$ and $\mathcal{D}\Omega$.

The quest for more expressive notions of type assignment for $\lambda\mu$ is part of an ongoing investigation into calculi for classical logic. In order to come to a characterisation of strong normalisation for Curien and Herbelin's (untyped) sequent calculus $\overline{\lambda}\mu\tilde{\mu}$ [11], Dougherty, Ghilezan and Lescanne presented System $\mathcal{M}^{\cap\cup}$ [14, 15], that defines a notion of intersection and union typing for that calculus. However, in [4] van Bakel showed that this system is not closed under conversion, an essential property of Coppo-Dezani systems; in fact, it is shown that it is *impossible* to define a notion of type assignment for $\overline{\lambda}\mu\tilde{\mu}$ that satisfies that property. In [5] van Bakel brought intersection (and union) types to the context of the (untyped) $\lambda\mu$-calculus, and showed type preservation with respect to $\lambda\mu$-conversion. However, union types are no longer dual to intersection types and play only a marginal role, as was also the intention of [15]. In particular, the normal $(\cup I)$ and $(\cup E)$ rules as used in [6], which are known to create the same soundness problem in the context of the λ-calculus, are not allowed. Moreover, although one can link intersection types with the logical connector *and*, the union types used in [5] bear *no* relation with *or*; one could argue that therefore *union* might perhaps not be the right name to use for this type constructor.

In the view of the above mentioned failure, the result of [5] came as a surprise, and led automatically to the question we answer here: does a filter semantics for $\lambda\mu$ exist?

Building on Girard's ideas [23] and Ong and Stewart's work [25], in [30, 21] the Streicher and Reus proposed a model of both typed and untyped λ-calculi embodying some idea of continuations, including a version of pure $\lambda\mu$. Their model is based on the solution of the domain equations $D = C \to R$ and $C = D \times C$, where R is an arbitrary domain of 'results' (we call the triple (R, D, C) a $\lambda\mu$-model). With respect to this, here we adapt the term-interpretation map of Streicher and Reus [30] to Parigot's original calculus (it is not difficult to do the same for de Groote's $\Lambda\mu$); we deviate from the variant studied in [30], where also continuation terms are included in the syntax.

Following Abramski's approach [1], we reconstruct the initial/final solution of these equations in the category of ω-algebraic lattices by describing compact points and the ordering of these domains as Lindenbaum-Tarski algebras of certain intersection type theories. Types are generated from type variables, the trivial type ω, the connective $\wedge$, plus the domain specific type constructors $\times$ and $\to$. This way, we obtain an extension of the type theory used in [8] which is a *natural equated* intersection type theory in terms of [3] and hence isomorphic to the inverse limit construction of a D_∞ λ-model (as an aside, we observe that this perfectly matches with Theorem 3.1 in [30]). The thus obtained type theory and term interpretation guide the definition of an intersection type assignment system. We prove this to be invariant under conversion, and sound and complete with respect to the validity of type judgements in any $\lambda\mu$-model; this is shown through the filter model which, together with the system, is the main contribution of this paper.

At this point we have to stress the quite different meaning of our types and type assignment system with respect to the system originally presented by Parigot. The types we use here are *not* logical formulas, and the $\wedge$ connective is not conjunction. As is the case with ordinary intersection types, $\wedge$ is not even the left adjoint of $\to$, which instead is the case for the $\times$ connective. Nonetheless, we show there exists a strong connection between Parigot's type assignment and ours. In fact, we show that any typed term in Parigot's system has a non-trivial type (i.e. one that cannot be equated to ω) in our system, and that this is true of all the subjects in its derivation. We interpret this result as evidence that terms that actually represent logical proofs do have a computational meaning in a precise sense. Assuming the model captures relevant computational properties, this might provide a characterisation of strong normalisation for $\lambda\mu$.

Due to space restrictions, proofs are omitted from this paper.

2 The $\lambda\mu$ calculus

In this section we briefly recall Parigot's pure $\lambda\mu$-calculus introduced in [27], slightly changing the notation.

Definition 2.1 (TERM SYNTAX [27]) The sets Trm of *terms* and Cmd of *commands* are defined inductively by the following grammar (where $x \in Var$, a set of *term variables*, and $\alpha \in Name$, a set of *names*, that are both denumerable):

$$\begin{array}{rcll} M, N & ::= & x \mid \lambda x.M \mid MN \mid \mu\alpha.Q & \text{(terms)} \\ Q & ::= & [\alpha]M & \text{(commands)} \end{array}$$

As usual, we consider λ and μ to be binders; we adopt Barendregt's convention on terms, and will assume that free and bound variables are different.

In [27] terms and commands are called unnamed and named terms respectively. In the same place names are called μ-variables, while they are better understood as *continuation variables* (see [30]), but this would be a direct link to the interpretation; we prefer a more neutral terminology.

Definition 2.2 (Substitution [27]) Substitution takes three forms:

term substitution:	$M[N/x]$	(N is substituted for x in M, avoiding capture)
renaming:	$Q[\alpha/\beta]$	(every free occurrence of β in Q is replaced by α)
structural substitution:	$T[\alpha \Leftarrow L]$	(every subterm $[\alpha]N$ of M is replaced by $[\alpha]NL$)

where $M,N,L \in \mathsf{Trm}$, $Q \in \mathsf{Cmd}$ and $T \in \mathsf{Trm} \cup \mathsf{Cmd}$. More precisely, $T[\alpha \Leftarrow L]$ is defined by:

$$([\alpha]M)[\alpha \Leftarrow L] \equiv [\alpha](M[\alpha \Leftarrow L])L$$

whereas in all the other cases it is defined by:

$$\begin{array}{rcl} x[\alpha \Leftarrow L] & \equiv & x \\ (\lambda x.M)[\alpha \Leftarrow L] & \equiv & \lambda x.M[\alpha \Leftarrow L] \\ (MN)[\alpha \Leftarrow L] & \equiv & (M[\alpha \Leftarrow L])(N[\alpha \Leftarrow L]) \\ (\mu\beta.Q)[\alpha \Leftarrow L] & \equiv & \mu\beta.Q[\alpha \Leftarrow L] \\ ([\beta]M)[\alpha \Leftarrow L] & \equiv & [\beta]M[\alpha \Leftarrow L] \end{array}$$

Definition 2.3 (Reduction [27]) The reduction relation $T \to_\mu S$, where $(T,S) \subseteq (\mathsf{Trm} \times \mathsf{Trm}) \cup (\mathsf{Cmd} \times \mathsf{Cmd})$ is defined as the compatible closure of the following rules :

$$\begin{array}{rrcll} (\beta): & (\lambda x.M)N & \to & M[N/x] & \\ (\mu): & (\mu\beta.Q)N & \to & \mu\beta.Q[\beta \Leftarrow N] & \\ (ren): & [\alpha]\mu\beta.Q & \to & Q[\alpha/\beta] & \\ (\mu\eta): & \mu\alpha.[\alpha]M & \to & M & \text{if } \alpha \notin fn(M) \end{array}$$

Parigot's original paper [27] just mentions rule $(\mu\eta)$, while proving confluence of the reduction relation axiomatised by rules $(\beta),(\mu),(ren)$ only. The full reduction relation in Def 2.3 has been proved confluent by Py [28].

Theorem 2.4 (Confluence of $\to_\mu$ [28]) *The reduction relation $\to_\mu$ is confluent.*

Because of Thm. 2.4 the convertibility relation $=_\mu$ determined by $\to_\mu$ is consistent in the usual sense that different normal forms are not equated. If we add the rule

$$(\eta): \quad \lambda x.Mx \;\to\; M \quad \text{if } x \notin fv(M)$$

of the λ-calculus, we obtain a non-confluent reduction relation, that we call $\to_{\mu\eta}$; see [28]§2.1.6 for an example of non-confluence and possible repairs. However, the convertibility relation $=_{\mu\eta}$ induced by $\to_{\mu\eta}$ is consistent (namely non-trivial) by a semantic argument (see Sect. 3). The theory of $=_{\mu\eta}$ is interesting because it validates the untyped version of Ong's equation (ζ) in [26], which has the following form:

$$(\zeta): \quad \mu\alpha.Q \;=\; \lambda x.\mu\alpha.Q[\alpha \Leftarrow x]$$

where $x \notin fv(Q)$. Indeed, for a fresh x we have:

$$\mu\alpha.Q \;\;{}_{\eta}\!\leftarrow\; \lambda x.(\mu\alpha.Q)x \;\to_\mu\; \lambda x.\mu\alpha.Q[\alpha \Leftarrow x]$$

So it is possible to define more reduction rules, but Parigot refrained from that since he aimed at defining a confluent reduction system.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{x{:}(A\to B)\to A \vdash x : (A\to B)\to A \mid \alpha{:}A} \qquad \dfrac{\dfrac{\dfrac{\overline{x{:}(A\to B)\to A, y{:}A \vdash y : A \mid \alpha{:}A, \beta{:}B}}{x{:}(A\to B)\to A, y{:}A \vdash [\alpha]y : \bot \mid \alpha{:}A}\,(\bot)}{x{:}(A\to B)\to A, y{:}A \vdash \mu\beta.[\alpha]y : B \mid \alpha{:}A}\,(\mu)}{x{:}(A\to B)\to A \vdash \lambda y.\mu\beta.[\alpha]y : A\to B \mid \alpha{:}A}\,(\to I)}{x{:}(A\to B)\to A \vdash x(\lambda y.\mu\beta.[\alpha]y) : A \mid \alpha{:}A}\,(\to E)}{x{:}(A\to B)\to A \vdash [\alpha](x(\lambda y.\mu\beta.[\alpha]y)) : \bot \mid \alpha{:}A}\,(\bot)}{x{:}(A\to B)\to A \vdash \mu\alpha.[\alpha](x(\lambda y.\mu\beta.[\alpha]y)) : A \mid}\,(\mu)}{\vdash \lambda x.\mu\alpha.[\alpha](x(\lambda y.\mu\beta.[\alpha]y)) : ((A\to B)\to A)\to A \mid}\,(\to I)$$

Figure 1: A proof of Peirce's Law (due to Ong and Stewart [25])

With $\lambda\mu$ Parigot created a multi-conclusion typing system which corresponds to classical logic; the derivable statements have the shape $\Gamma \vdash M : A \mid \Delta$, where A is the main conclusion of the statement, expressed as the *active* conclusion, and Δ contains the alternative conclusions. The reduction rules for the terms that represent the proofs correspond to proof contractions.

Parigot's type assignment for $\lambda\mu$ is defined by the following natural deduction system; there is a *main*, or *active*, conclusion, labelled by a term of this calculus, and the alternative conclusions are labelled names.

Definition 2.5 (Typing rules for $\lambda\mu$ [27]) Types are those of the simply typed λ-calculus, extended with the type constant $\bot$ (essentially added to express negation), *i.e.*:

$$A, B \ ::= \ \varphi \mid \bot \mid A \to B \quad (A \neq \bot)$$

The type assignment rules are:

$$(Ax) : \dfrac{}{\Gamma \vdash x : A \mid \Delta}\,(x{:}A \in \Gamma) \qquad (\bot) : \dfrac{\Gamma \vdash M : B \mid \beta{:}B, \Delta}{\Gamma \vdash [\beta]M : \bot \mid \beta{:}B, \Delta} \qquad (\mu) : \dfrac{\Gamma \vdash Q : \bot \mid \alpha{:}A, \Delta}{\Gamma \vdash \mu\alpha.Q : A \mid \Delta}$$

$$(\to I) : \dfrac{\Gamma, x{:}A \vdash M : B \mid \Delta}{\Gamma \vdash \lambda x.M : A\to B \mid \Delta} \qquad (\to E) : \dfrac{\Gamma \vdash M : A\to B \mid \Delta \quad \Gamma \vdash N : A \mid \Delta}{\Gamma \vdash MN : B \mid \Delta}$$

We write $\Gamma \vdash_{\mathrm{P}} N : A \mid \Delta$ for statements derivable using these rules.

We can think of $[\alpha]M$ as storing the type of M amongst the alternative conclusions by giving it the name α.

Example 2.6 As an example illustrating the fact that this system is more powerful than the system for the λ-calculus, Fig. 1 contains a proof of Peirce's Law.

3 Semantics

The semantics considered here is due to Streicher and Reus [30]. Their idea is to work in the category $\mathcal{N}_R$ of 'negated' domains of the shape $A \to R$, where R is a parameter for the domain of results. In such a category, continuations are directly modelled and treated as the fundamental concept, providing a semantics both to Felleisen's $\lambda\mathcal{C}$-calculus and to a variant of $\lambda\mu$, with three sorts of terms, instead of two.

In this section we adapt such a semantics to Parigot's original $\lambda\mu$, which does not have continuation terms $M_1 :: \ldots :: M_k$. We rephrase the model definition in the setting of ordinary categories of domains, getting something similar to the Hindley-Longo 'syntactical models', but without pretending to achieve the general definition of what a $\lambda\mu$-model is, an issue which is dealt with in [26, 20].

Definition 3.1 ($\lambda\mu$-MODEL) We say that a triple (R, D, C) is a $\lambda\mu$-model in a category of domains $\mathcal{D}$ if $R \in \mathcal{D}$ is a fixed domain of *results* and D, C (called domains of *denotations* and of *continuations* respectively), are solutions in $\mathcal{D}$ of the equations:

$$\begin{cases} D &= C \to R \\ C &= D \times C \end{cases} \tag{1}$$

In the terminology of [30] elements of D are *denotations*, while those of C are *continuations*. We refer to these equations as the *continuation domain equations*.

Definition 3.2 (TERM INTERPRETATION) Let (R, D, C) be a $\lambda\mu$-model, and $\mathsf{Env} = (Var \to D) + (Name \to C)$. The interpretation mappings $⟦\cdot⟧^D : \mathsf{Trm} \to \mathsf{Env} \to D$ and $⟦\cdot⟧^C : \mathsf{Cmd} \to \mathsf{Env} \to C$ are mutually defined by the following equations, where $e \in \mathsf{Env}$ and $k \in C$:

$$\begin{array}{lcll}
⟦x⟧^D\, e\, k &=& e\, x\, k & \\
⟦\lambda x.M⟧^D\, e\, k &=& ⟦M⟧^D\, e[x := d]\, k' & \text{where } k = \langle d, k' \rangle \\
⟦MN⟧^D\, e\, k &=& ⟦M⟧^D\, e\, \langle ⟦N⟧^D\, e, k \rangle & \\
⟦\mu\alpha.Q⟧^D\, e\, k &=& d\, k' & \text{where } \langle d, k' \rangle = ⟦Q⟧^C\, e[\alpha := k] \\
⟦[\alpha]M⟧^C\, e &=& \langle ⟦M⟧^D\, e, e\, \alpha \rangle &
\end{array}$$

This definition has (of course), a strong similarity with Bierman's interpretation of $\lambda\mu$ [9]; however, he considers a *typed* version.

In the second equation the assumption $k = \langle d, k' \rangle$ is not restrictive: in particular, if $k = \bot_C = \langle \bot_D, \bot_C \rangle$ then $d = \bot_D$ and $k' = k = \bot_C$. We shall omit the superscripts in $⟦\cdot⟧^C$ and $⟦\cdot⟧^D$ when clear from the context.

Remark 3.3 Let us recall $\Lambda\mu$, the variant of $\lambda\mu$ deviced by de Groote [18] (see also [19]), where there is no distinction between terms and commands:

$$\Lambda\mu\text{-}\mathsf{Trm}: \qquad M, N ::= x \mid \lambda x.M \mid MN \mid \mu\alpha.M \mid [\alpha]M$$

Then, given a solution D, C of the continuation domain equations, it is possible to define a similar interpretation map $⟦\cdot⟧^D : \Lambda\mu\text{-}\mathsf{Trm} \to \mathsf{Env} \to D$, where the first three clauses are the same as in Def. 3.2, while the last two become:

$$\begin{array}{lcl}
⟦\mu\alpha.M⟧^D\, e\, k &=& ⟦M⟧^D\, e[\alpha := k]\, k \\
⟦[\alpha]M⟧^D\, e\, k &=& ⟦M⟧^D\, e\, (e\, \alpha)
\end{array}$$

The distinctive feature of $\Lambda\mu$ is that in $\mu\alpha.M$ the subterm M need not be a named term (of the shape $[\alpha]L$), and that there exist terms of the form $\lambda x.[\alpha]M$, which play a key role in [18] to inhabit the type $\neg\neg A \to A$ with a closed term.

The analogous of Thm. 3.7 below can be established for $\Lambda\mu$ and the above semantics, with the proviso that the renaming reduction rule:

$$[\alpha]\mu\beta.M \to M[\alpha/\beta]$$

is unsound for arbitrary M. On the contrary, renaming is sound in the expected contexts:

$$[\alpha]\mu\beta.[\gamma]M \to ([\gamma]M)[\alpha/\beta] \quad \text{and} \quad \mu\alpha[\alpha]\mu\beta.M \to \mu\alpha.M[\alpha/\beta]$$

where the second one is essentially the same as Parigot's. The form of the renaming rule is a delicate point, for which we refer the reader to [19].

Since Def. 3.2 does not coincide exactly with Streicher and Reus's, we check below that it actually models $\lambda\mu$ convertibility.

Proposition 3.4 *i)* $[\![M[N/x]]\!]\, e = [\![M]\!]\, e[x := [\![N]\!]\, e]$
ii) $[\![M[\alpha/\beta]]\!]\, e = [\![M]\!]\, e[\beta := e\,\alpha]$

The semantics satisfies the following "swapping continuations" equation[1]:

Lemma 3.5 $[\![\mu\alpha.[\beta]M]\!]\, e\, k = [\![M]\!]\, e[\alpha := k]\ (e[\alpha := k]\beta)$.

We can now show:

Lemma 3.6 $[\![M[\alpha \Leftarrow N]]\!]\, e\, k = [\![M]\!]\, e[\alpha := \langle [\![N]\!]\, e, e\,\alpha\rangle]\, k$.

Theorem 3.7 (Soundness of $\lambda\mu$) $\vdash_{\lambda\mu} M = N \Rightarrow [\![M]\!] = [\![N]\!]$

The soundness property holds also in case rule (η) is taken into account. The proof of Thm. 3.7 can in fact be easily extended with a further induction case.

Proposition 3.8 (Soundness of $\lambda\mu\eta$) *The (η) rule is sound with respect to the semantics, i.e.* $[\![\lambda x.Mx]\!] = [\![M]\!]$ *if*

4 The filter domain

We look for a solution of the continuation domain equations (1) in the category **ALG** of ω-algebraic lattices and Scott-continuous functions (see the classic [17]), described as a domain of filters generated by a suitable intersection type theory (below, a *domain* is an object of **ALG**). Filter models appeared first in the theory of intersection type assignment in [8] (see also [2]). A general theory of Stone duality for domains is treated in [1]; more recent contributions for filter models and recursive domain equations can be found in [13, 3].

Definition 4.1 (Intersection Type Language and Theory) An *intersection type language* is a countable set of types $\mathcal{L}$ such that: $\omega \in \mathcal{L}$ and $\sigma, \tau \in \mathcal{L} \Rightarrow \sigma \wedge \tau \in \mathcal{L}$.

An *intersection type theory* $\mathcal{T}$ is a collection of inequalities among types in a language $\mathcal{L}$, closed under the following axioms and rules:

$$\frac{}{\sigma \leq \sigma} \qquad \frac{}{\sigma\wedge\tau \leq \sigma} \qquad \frac{}{\sigma\wedge\tau \leq \tau} \qquad \frac{}{\sigma \leq \omega} \qquad \frac{\sigma \leq \tau \quad \tau \leq \rho}{\sigma \leq \rho} \qquad \frac{\rho \leq \sigma \quad \rho \leq \tau}{\rho \leq \sigma\wedge\tau}$$

We write $\sigma \leq_{\mathcal{T}} \tau$ for $\sigma \leq \tau \in \mathcal{T}$, or just $\sigma \leq \tau$ when $\mathcal{T}$ is understood. The inequality $\leq$ is a preorder over $\mathcal{L}$ and $\mathcal{L}/_{\leq}$ is an inf-semilattice with top $[\omega]$ (the equivalence class of ω).

Let $\mathcal{F}$ be the set of subsets $F \subseteq \mathcal{L}$ which are non-empty, upward closed and closed under finite intersection. Then one has $\mathcal{F} \simeq \mathsf{Filt}(\mathcal{L}/_{\leq})$, the set of filters over $\mathcal{L}/_{\leq}$, so that the elements of $\mathcal{F}$ are called *filters* as well. A *principal filter* is a set $\uparrow\sigma = \{\tau \in \mathcal{L} \mid \sigma \leq \tau\}$; we write $\mathcal{F}_p$ for the set of principal filters; clearly $\mathcal{F}_p \subseteq \mathcal{F}$.

The poset $(\mathcal{F}, \subseteq)$ is an ω-algebraic lattice, whose set of *compact points* $\mathcal{K}(\mathcal{F})$ is $\mathcal{F}_p$. By this, $\mathcal{F}$ is called a *filter domain*. On the other hand, any ω-algebraic lattice X can be presented as a filter domain, due to the isomorphism $X \simeq \mathsf{Filt}(\mathcal{K}^{op}(X))$ where $\mathcal{K}^{op}(X)$ is $\mathcal{K}(X)$ (the set of compact elements in X) with the inverse ordering. This is an instance of Stone duality in the case of the category **ALG**, which is a particular and simpler case than the general ones studied in [1].

[1]The equation in [30] is actually $[\![\mu\alpha.[\beta]M]\!]\, e\, k = [\![M]\!]\, e[\alpha := k]\ (e\,\beta)$, but this is a typo.

Observing that

$$\sigma \leq \tau \iff \uparrow\tau \subseteq \uparrow\sigma$$

we have that $\mathcal{F}_p \simeq \mathcal{L}/_{\leq^{op}}$ considered as sup-semilattices, where $\uparrow\sigma \sqcup \uparrow\tau = \uparrow(\sigma \wedge \tau)$. Suppose that, given $X \in \mathbf{ALG}$, we have established the isomorphism $\mathcal{L}/_{\leq^{op}} \simeq \mathcal{K}(X)$ (as sup-semilattices) or equivalently $\mathcal{L}/_{\leq} \simeq \mathcal{K}^{op}(X)$ (as inf-semilattices), then we obtain

$$\mathcal{F} \simeq \mathsf{Filt}(\mathcal{L}/_{\leq}) \simeq \mathsf{Filt}(\mathcal{K}^{op}(X)) \simeq X.$$

Lemma 4.2 Let $\mathcal{T}$ be an intersection type theory over the language $\mathcal{L}$ and $(\mathcal{L}, \leq)$ the induced preorder. If X is a domain and $\Theta : \mathcal{L} \to \mathcal{K}(X)$ a surjective map such that

$$\forall \sigma, \tau \in \mathcal{L}\,[\sigma \leq \tau \iff \Theta(\tau) \sqsubseteq \Theta(\sigma)]$$

then $\mathcal{L}/_{\leq} \simeq \mathcal{K}^{op}(X)$ as inf-semilattices.

Definition 4.3 (Type languages) Fix a domain R of results. For $A \in \{R, C, D\}$, we define the languages of intersection types $\mathcal{L}_A$:

$$\begin{array}{ll} \mathcal{L}_R: & \rho ::= v \mid \omega \mid \rho \wedge \rho \\ \mathcal{L}_C: & \kappa ::= \omega \mid \delta \times \kappa \mid \kappa \wedge \kappa \\ \mathcal{L}_D: & \delta ::= v \mid \omega \mid \kappa \to \rho \mid \delta \wedge \delta \end{array}$$

where the type constants $v = v_a$ are in one-to-one correspondence with $a \in \mathcal{K}(R)$.

Definition 4.4 (Type theories) The theories $\mathcal{T}_R, \mathcal{T}_C$ and $\mathcal{T}_D$ are the least intersection type theories closed under the following axioms and rules, inducing the preorders $\leq_R, \leq_C$ and $\leq_D$ over $\mathcal{L}_R, \mathcal{L}_C$ and $\mathcal{L}_D$ respectively, where, for $A \in \{R, C, D\}$, $\sigma =_A \tau$ is defined by $\sigma \leq_A \tau \leq_A \sigma$:

$$\frac{}{v_\perp =_R \omega} \quad \frac{}{v_{a \sqcup b} =_R v_a \wedge v_b} \quad \frac{}{\omega \leq_C \omega \times \omega} \quad \frac{}{\omega \leq_D \omega \to \omega} \quad \frac{}{v =_D \omega \to v}$$

$$\frac{}{(\delta_1 \times \kappa_1) \wedge (\delta_2 \times \kappa_2) \leq_C (\delta_1 \wedge \delta_2) \times (\kappa_1 \wedge \kappa_2)} \quad \frac{}{(\kappa \to \delta_1) \wedge (\kappa \to \delta_2) \leq_D \kappa \to (\delta_1 \wedge \delta_2)}$$

$$\frac{\delta_1 \leq_D \delta_2 \quad \kappa_1 \leq_C \kappa_2}{\delta_1 \times \kappa_1 \leq_C \delta_2 \times \kappa_2} \quad \frac{b \sqsubseteq a \in \mathcal{K}(R)}{v_a \leq_R v_b} \quad \frac{\kappa_2 \leq_C \kappa_1 \quad \rho_1 \leq_R \rho_2}{\kappa_1 \to \rho_1 \leq_D \kappa_2 \to \rho_2}$$

The filter-domain induced by $\mathcal{T}^A$ is $\mathcal{F}^A$, $\uparrow\rho$ is the principal filter generated by ρ, and $\mathcal{F}^A_p$ is the set of principal filters in $\mathcal{F}^A$. Define $\Theta_R : \mathcal{L}_R \to \mathcal{K}(R)$ by:

$$\Theta_R(v_a) = a, \quad \Theta_R(\omega) = \perp, \quad \Theta_R(\rho_1 \wedge \rho_2) = \Theta_R(\rho_1) \sqcup \Theta_R(\rho_2).$$

Lemma 4.5 $\mathcal{F}^R \simeq R$, and the isomorphism is the continuous extension of the mapping $\uparrow\rho \mapsto \Theta_R(\rho)$ from $\mathcal{F}_p$ to $\mathcal{K}(R)$.

Definition 4.6 We define the following maps:

$$\begin{array}{ll} F : \mathcal{F}^D \to [\mathcal{F}^C \to \mathcal{F}^R] & F\,d\,k = \{\rho \in \mathcal{L}_R \mid \exists \kappa \to \rho \in d\,[\kappa \in k]\} \\ G : [\mathcal{F}^C \to \mathcal{F}^R] \to \mathcal{F}^D & G\,f = \uparrow\{\bigwedge_{i \in I} \kappa_i \to \rho_i \in \mathcal{L}_D \mid \forall i \in I\,[\rho_i \in f(\uparrow\kappa_i)]\} \\ H : \mathcal{F}^C \to (\mathcal{F}^D \times \mathcal{F}^C) & H\,k = \langle \{\delta \in \mathcal{L}_D \mid \delta \times \kappa \in k\}, \{\kappa \in \mathcal{L}_D \mid \delta \times \kappa \in k\} \rangle \\ K : (\mathcal{F}^D \times \mathcal{F}^C) \to \mathcal{F}^C & K\langle d, k \rangle = \uparrow\{\delta \times \kappa \in \mathcal{L}_C \mid \delta \in d \;\&\; \kappa \in k\} \end{array}$$

where $[\_ \to \_]$ is the space of continuous functions.

Theorem 4.7 *$\mathcal{F}^D \simeq [\mathcal{F}^C \to \mathcal{F}^R]$ and $\mathcal{F}^C \simeq \mathcal{F}^D \times \mathcal{F}^C$.*

Combining Lem. 4.5 with Thm. 4.7, we conclude that the filter-domains $\mathcal{F}^R, \mathcal{F}^D$ and $\mathcal{F}^C$ are solutions of the continuation equations in **ALG**. However, a closer look at their structure exhibits the choice of the type languages and theories in Def. 4.3 and 4.4.

Theorem 4.8 (SOLUTION OF CONTINUATION DOMAIN EQUATIONS [30]) *Let $\mathcal{D}$ be a category of domains, and R a fixed object of $\mathcal{D}$. If C and D are initial/final solutions of the domain equations (1) in the category $\mathcal{D}$ then*

$$D \simeq [D \to D] \quad and \quad D \simeq R_\infty,$$

where R_∞ is the inverse limit with respect to $R_0 = R$ and $R_{n+1} = R_n \to R_n$.

The solution of the continuation domain equations in Thm. 4.8 is obtained by the inverse limit technique, considering:

$$\begin{array}{lcl} C_0 & = & \{\perp\} \\ D_n & = & [C_n \to R] \\ C_{n+1} & = & D_n \times C_n \end{array}$$

so that in particular $D_0 = [C_0 \to R] \simeq R$. Following [13], to describe C and D as filter-domains we stratify the languages $\mathcal{L}_A$ according to the rank function:

$$\begin{array}{rcl} rk(v) = rk(\omega) & = & 0 \\ rk(\sigma\times\tau) = rk(\sigma\wedge\tau) & = & \max\{rk(\sigma), rk(\tau)\} \\ rk(\sigma\to\tau) & = & \max\{rk(\sigma), rk(\tau)\} + 1 \end{array}$$

Let $\mathcal{L}_{A_n} = \{\sigma \in \mathcal{L}_A \mid rk(\sigma) \leq n\}$, then $\leq_{A_n}$ is $\leq_A$ restricted to $\mathcal{L}_{A_n}$ and $\mathcal{F}^{A_n}$ is the set of filters over $\mathcal{L}_{A_n}$.

Recall that if $a \in \mathcal{K}(A)$ and $b \in \mathcal{K}(B)$ then the *step function* $(a \Rightarrow b) : A \to B$ is defined by $(a \Rightarrow b)(x) = b$ if $a \sqsubseteq x$, and $(a \Rightarrow b)(x) = \perp$ otherwise. The function $(a \Rightarrow b)$ is continuous and compact in the space $[A \to B]$, whose compact elements are all finite sups of step functions.

Definition 4.9 (TYPE INTERPRETATION) The mappings $\Theta_{C_n} : \mathcal{L}_{C_n} \to \mathcal{K}(C_n)$ and $\Theta_{D_n} : \mathcal{L}_{D_n} \to \mathcal{K}(D_n)$ are defined through mutual induction by:

$$\begin{array}{lcl} \Theta_{C_0}(\kappa) & = & \perp \\ \Theta_{D_n}(v) & = & (\perp \Rightarrow \Theta_R(v)) = \lambda\!\!\lambda k \in C_n.\Theta_R(v) \\ \Theta_{D_n}(\kappa\to\rho) & = & (\Theta_{C_n}(\kappa) \Rightarrow \Theta_R(\rho)) \\ \Theta_{C_{n+1}}(\delta\times\kappa) & = & \langle \Theta_{D_n}(\delta), \Theta_{C_n}(\kappa)\rangle \end{array}$$

Finally, for $A_n = C_n, D_n$:

$$\begin{array}{lcl} \Theta_{A_n}(\omega) & = & \perp \\ \Theta_{A_n}(\sigma\wedge\tau) & = & \Theta_{A_n}(\sigma) \sqcup \Theta_{A_n}(\tau) \end{array}$$

Proposition 4.10 The filter domains $\mathcal{F}^D \simeq D$ and $\mathcal{F}^C \simeq C$ are the initial/final solutions of the continuation equations in **ALG**. *The isomorphisms are given in terms of the mappings Θ_C and Θ_D where $\Theta_C(\kappa) = \Theta_{C_{rk(\kappa)}}(\kappa)$, and similarly for Θ_D.*

Remark 4.11 Thm. 4.7 and Prop. 4.10 suggest that $\mathcal{F}^D$ in Thm. 4.7 is a λ-model (see [24]). On the other hand, by Thm. 4.8 it is also isomorphic to a $R_\infty \simeq [R_\infty \to R_\infty]$ model. To see this from the point of view of the intersection type theory, consider the extension $\mathcal{L}_{D'} = \cdots \mid \delta\to\delta$ of $\mathcal{L}_D$, adding to $\mathcal{T}_D$ the equation $\delta\times\kappa\to\rho = \delta\to\kappa\to\rho$. In the intersection type theory $\mathcal{T}_{D'}$, the following rules are derivable:

$$\frac{}{(\delta\to\delta_1)\wedge(\delta\to\delta_2) \leq_{D'} \delta\to(\delta_1\wedge\delta_2)} \qquad \frac{\delta_1' \leq_{D'} \delta_1 \quad \delta_2 \leq_{D'} \delta_2'}{\delta_1\to\delta_2 \leq_{D'} \delta_1'\to\delta_2'}$$

By this, $\mathcal{T}_{D'}$ is a *natural equated* intersection type theory in terms of [3], and hence $\mathcal{F}^{D'} \simeq [\mathcal{F}^{D'} \to \mathcal{F}^{D'}]$ (see [3], Cor. 28(4)). By an argument similar to Prop. 4.10 we can show that

$\mathcal{F}^{D'} \simeq R_\infty$, so that by the same Prop. and Thm. 4.8, we have $\mathcal{F}^D \simeq \mathcal{F}^{D'}$.

We end this section by defining the interpretation of types in $\mathcal{L}_A$ as subsets of A.

Definition 4.12 (Semantics) For $A \in \{R, C, D\}$ define $\llbracket \cdot \rrbracket^A : \mathcal{L}_A \to \mathcal{P}(A)$:

$$\begin{array}{lcllcl}
\llbracket v_a \rrbracket^R & = & \uparrow a = \{ r \in R \mid a \sqsubseteq r \} & \llbracket \omega \rrbracket^A & = & A \\
\llbracket \delta \times \kappa \rrbracket^C & = & \llbracket \delta \rrbracket^D \times \llbracket \kappa \rrbracket^C & \llbracket \tau_1 \wedge \tau_2 \rrbracket^A & = & \llbracket \tau_1 \rrbracket^A \cap \llbracket \tau_2 \rrbracket^A \\
\llbracket \kappa \to \rho \rrbracket^D & = & \{ d \in D \mid \forall k \in \llbracket \kappa \rrbracket^C [d(k) \in \llbracket \rho \rrbracket^R \}] & & & \\
\llbracket v_a \rrbracket^D & = & \llbracket \omega \to v_a \rrbracket^D = \{ d \in D \mid \forall k \in C [d(k) \in \llbracket v_a \rrbracket^R] \} & & &
\end{array}$$

Theorem 4.13 *(For $A \in \{R, C, D\}$) : a) $\llbracket \sigma \rrbracket^A = \uparrow \Theta_A(\sigma)$.*
b) $\forall \sigma, \tau \in \mathcal{L}_A [\sigma \leq_A \tau \iff \llbracket \sigma \rrbracket^A \subseteq \llbracket \tau \rrbracket^A]$.

5 An intersection type assignment system and filter model

We now define a type assignment system for pure $\lambda\mu$, following a construction analogous to that of filter-models for the λ-calculus. The central idea is that types completely determine the meaning of terms; this is achieved by tailoring the type assignment to the term interpretation in the filter description of a model, usually called the *filter*-model. In fact, by instantiating D to $\mathcal{F}^D$ and C to $\mathcal{F}^C$, for closed M the interpretation of a type statement $M : \delta$ (which is in general $\llbracket M \rrbracket^D \in \llbracket \delta \rrbracket^D$) becomes $\delta \in \llbracket M \rrbracket^D$, since $\Theta_D(\delta) = \{ d \in \mathcal{F}^D \mid \delta \in d \}$. Representing the environment by a pair of sets of assumptions Γ and Δ, the treatment can be extended to open terms.

A set $\Gamma = \{ x_1{:}\delta_1, \ldots, x_n{:}\delta_n \}$ is a *basis* or *variable environment* where $\delta_i \in \mathcal{L}_D$ for all $1 \leq i \leq n$, and the x_i are distinct. Similarly, a set $\Delta = \{ \alpha_1{:}\kappa_1, \ldots, \alpha_m{:}\kappa_m \}$ is a *name context*, where $\kappa_j \in \mathcal{L}_C$ for all $1 \leq j \leq m$. A *judgement* is an expression of the form $\Gamma \vdash M : \sigma \mid \Delta$. We write $\Gamma, x{:}\delta$ for the set $\Gamma \cup \{ x{:}\delta \}$, and $\alpha{:}\kappa, \Delta$ for $\{ \alpha{:}\kappa \} \cup \Gamma$.

Definition 5.1 (Intersection type assignment for $\lambda\mu$)

$$(Ax){:}\ \frac{}{\Gamma, x{:}\delta \vdash x{:}\delta \mid \Delta} \quad (\times){:}\ \frac{\Gamma \vdash M{:}\delta \mid \alpha{:}\kappa, \Delta}{\Gamma \vdash [\alpha]M{:}\delta \times \kappa \mid \alpha{:}\kappa, \Delta} \quad (\mu){:}\ \frac{\Gamma \vdash Q{:}(\kappa' \to \rho) \times \kappa' \mid \alpha{:}\kappa, \Delta}{\Gamma \vdash \mu\alpha.Q{:}\kappa \to \rho \mid \Delta}$$

$$(\to E){:}\ \frac{\Gamma \vdash M{:}\delta \times \kappa \to \rho \mid \Delta \quad \Gamma \vdash N{:}\delta \mid \Delta}{\Gamma \vdash MN{:}\kappa \to \rho \mid \Delta} \qquad (\to I){:}\ \frac{\Gamma, x{:}\delta \vdash M{:}\kappa \to \rho \mid \Delta}{\Gamma \vdash \lambda x.M{:}\delta \times \kappa \to \rho \mid \Delta}$$

$$(\wedge){:}\ \frac{\Gamma \vdash M{:}\sigma \mid \Delta \quad \Gamma \vdash M{:}\tau \mid \Delta}{\Gamma \vdash M{:}\sigma \wedge \tau \mid \Delta} \quad (\omega){:}\ \frac{}{\Gamma \vdash M{:}\omega \mid \Delta} \quad (\leq){:}\ \frac{\Gamma \vdash M{:}\sigma \mid \Delta}{\Gamma \vdash M{:}\tau \mid \Delta}\ (\sigma \leq \tau)$$

where $\delta \in \mathcal{L}_D$, $\kappa, \kappa' \in \mathcal{L}_C$ and $\rho \in \mathcal{L}_R$.

To understand the above rules, we read them backward from the conclusion to the premises. To conclude that MN is a function mapping a continuation with type κ to a value of type ρ, rule $(\to E)$ requires that the continuation fed to M has type $\delta \times \kappa$, where δ is the type of N. This mimics the storage of (the denotation of) N into the continuation fed to MN, which is treated as a stack. Rule $(\to I)$ is just the rule symmetric to $(\to E)$.

Note that the (perhaps unusual) rules $(\to E)$ and $(\to I)$ are instances of the usual ones of the simply typed λ-calculus, observing that $\delta \times \kappa \to \rho \in \mathcal{L}_D$ is equivalent to $\delta \to (\kappa \to \rho) \in \mathcal{L}_{D'}$ so that, admitting types in $\mathcal{L}_{D'}$, the following rules are admissible:

$$(\to E'){:}\ \frac{\Gamma \vdash M : \delta \to (\kappa \to \rho) \mid \Delta \quad \Gamma \vdash N : \delta \mid \Delta}{\Gamma \vdash MN : \kappa \to \rho \mid \Delta} \qquad (\to I'){:}\ \frac{\Gamma, x : \delta \vdash M : \kappa \to \rho \mid \Delta}{\Gamma \vdash \lambda x.M : \delta \to (\kappa \to \rho) \mid \Delta}$$

In rule $(\times)$, we conclude that the command $[\alpha]M$ has type $\delta \times \kappa$ because $[\![[\alpha]M]\!]^{C}\, e = \langle [\![M]\!]^{D}\, e, e\,\alpha \rangle$, so M must have type δ and the continuation type κ has to be the same as that given to α in the name context.

Rule (μ) expresses (a part of) the concept of swapping the current continuation by stating that $\mu\alpha.Q$ is able to produce a result of type ρ when applied to a continuation of type κ; type ρ is of the result obtained from Q and the (in general different) continuation of type κ'. Nonetheless, κ cannot be ignored, and indeed it contributes to the typing of the command Q since it appears in the name environment; this corresponds to the fact that in the semantics $[\![\mu\alpha.Q]\!]^{D}\, e\, k = d\, k'$ where $\langle d, k' \rangle = [\![Q]\!]^{C}\, e[\alpha := k]$, *i.e.* Q is evaluated in the modified environment $e[\alpha := k]$. We clarify the interaction between $(\times)$ and (μ) in the next example, where this is compared to typing in Parigot's system.

Example 5.2 Consider the following derivations in, respectively, Parigot's system and ours.

$$\dfrac{\dfrac{\Gamma \vdash M : A \mid \beta{:}A, \alpha{:}B, \Delta}{\Gamma \vdash [\beta]M : \bot \mid \beta{:}A, \alpha{:}B, \Delta}\,(\bot)}{\Gamma \vdash \mu\alpha.[\beta]M : B \mid \beta{:}A, \Delta}\,(\mu) \qquad \dfrac{\dfrac{\Gamma \vdash M : \kappa \to \rho \mid \beta{:}\kappa, \alpha{:}\kappa', \Delta}{\Gamma \vdash [\beta]M : (\kappa \to \rho) \times \kappa \mid \beta{:}\kappa, \alpha{:}\kappa', \Delta}\,(\times)}{\Gamma \vdash \mu\alpha.[\beta]M : \kappa' \to \rho \mid \beta{:}\kappa, \Delta}\,(\mu)$$

Both derivations express that $\mu\alpha.[\beta]M$ constitutes a *context switch*; in our case the computational side of the switch is (very elegantly) made apparent: before the switch, the type $\kappa \to \rho$ expresses that M is a term working with the input-stream κ of terms, and returns ρ; after the context switch, the type $\kappa' \to \rho$ for $\mu\alpha.[\beta]M$ now expresses that the input-stream κ' is taken instead.

The following lemma provides a characterisation of derivability in our system.

Lemma 5.3 (Generation lemma) *If $\Gamma \vdash M : \delta \mid \Delta$, then either δ is an intersection, or ω, or:*

$$\begin{array}{lcl}
\Gamma, x{:}\delta' \vdash x : \delta \mid \Delta & \iff & \delta' \leq \delta \\
\Gamma \vdash \lambda x.M : \delta \times \kappa \to \rho \mid \Delta & \iff & \Gamma, x{:}\delta \vdash M : \kappa \to \rho \mid \Delta \\
\Gamma \vdash MN : \kappa \to \rho \mid \Delta & \iff & \exists \delta\, [\, \Gamma \vdash M : \delta \times \kappa \to \rho \mid \Delta \ \&\ \Gamma \vdash N : \delta \mid \Delta \,] \\
\Gamma \vdash \mu\alpha.Q : \kappa \to \rho \mid \Delta & \iff & \exists \kappa'\, [\, \Gamma \vdash Q : (\kappa' \to \rho) \times \kappa' \mid \alpha{:}\kappa, \Delta \,]. \\
\Gamma \vdash [\alpha]M : \delta \times \kappa \mid \Delta & \iff & \alpha{:}\kappa \in \Delta \ \&\ \Gamma \vdash M : \delta \mid \Delta.
\end{array}$$

We will now show that our notion of type assignment is closed under conversion. The proofs of the properties of subject expansion and subject reduction are relatively standard; the main theorems follow from the relative substitution lemmas.

Lemma 5.4 (Term substitution lemma) *$\Gamma \vdash M[L/x] : \sigma \mid \Delta$ if and only if there exists δ such that $\Gamma, x{:}\delta \vdash M : \sigma \mid \Delta$ and $\Gamma \vdash L : \delta \mid \Delta$.*

A similar lemma for structural substitution is not as easily formulated. This is mainly due to the fact that the μ-bound variable gets 're-used' in the result of the substitution, but with a different type, as can be observed from the following example.

Example 5.5 First observe that $(\mu\alpha.[\beta]\mu\gamma.[\alpha]x)N$ reduces to $(\mu\alpha.[\beta]\mu\gamma.[\alpha]xN)$. We can type the first term in our system as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{x{:}\delta\times\kappa\to\rho \vdash x:\delta\times\kappa\to\rho \mid \alpha{:}\delta\times\kappa,\beta{:}\kappa_1,\gamma{:}\kappa_1}}{x{:}\delta\times\kappa\to\rho \vdash [\alpha]x:(\delta\times\kappa\to\rho)\times\delta\times\kappa \mid \alpha{:}\delta\times\kappa,\beta{:}\kappa_1,\gamma{:}\kappa_1}}{x{:}\delta\times\kappa\to\rho \vdash \mu\gamma.[\alpha]x:\kappa_1\to\rho \mid \alpha{:}\delta\times\kappa,\beta{:}\kappa_1}}{x{:}\delta\times\kappa\to\rho \vdash [\beta]\mu\gamma.[\alpha]x:(\kappa_1\to\rho)\times\kappa_1 \mid \alpha{:}\delta\times\kappa,\beta{:}\kappa_1}}{x{:}\delta\times\kappa\to\rho \vdash \mu\alpha.[\beta]\mu\gamma.[\alpha]x:\delta\times\kappa\to\rho \mid \beta{:}\kappa_1} \qquad \Gamma\vdash N:\delta\mid\Delta}{\Gamma,x{:}\delta\times\kappa\to\rho \vdash (\mu\alpha.[\beta]\mu\gamma.[\alpha]x)N:\kappa\to\rho \mid \beta{:}\kappa_1,\Delta}$$

Using the information of this derivation, the derivation for $(\mu\alpha.[\beta]\mu\gamma.[\alpha]xN)$ implicitly constructed by the reduction is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{x{:}\delta\times\kappa\to\rho \vdash x:\delta\times\kappa\to\rho \mid \alpha{:}\kappa,\beta{:}\kappa_1,\gamma{:}\kappa_1} \qquad \Gamma\vdash N:\delta\mid\Delta}{x{:}\delta\times\kappa\to\rho \vdash xN:\kappa\to\rho \mid \alpha{:}\kappa,\beta{:}\kappa_1,\gamma{:}\kappa_1}}{x{:}\delta\times\kappa\to\rho \vdash [\alpha]xN:(\kappa\to\rho)\times\kappa \mid \alpha{:}\kappa,\beta{:}\kappa_1,\gamma{:}\kappa_1}}{x{:}\delta\times\kappa\to\rho \vdash \mu\gamma.[\alpha]xN:\kappa_1\to\rho \mid \alpha{:}\kappa,\beta{:}\kappa_1}}{x{:}\delta\times\kappa\to\rho \vdash [\beta]\mu\gamma.[\alpha]xN:(\kappa_1\to\rho)\times\kappa_1 \mid \alpha{:}\kappa,\beta{:}\kappa_1}}{x{:}\delta\times\kappa\to\rho \vdash \mu\alpha.[\beta]\mu\gamma.[\alpha]xN:\kappa\to\rho \mid \beta{:}\kappa_1}}{\Gamma,x{:}\delta\times\kappa\to\rho \vdash (\mu\alpha.[\beta]\mu\gamma.[\alpha]xN):\kappa\to\rho \mid \beta{:}\kappa_1,\Delta}$$

Notice that here the type for α has changed from $\delta\times\kappa$ to κ.

Lemma 5.6 (Structural substitution lemma) *$\Gamma\vdash M[\alpha\Leftarrow L]:\sigma \mid \alpha{:}\kappa,\Delta$ if and only if there exists δ such that $\Gamma\vdash L:\delta \mid \Delta$, and $\Gamma\vdash M:\sigma\mid\alpha{:}\delta\times\kappa,\Delta$.*

Using these two substitution results, the following theorems are easy to show.

Theorem 5.7 (Subject expansion) *If $M\to N$, and $\Gamma\vdash N:\delta\mid\Delta$, then $\Gamma\vdash M:\delta\mid\Delta$.*

Theorem 5.8 (Subject reduction) *If $M\to N$, and $\Gamma\vdash M:\delta\mid\Delta$, then $\Gamma\vdash N:\delta\mid\Delta$*

We define satisfaction in a $\lambda\mu$-model (R,D,C) and validity in the standard way.

Definition 5.9 (Satisfaction and Validity) Let $\mathcal{M}=(R,D,C)$. We define:

$$\begin{array}{lcl} e\models_{\mathcal{M}}\Gamma,\Delta & \iff & \forall x{:}\delta\in\Gamma\,[e(x)\in[\![\delta]\!]^D] \;\&\; \forall\alpha{:}\kappa\in\Delta\,[e(\alpha)\in[\![\kappa]\!]^C] \\ \Gamma\models_{\mathcal{M}} M:\delta\mid\Delta & \iff & \forall e\in\mathsf{Env}\,[e\models_{\mathcal{M}}\Gamma,\Delta\Rightarrow[\![M]\!]\,e\in[\![\delta]\!]^D] \\ \Gamma\models M:\delta\mid\Delta & \iff & \forall\mathcal{M}\,[\Gamma\models_{\mathcal{M}} M:\delta\mid\Delta] \end{array}$$

We can now show soundness and completeness for our notion of intersection type assignment with respect to the filter semantics:

Theorem 5.10 (Soundness) $\Gamma\vdash M:\delta\mid\Delta \;\Rightarrow\; \Gamma\models M:\delta\mid\Delta$.

Lemma 5.11 (Filter model) *Let $A\in\{D,C\}$. Given an environment $e\in(Var\to\mathcal{F}^D)+(Name\to\mathcal{F}^C)$, we have*

$$[\![M]\!]^{\mathcal{F}^A}e \;=\; \{\sigma\in\mathcal{L}_A \mid \exists\Gamma,\Delta[e\models\Gamma,\Delta \;\&\; \Gamma\vdash M:\sigma\mid\Delta]\}$$

Theorem 5.12 (Completeness) $\Gamma\models M:\delta\mid\Delta \;\Rightarrow\; \Gamma\vdash M:\delta\mid\Delta$.

6 Type preservation from Parigot's Type System to the Intersection Type System

We conclude this paper by showing that typed $\lambda\mu$-terms have a non-trivial intersection typing in our system, i.e. they can be assigned a type different from ω.

Example 6.1 In Parigot's system for $\lambda\mu$, we can derive $x{:}A \vdash_{\mathrm{P}} \mu\alpha.[\beta]\mu\gamma.[\alpha]x : A \mid \beta{:}C$; We can type the same pure $\lambda\mu$-term in our system as well (where $\Gamma = x{:}\delta\times\kappa\rightarrow\rho$):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x{:}A \vdash x : A \mid \alpha{:}A,\beta{:}C,\gamma{:}C}\,(Ax)}{x{:}A \vdash [\alpha]x : \perp \mid \alpha{:}A,\beta{:}C,\gamma{:}C}\,(\perp)}{x{:}A \vdash \mu\gamma.[\alpha]x : C \mid \alpha{:}A,\beta{:}C}\,(\mu)}{x{:}A \vdash [\beta]\mu\gamma.[\alpha]x : \perp \mid \alpha{:}A,\beta{:}C}\,(\perp)}{x{:}A \vdash \mu\alpha.[\beta]\mu\gamma.[\alpha]x : A \mid \beta{:}C}\,(\mu)
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash x : \delta\times\kappa\rightarrow\rho \mid \alpha{:}\delta\times\kappa,\beta{:}\kappa_1,\gamma{:}\kappa_1}\,(Ax)}{\Gamma \vdash [\alpha]x : (\delta\times\kappa\rightarrow\rho)\times\delta\times\kappa \mid \alpha{:}\delta\times\kappa,\beta{:}\kappa_1,\gamma{:}\kappa_1}\,(\times)}{\Gamma \vdash \mu\gamma.[\alpha]x : \kappa_1\rightarrow\rho \mid \alpha{:}\delta\times\kappa,\beta{:}\kappa_1}\,(\mu)}{\Gamma \vdash [\beta]\mu\gamma.[\alpha]x : (\kappa_1\rightarrow\rho)\times\kappa_1 \mid \alpha{:}\delta\times\kappa,\beta{:}\kappa_1}\,(\times)}{\Gamma \vdash \mu\alpha.[\beta]\mu\gamma.[\alpha]x : \delta\times\kappa\rightarrow\rho \mid \beta{:}\kappa_1}\,(\mu)$$

We can also type terms that are not typeable in Parigot's system:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{x{:}\kappa\rightarrow\rho \vdash x : \kappa\rightarrow\rho \mid \beta{:}\kappa',\gamma{:}\kappa\wedge((\kappa\rightarrow\rho)\times\kappa')}\,(Ax)}{x{:}\kappa\rightarrow\rho \vdash [\gamma]x : (\kappa\rightarrow\rho)\times(\kappa\wedge((\kappa\rightarrow\rho)\times\kappa')) \mid \beta{:}\kappa',\gamma{:}\kappa\wedge((\kappa\rightarrow\rho)\times\kappa')}\,(\times)}{x{:}\kappa\rightarrow\rho \vdash [\gamma]x : (\kappa\rightarrow\rho)\times\kappa \mid \beta{:}\kappa',\gamma{:}\kappa\wedge((\kappa\rightarrow\rho)\times\kappa')}\,(\leq)}{x{:}\kappa\rightarrow\rho \vdash \mu\beta.[\gamma]x : \kappa'\rightarrow\rho \mid \gamma{:}\kappa\wedge((\kappa\rightarrow\rho)\times\kappa')}\,(\mu)}{\vdash \lambda x.\mu\beta.[\gamma]x : (\kappa\rightarrow\rho)\times\kappa'\rightarrow\rho \mid \gamma{:}\kappa\wedge((\kappa\rightarrow\rho)\times\kappa')}\,(\rightarrow I)}{\vdash [\gamma](\lambda x.\mu\beta.[\gamma]x) : ((\kappa\rightarrow\rho)\times\kappa'\rightarrow\rho)\times(\kappa\wedge((\kappa\rightarrow\rho)\times\kappa')) \mid \gamma{:}\kappa\wedge((\kappa\rightarrow\rho)\times\kappa')}\,(\times)}{\vdash [\gamma](\lambda x.\mu\beta.[\gamma]x) : ((\kappa\rightarrow\rho)\times\kappa'\rightarrow\rho)\times((\kappa\rightarrow\rho)\times\kappa') \mid \gamma{:}\kappa\wedge((\kappa\rightarrow\rho)\times\kappa')}\,(\leq)}{\vdash \mu\gamma.[\gamma](\lambda x.\mu\beta.[\gamma]x) : (\kappa\wedge((\kappa\rightarrow\rho)\times\kappa'))\rightarrow\rho \mid}\,(\mu)$$

Notice that $\mu\gamma.[\gamma](\lambda x.\mu\beta.[\gamma]x)$ is not typeable in Parigot's system, since the two occurrences of $[\gamma]$ need to be typed differently, with non-unifiable types.

We can strengthen the first observation and show that we can faithfully embed Parigot's system into ours; to this purpose, we define first an interpretation of Parigot's types into our intersection types.

Definition 6.2 We change the definition of types for Parigot's system slightly by using

$$A,B \ ::= \ \varphi \mid \perp_A \mid A\rightarrow B \quad (A \neq \perp_C)$$

and change the rule $(\perp)$ into a rule that registers to what type the contradiction was established:

$$(\perp_B) : \ \dfrac{\Gamma \vdash M : B \mid \beta{:}B,\Delta}{\Gamma \vdash [\beta]M : \perp_B \mid \beta{:}B,\Delta}$$

Taking an arbitrary $a \neq \perp \in R$, we define $\overline{A}$ and $\underline{A}$ simultaneously through:

$$\begin{array}{rcll}
\overline{\varphi} & = & (v_a\times\omega)\rightarrow v_a & \textit{for all } \varphi \\
\overline{\perp_A} & = & (\underline{A}\rightarrow v_a)\times\underline{A} & \\
\overline{A\rightarrow B} & = & \overline{A}\times\kappa\rightarrow v_a & \textit{where } \overline{B} = \kappa\rightarrow v_a
\end{array}
\qquad
\begin{array}{rcll}
\underline{\varphi} & = & v_a\times\omega & \text{for all } \varphi \\
\underline{A\rightarrow B} & = & (\underline{A}\rightarrow v_a)\times\underline{B} &
\end{array}$$

We also define $\overline{\Gamma} = \{\, x{:}\overline{A} \mid x{:}A \in \Gamma \,\}$ and $\underline{\Delta} = \{\, \alpha{:}\underline{A} \mid \alpha{:}A \in \Delta \,\}$.

We can now show:

Theorem 6.3 (Type preservation) *If $\Gamma \vdash_{\mathrm{P}} M : A \mid \Delta$, then $\overline{\Gamma} \vdash M : \overline{A} \mid \underline{\Delta}$*

Example 6.4 To illustrate this result, consider the derivation in $\vdash_{\mathrm{P}}$ on the left, which gets translated into the one on the right (where $\Gamma = x{:}(v_a \times \omega) \to v_a$):

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x{:}\varphi \vdash x : \varphi \mid \alpha{:}\varphi, \beta{:}\varphi'}}{x{:}\varphi \vdash [\alpha]x : \bot_\varphi \mid \alpha{:}\varphi, \beta{:}\varphi'}}{x{:}\varphi \vdash \mu\beta.[\alpha]x : \varphi' \mid \alpha{:}\varphi}}{\vdash \lambda x.\mu\beta.[\alpha]x : \varphi \to \varphi' \mid \alpha{:}\varphi} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash x : (v_a \times \omega) \to v_a \mid \alpha{:}v_a \times \omega, \beta{:}v_a \times \omega}}{\Gamma \vdash [\alpha]x : ((v_a \times \omega) \to v_a) \times (v_a \times \omega) \mid \alpha{:}v_a \times \omega, \beta{:}v_a \times \omega}}{\Gamma \vdash \mu\beta.[\alpha]x : (v_a \times \omega) \to v_a \mid \alpha{:}v_a \times \omega}}{\vdash \lambda x.\mu\beta.[\alpha]x : ((v_a \times \omega) \to v_a) \times (v_a \times \omega) \to v \mid \alpha{:}v_a \times \omega}$$

Conclusions and Future Work

We have presented a filter model for the $\lambda\mu$-calculus which is an instance of Streicher and Reus's continuation model, and a type assignment system such that the set of types that can be given to a term coincides with its denotation in the model. The type theory and the assignment system can be viewed as the logic for reasoning about the computational meaning of $\lambda\mu$-terms, much as it is the case for λ-calculus. We expect that significant properties of pure $\lambda\mu$ can be characterised via their typing, and we see the characterisation of the strongly normalisable pure terms as the first challenge.

We have also shown that $\lambda\mu$-terms which are typeable in Parigot's first order type system have non-trivial typing in our system in a strong sense. This opens the possibility of a new proof of strong normalisation for typed $\lambda\mu$ and, by using a variant of the system, of de Groote's typeable $\Lambda\mu$-terms.

The investigation of other significant properties of the calculi should also be possible with the same tools, like confluence, standardisation, solvable terms, etc. More significantly, for the $\lambda\mu$-calculus we are interested in the use of the type system for interpreting relevant combinators, like in the case of de Groote's encoding of Felleisen's $\mathcal{C}$ operator, whose types are essentially those of (the η-expansion of) identity, and, in general, for investigating the computational behaviour of combinators representing proofs of non-constructive principles. To this aim a study of the principal typing with respect to the present system would be of great help.

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Appendix A Proofs

Lemma A.1 For all $e \in \mathsf{Env}$:
i) $x \notin fv(M) \Rightarrow \forall d \in D\,[\,\ulcorner M \lrcorner\, e = \ulcorner M \lrcorner\, e[x := d]\,]$.
ii) $\alpha \notin fn(M) \Rightarrow \forall k \in C\,[\,\ulcorner M \lrcorner\, e = \ulcorner M \lrcorner\, e[\alpha := k]\,]$.

Proof: Easy. ■

Lemma A.2
i) The (β) *rule is sound w.r.t. the semantics, i.e.* $\ulcorner(\lambda x.M)N\lrcorner^D = \ulcorner M[N/x]\lrcorner^D$
ii) The (μ) *rule is sound w.r.t. the semantics, i.e.* $\ulcorner(\mu\alpha.Q)N\lrcorner^D = \ulcorner \mu\alpha.Q[\alpha \Leftarrow N]\lrcorner^D$
iii) The $(\mu\eta)$ *rule is sound w.r.t. the semantics, i.e.* $\ulcorner \mu\alpha.[\alpha]M\lrcorner = \ulcorner M\lrcorner$ *if* $\alpha \notin fn(M)$.
iv) Let $e \in \mathsf{Env}$. *Then* $\ulcorner[\alpha]\mu\beta.[\gamma]M\lrcorner^C e = \langle \lambda\!\!\!\lambda h.\ulcorner M\lrcorner^D\, e[\beta := h], (e[\beta := h]\,\gamma)\rangle(e\,\alpha)$
v) Let $e \in \mathsf{Env}$. *Then* $\ulcorner([\gamma]M)[\alpha/\beta]\lrcorner^C e = \langle \lambda\!\!\!\lambda h.\ulcorner M\lrcorner^D\, e[\beta := e\,\alpha]\,h, (e[\beta := e\,\alpha]\,\gamma)\rangle$

Proof: In order to show the soundness of (β), (μ) and $(\mu\eta)$, let $e \in \mathsf{Env}$ and $k \in C$.

$$
\begin{array}{lll}
((\lambda x.M)N = M[N/x]) : \quad \ulcorner(\lambda x.M)N\lrcorner\, e\, k & = & \\
\quad \ulcorner\lambda x.M\lrcorner\, e\, \langle \ulcorner N\lrcorner\, e, k\rangle & = & \\
\quad \ulcorner\lambda x.M\lrcorner\, e[x := \ulcorner N\lrcorner\, e]\, k & = & \text{by } (i)) \text{ of Lem. 3.4} \\
\quad \ulcorner M[N/x]\lrcorner\, e\, k & &
\end{array}
$$

$$
\begin{array}{lll}
((\mu\alpha.[\beta]M)N = \mu\alpha.[\beta]M[\alpha \Leftarrow N] \textit{ with } \alpha \neq \beta) : & & \\
\quad \ulcorner(\mu\alpha.[\beta]M)N\lrcorner\, e\, k & = & \\
\quad \ulcorner\mu\alpha.[\beta]M\lrcorner\, e\, \langle \ulcorner N\lrcorner\, e, k\rangle & = & (3.5) \\
\quad \ulcorner M\lrcorner\, e[\alpha := \langle \ulcorner N\lrcorner\, e, k\rangle]\, (e[\alpha := \langle \ulcorner N\lrcorner\, e, k\rangle]\,\beta) & = & (\alpha \neq \beta) \\
\quad \ulcorner M\lrcorner\, e[\alpha := \langle \ulcorner N\lrcorner\, e, k\rangle]\, (e\,\beta) & = & (A.1\ \&\ \alpha \neq \beta) \\
\quad \ulcorner M\lrcorner\, e[\alpha := \langle \ulcorner N\lrcorner\, e[\alpha := k], k\rangle]\, (e[\alpha := k]\,\beta) & = & \\
\quad \ulcorner M\lrcorner\, e[\alpha := k][\alpha := \langle \ulcorner N\lrcorner\, e[\alpha := k], e[\alpha := k]\,\alpha\rangle]\, (e[\alpha := k]\,\beta) & = & (3.6) \\
\quad \ulcorner M[\alpha \Leftarrow N]\lrcorner\, e[\alpha := k]\, (e[\alpha := k]\,\beta) & = & (3.5) \\
\quad \ulcorner\mu\alpha.[\beta](M[\alpha \Leftarrow N])\lrcorner\, e\, k & = & (\alpha \neq \beta) \\
\quad \ulcorner\mu\alpha.([\beta]M[\alpha \Leftarrow N])\lrcorner\, e\, k & &
\end{array}
$$

$$
\begin{array}{lll}
((\mu\alpha.[\alpha]M)N = \mu\alpha.[\alpha]M[\alpha \Leftarrow N]) : \quad \ulcorner(\mu\alpha.[\alpha]M)N\lrcorner\, e\, k & = & \\
\quad \ulcorner\mu\alpha.[\alpha]M\lrcorner\, e\, \langle \ulcorner N\lrcorner\, e, k\rangle & = & (3.5) \\
\quad \ulcorner M\lrcorner\, e[\alpha := \langle \ulcorner N\lrcorner\, e, k\rangle]\, (e[\alpha := \langle \ulcorner N\lrcorner\, e, k\rangle]\,\alpha) & = & \\
\quad \ulcorner M\lrcorner\, e[\alpha := \langle \ulcorner N\lrcorner\, e, k\rangle]\, \langle \ulcorner N\lrcorner\, e, k\rangle & = & \\
\quad \ulcorner M\lrcorner\, e[\alpha := k][\alpha := \langle \ulcorner N\lrcorner\, e, k\rangle]\, \langle \ulcorner N\lrcorner\, e, k\rangle & = & (A.1) \\
\quad \ulcorner M\lrcorner\, e[\alpha := k][\alpha := \langle \ulcorner N\lrcorner\, e[\alpha := k], k\rangle]\, \langle \ulcorner N\lrcorner\, e[\alpha := k], k\rangle & = & (3.6) \\
\quad \ulcorner M[\alpha \Leftarrow N]\lrcorner\, e[\alpha := k]\, \langle \ulcorner N\lrcorner\, e[\alpha := k], k\rangle & = & \\
\quad \ulcorner M[\alpha \Leftarrow N]N\lrcorner\, e[\alpha := k]\, k & = & (3.5) \\
\quad \ulcorner\mu\alpha.[\alpha](M[\alpha \Leftarrow N])N\lrcorner\, e\, k & = & \\
\quad \ulcorner\mu\alpha.([\alpha]M)[\alpha \Leftarrow N]\lrcorner\, e\, k & &
\end{array}
$$

$$
\begin{array}{lll}
(\mu\alpha.[\alpha]M \textit{ with } \alpha \notin fn(M)) : \quad \ulcorner\mu\alpha.[\alpha]M\lrcorner\, e\, k & = & (3.5) \\
\quad \ulcorner M\lrcorner\, e[\alpha := k]\, (e[\alpha := k]\,\alpha) & = & \\
\quad \ulcorner M\lrcorner\, e[\alpha := k]\, k & = & (\alpha \notin M\ \&\ A.1(ii)) \\
\quad \ulcorner M\lrcorner\, e\, k & &
\end{array}
$$

$$
\begin{array}{llll}
(iv)) : & \text{Let } e \in \mathsf{Env}. & \ulcorner[\alpha]\mu\beta.[\gamma]M\lrcorner^C\, e & = \\
& & \langle \ulcorner\mu\beta.[\gamma]M\lrcorner^D\, e, e\,\alpha\rangle & = \\
& & \langle \lambda\!\!\!\lambda h.\ulcorner\mu\beta.[\gamma]M\lrcorner^D e\, h, e\,\alpha\rangle & = \\
& & \langle \lambda\!\!\!\lambda h.\ulcorner\mu\beta.[\gamma]M\lrcorner^D\, e\, h, e\,\alpha\rangle & = \ (3.5) \\
& & \langle \lambda\!\!\!\lambda h.\ulcorner M\lrcorner^D\, e[\beta := h]\, (e[\beta := h]\,\gamma), e\,\alpha\rangle &
\end{array}
$$

$$(v)): \quad \text{Let } e \in \mathsf{Env}. \quad \begin{array}{ll} \llbracket([\gamma]M)[\alpha/\beta]\rrbracket^C e & = \quad (3.4(ii))) \\ \llbracket([\gamma]M)[\alpha/\beta]\rrbracket^C e[\beta := e\,\alpha] & = \\ \langle \llbracket M\rrbracket^D e[\beta := e\,\alpha], e[\beta := e\,\alpha]\,\gamma\rangle & = \\ \langle \lambda\!\!\!\lambda h. \llbracket M\rrbracket^D e[\beta := e\,\alpha]h, e[\beta := e\,\alpha]\,\gamma\rangle & \end{array}$$

■

Theorem 3.7 (SOUNDNESS OF $\lambda\mu$) $\vdash_{\lambda\mu} M = N \Rightarrow \llbracket M\rrbracket = \llbracket N\rrbracket$

Proof: By induction on the structure of terms.

$(M \equiv x)$: Trivial

$(M \equiv \lambda x.L)$: Then $L \in \mathsf{Trm}$ and $N = \lambda x.L'$, where $L = L'$. In such a case the thesis follows easily by induction.

$(M \equiv LH)$: Then $L, H \in \mathsf{Trm}$. We distinguish three cases according to the shape of L:

$(L \equiv \lambda x.L' \textit{ and } N = L'[H/x])$: Immediate by Lem. A.2($i$)).

$(L \equiv \mu\alpha.Q \textit{ and } N = \mu\alpha.Q[\alpha \Leftarrow L])$: Immediate by Lem. A.2($ii$)).

$(\textit{Otherwise})$: Then $N \equiv L'H'$ with $L = L'$ and $H = H'$, where $L', H' \in \mathsf{Trm}$. In such a case the thesis follows easily by induction.

$(M \equiv \mu\delta.Q)$: Then $Q \in \mathsf{Cmd}$. We consider three different cases:

$(Q \equiv [\alpha]\mu\beta.[\gamma]L \textit{ and } N \equiv \mu\delta.Q' \textit{ where } Q' \equiv ([\gamma]L)[\alpha/\beta])$: (i.e. $Q \to Q'$ is an instance of rule (ren)).

Then $Q, Q' \in \mathsf{Cmd}$. Let now $e \in \mathsf{Env}$ and $k \in C$.

By definition of interpretation of terms, we get $\llbracket M\rrbracket^D\, e\, k = d\,k'$, where $\langle d, k'\rangle = \llbracket Q\rrbracket^C\, e\,[\delta := k]$.

We can then apply Lem. A.2(iv)), obtaining

$$\llbracket Q\rrbracket^C e[\delta := k] = \langle \lambda\!\!\!\lambda h. \llbracket L\rrbracket^D e'\ (e'\,\gamma), e[\delta := k]\,\alpha\rangle$$

where $e' = e[\delta := k, \beta := h]$, and hence

$$\llbracket M\rrbracket^D e\,k = \llbracket L\rrbracket^D e_1\ (e_1\,\gamma)$$

where $e_1 = e[\delta := k, \beta := (e[\delta := k]\,\alpha)]$.

By definition of interpretation of terms $\llbracket N\rrbracket^D\, e\, k = d'\,k''$, where $\langle d', k''\rangle = \llbracket Q\rrbracket^C e[\delta := k]$.

We can then apply Lem. A.2(v)), obtaining

$$\llbracket Q'\rrbracket^C e[\delta := k] = \langle \lambda\!\!\!\lambda h. \llbracket L\rrbracket^D e_2\, h, (e_2\,\gamma)\rangle$$

where $e_2 = e[\delta := k, \beta := (e[\delta := k]\,\alpha)]$.

and hence

$$\llbracket N\rrbracket^D e\,k = \llbracket L\rrbracket^D e_2\ (e_2\,\gamma)$$

Since $e_1 = e_2$, we immediately get

$$\llbracket M\rrbracket^D e\,k = \llbracket N\rrbracket^D e\,k$$

$(Q \equiv [\delta]L \textit{ with } \delta \notin fn(L) \textit{ and } N \equiv L)$: Immediate by Lem. A.2($iii$)).

$(Q \equiv [\alpha]L \textit{ and } N \equiv \mu\delta.[\alpha]L' \textit{ where } L = L')$: Then $L' \in \mathsf{Trm}$. In such a case the result follows easily by induction. ■

Lemma 3.5 $\llbracket \mu\alpha.[\beta]M\rrbracket\ e\,k = \llbracket M\rrbracket\ e[\alpha := k]\ (e[\alpha := k]\beta)$.

Proof: Easy by definition of interpretation of terms. ■

Lemma 3.6 $\ulcorner M[\alpha \Leftarrow N] \lrcorner\, e\, k = \ulcorner M \lrcorner\, e[\alpha := \langle \ulcorner N \lrcorner\, e, (e\,\alpha)\rangle]\, k.$

Proof: By induction on the structure of M.

$(M \equiv x)$: Then $x[\alpha \Leftarrow N] \equiv x$ and by the simple fact that $x \neq \alpha$, we have:

$$\ulcorner x \lrcorner\, e\, k = e\, x\, k = e[\alpha := \langle \ulcorner N \lrcorner\, e, (e\,\alpha)\rangle]\, x\, k = \ulcorner x \lrcorner\, e[\alpha := \langle \ulcorner N \lrcorner\, e, (e\,\alpha)\rangle]\, k$$

$(M \equiv \lambda x.L)$: Then $(\lambda x.L)[\alpha \Leftarrow N] \equiv \lambda x.L[\alpha \Leftarrow N]$ and we have:

$$\begin{array}{lcl} \ulcorner \lambda x.L[\alpha \Leftarrow N] \lrcorner\, e\, k & = & \text{where } k = \langle d, k' \rangle \\ \ulcorner L[\alpha \Leftarrow N] \lrcorner\, e[x := d]\, k' & = & (IH) \\ \ulcorner L \lrcorner\, e[x := d, \alpha := \langle \ulcorner N \lrcorner\, e, e\,\alpha\rangle]\, k' & = & \\ \ulcorner \lambda x.L \lrcorner\, e[\alpha := \langle \ulcorner N \lrcorner\, e, e\,\alpha\rangle]\langle d, k' \rangle & = & \\ \ulcorner \lambda x.L \lrcorner\, e[\alpha := \langle \ulcorner N \lrcorner\, e, e\,\alpha\rangle]\, k & & \end{array}$$

$(M \equiv LH)$: Then $(LH)[\alpha \Leftarrow N] \equiv (L[\alpha \Leftarrow N])(H[\alpha \Leftarrow N])$ and:

$$\begin{array}{lcl} \ulcorner (L[\alpha \Leftarrow N])(H[\alpha \Leftarrow N]) \lrcorner\, e\, k & = & \\ \ulcorner (L[\alpha \Leftarrow N]) \lrcorner\, e\, \langle \ulcorner (H[\alpha \Leftarrow N]) \lrcorner e, k \rangle & = & (IH) \\ \ulcorner L \lrcorner\, e[\alpha := \langle \ulcorner N \lrcorner\, e, e\,\alpha\rangle]\, \langle \ulcorner H \lrcorner\, e[\alpha := \langle \ulcorner N \lrcorner\, e, e\,\alpha\rangle], k \rangle & = & \\ \ulcorner LH \lrcorner\, e[\alpha := \langle \ulcorner N \lrcorner\, e, e\,\alpha\rangle]\, k & & \end{array}$$

$(M \equiv \mu\beta.[\gamma]L$ *with* $\alpha \neq \beta$ *and* $\gamma \neq \alpha)$: Then $(\mu\beta.[\gamma]L)[\alpha \Leftarrow N] \equiv \mu\beta.[\gamma](L[\alpha \Leftarrow N])$, and

$$\begin{array}{lcl} \ulcorner \mu\beta.[\gamma]L[\alpha \Leftarrow N] \lrcorner\, e\, k & = & (\langle d, k' \rangle = \ulcorner [\gamma]L[\alpha \Leftarrow N] \lrcorner\, e[\beta := k]) \\ d(k') & = & (\langle d, k' \rangle = (IH)\, \ulcorner [\gamma]L \lrcorner\, e[\beta := k, \alpha := \langle \ulcorner N \lrcorner\, e, e\,\alpha\rangle]) \\ \ulcorner \mu\beta.[\gamma]L \lrcorner\, e[\alpha := \langle \ulcorner N \lrcorner\, e, e\,\alpha\rangle]\, k & & \end{array}$$

$(M \equiv \mu\beta.[\alpha]L$ *with* $\alpha \neq \beta)$: Then $(\mu\beta.[\alpha]L)[\alpha \Leftarrow N] \equiv \mu\beta.([\alpha]L)[\alpha \Leftarrow N]$; moreover we can safely assume $\beta \notin fn(N)$.
Then

$$\begin{array}{lcl} \ulcorner \mu\beta.([\alpha]L)[\alpha \Leftarrow N] \lrcorner\, e\, k & = & (\langle d, k' \rangle = \ulcorner ([\alpha]L)[\alpha \Leftarrow N] \lrcorner\, e[\beta := k]) \\ d(k') & = & (\langle d, k' \rangle = (HI)\, \ulcorner [\alpha]L \lrcorner\, e[\beta := k, \alpha := \langle \ulcorner N \lrcorner\, e[\beta := k], e[\beta := k]\,\alpha\rangle]) \\ \multicolumn{3}{l}{\ulcorner L \lrcorner\, e[\beta := k, \alpha := \langle \ulcorner N \lrcorner\, e[\beta := k], e[\beta := k]\,\alpha\rangle]\, \langle \ulcorner N \lrcorner\, e[\beta := k], e[\beta := k]\,\alpha \rangle =} \\ \multicolumn{3}{l}{\ulcorner L \lrcorner\, e[\beta := k, \alpha := \langle \ulcorner N \lrcorner\, e, e\,\alpha\rangle]\, \langle \ulcorner N \lrcorner\, e, e\,\alpha \rangle = (\text{by A.1 and } \beta \neq \alpha)} \\ d'(k'') & = & (\langle d', k'' \rangle = \langle \ulcorner [L \lrcorner e[\beta := k, \alpha := \langle \ulcorner N \lrcorner\, e, e\,\alpha\rangle], \langle \ulcorner N \lrcorner\, e, e\,\alpha\rangle\rangle) \\ d'(k'') & = & (\langle d', k'' \rangle = \ulcorner [\alpha]L \lrcorner e[\beta := k, \alpha := \langle \ulcorner N \lrcorner\, e, e\,\alpha\rangle]) \\ \ulcorner \mu\beta.[\alpha]L \lrcorner\, e[\alpha := \langle \ulcorner N \lrcorner\, e, e\,\alpha\rangle]\, k & & \end{array}$$

$(M \equiv \mu\alpha.[\gamma]L)$: Then $(\mu\alpha.[\gamma]L)[\alpha \Leftarrow N] \equiv \mu\alpha.[\gamma]L$ and

$$\begin{array}{lcl} \ulcorner \mu\alpha.[\gamma]L \lrcorner\, e\, k & = & \\ \multicolumn{3}{l}{\ulcorner \mu\beta.([\gamma]L)\{\beta/\alpha\} \lrcorner\, e\, k =} \\ d(k') & = & (\langle d, k' \rangle = \ulcorner ([\gamma]L)\{\beta/\alpha\} \lrcorner\, e[\beta := k] = \ulcorner ([\gamma]L)\{\beta/\alpha\}[\alpha \Leftarrow N] \lrcorner\, e[\beta := k]) \\ d(k') & = & (\langle d, k' \rangle = (HI)\, \ulcorner ([\gamma]L\{\beta/\alpha\}) \lrcorner\, e[\beta := k][\alpha := \langle \ulcorner N \lrcorner\, e[\beta := k], e[\beta := k]\,\alpha\rangle] \\ d(k') & = & (\langle d, k' \rangle = (3.4)\, \ulcorner [\gamma]L \lrcorner\, e[\beta := \langle \ulcorner N \lrcorner\, e[\beta := k], e[\beta := k]\,\alpha\rangle] = \\ \multicolumn{3}{l}{\ulcorner \mu\beta.[\gamma]L \lrcorner\, e[\beta := k][\alpha := \langle \ulcorner N \lrcorner\, e[\beta := k], e[\beta := k]\,\alpha\rangle]\, k =} \\ \multicolumn{3}{l}{\ulcorner \mu\alpha.[\gamma]L \lrcorner\, e[\alpha := \langle \ulcorner N \lrcorner\, e, e\,\alpha\rangle]\, k} \end{array}$$

■

[Soundness of $\lambda\mu\eta$]3.8 The (η) rule is sound w.r.t. the semantics, i.e.
$\ulcorner \lambda.Mx \lrcorner \;=\; \ulcorner M \lrcorner$ if $x \notin fv(M)$.

Proof: Let $e \in \mathsf{Env}$ and $k = \langle d, k' \rangle \in C$

$$\begin{array}{lcl}
\llbracket \lambda x.Mx \rrbracket\, e\, k & = & \\
\llbracket Mx \rrbracket\, e[x := d]\, k' & = & \\
\llbracket M \rrbracket\, e[x := d]\, \langle \llbracket x \rrbracket\, e[x := d], k' \rangle & = & (\text{since } \llbracket x \rrbracket\, e[x := d] = d \text{ and } k = \langle d, k' \rangle \\
\llbracket M \rrbracket\, e[x := d]\, k) & = & (\text{by } x \notin fv(M) \text{ and } A.1(i))) \\
\llbracket M \rrbracket\, e\, k & &
\end{array}$$

■

Lemma 4.2 Let $\mathcal{T}$ be an intersection type theory over the language $\mathcal{L}$ and $(\mathcal{L}, \leq)$ the induced preorder. If X is a domain and there is a surjective map $\Theta : \mathcal{L} \longrightarrow \mathcal{K}(X)$ such that

$$\forall \sigma, \tau \in \mathcal{L}.\ \sigma \leq \tau \Longleftrightarrow \Theta(\tau) \sqsubseteq \Theta(\sigma)$$

then $\mathcal{L}/_{\leq} \simeq \mathcal{K}^{op}(X)$ as inf-semilattices.

Proof: Obvious: the isomorphism is the map $\Theta' : \mathcal{L}/_{\leq} \longrightarrow \mathcal{K}^{op}(X)$ defined by $\Theta'([\sigma]) = \Theta(\sigma)$. Also we have that $\Theta'([\sigma \wedge \tau]) = \Theta(\sigma \wedge \tau) = \Theta(\sigma) \sqcap^{op} \Theta(\tau) = \Theta(\sigma) \sqcup \Theta(\tau)$, and $\Theta'([\omega]) = \Theta(\omega) = \bot$. ■

Lemma 4.5 $R \simeq \mathcal{F}^R$.

Proof: To use Lem. 4.2 we define $\Theta_R : \mathcal{L}_R \longrightarrow \mathcal{K}(R)$ by:

$$\Theta_R(\omega) = \bot, \quad \Theta_R(v_a) = a, \quad \Theta_R(\rho_1 \wedge \rho_2) = \Theta_R(\rho_1) \sqcup \Theta_R(\rho_2).$$

Θ is surjective since there is a type constant v_a for all $a \in \mathcal{K}(R)$. If $\rho_1 \leq_R \rho_2$, then we see that $\Theta(\rho_1) \sqsupseteq \Theta(\rho_2)$ by induction on the rules in definitions 4.1 and 4.4. In particular this implies that if $\rho_1 =_R \rho_2$ then $\Theta(\rho_1) = \Theta(\rho_2)$.

Viceversa suppose that $\Theta(\rho_1) \sqsupseteq \Theta(\rho_2)$. Observe that either $\rho_1 \equiv \omega$, in which case $\rho_1 =_R v_\bot$, or $\rho_1 \equiv \bigwedge_{i \in I} v_{a_i} =_R v_a$ where $a = \bigsqcup_{i \in I} a_i$. In both cases $\rho_1 = v_a$ for some $a \in \mathcal{K}(R)$, and similarly $\rho_2 =_R v_b$ for some b. Now

$$a = \Theta(v_a) = \Theta(\rho_1) \sqsupseteq \Theta(\rho_2) = \Theta(v_b) = b$$

which implies $\rho_1 =_R v_a \leq_R v_b =_R \rho_2$. ■

Lemma A.3 The functions F, G and H, K are well defined and monotonic.

Proof: Easy. Note that in the defintion of $G(f)$ we could avoid to take the upward closure, at the price of proving that if $\bigwedge_{i \in I} \kappa_i \to \rho_i \leq_D \delta$ then $\delta =_D \bigwedge_{j \in J} \kappa_j \to \rho_j$ and for all $J' \subseteq J$ there exists $I' \subseteq I$ such that if $\bigwedge_{j \in J'} \kappa_j \leq_C \bigwedge_{i \in I'} \kappa_i$ then $\bigwedge_{i \in I'} \rho_j \leq_R \bigwedge_{j \in J'} \rho_j$. ■

Theorem 4.7 *$\mathcal{F}^D \simeq [\mathcal{F}^C \to \mathcal{F}^R]$ and $\mathcal{F}^C \simeq \mathcal{F}^D \times \mathcal{F}^C$.*

Proof: By Lem. A.3 F, G, H and K are morphisms of posets. On the other hand, it is easy to verify that $F \circ G = \mathsf{Id}_{[\mathcal{F}^C \to \mathcal{F}^R]}$ and $G \circ F = \mathsf{Id}_{\mathcal{F}^D}$, and similarly that $H \circ K = \mathsf{Id}_{\mathcal{F}^D \times \mathcal{F}^C}$ and $K \circ H = \mathsf{Id}_{\mathcal{F}^C}$, so that they are bijective, and therefore isomorphisms in **ALG**. ■

Lemma A.4 $\forall n.\ C_n \simeq \mathcal{F}^{C_n}$ and $D_n \simeq \mathcal{F}^{D_n}$.

Proof: It is routine to prove, by induction over n, that both Θ_{C_n} and Θ_{D_n} satisfy the hypothesis of Lem. 4.2. ■

4.10 The filter domains $\mathcal{F}^D$ and $\mathcal{F}^C$ are the initial/final solutions of the continuation equations in **ALG**.

Proof: By Lem. A.4 and the fact that for $A = C, D$:

$$\mathcal{K}(\mathcal{F}^A) = \mathcal{F}_p^A = \bigcup_n \mathcal{F}_p^{A_n} = \bigcup_n \mathcal{K}(\mathcal{F}^{A_n}).$$

■

Theorem 4.13 *For* $A \in \{R, C, D\}$*:*
i) $\llbracket\sigma\rrbracket^A = \uparrow\Theta_A(\sigma)$
ii) $\forall \sigma, \tau \in \mathcal{L}_A.\ \sigma \leq_A \tau \iff \llbracket\sigma\rrbracket^A \subseteq \llbracket\tau\rrbracket^A$

Proof: Part (*ii.a*)) is proven by a straightforward induction over the definition of Θ_A.

Concerning (*ii.b*)), the *only if* part is proved by induction over the definition of $\leq_A$, starting with the base case where: $v_a \leq_R v_b \iff a \sqsupseteq b \iff \uparrow a \subseteq \uparrow b$.

For the *if* part, use part (*ii.a*)) and the fact that Θ_A preserves and respects $\leq_A$ w.r.t. $\sqsupseteq$ in the appropriate domain A. ■

Theorem 5.10 (Soundness) $\Gamma \vdash M : \delta \mid \Delta \ \Rightarrow\ \Gamma \models M : \delta \mid \Delta$

Proof: By induction on the structure of derivations.

(Ax): Let $e \models \Gamma, x{:}\delta, \Delta$. Then $e(x) \in \llbracket\delta\rrbracket^D$. Hence, by definition of interpretation of terms (Def. 3.2), we get $\llbracket x\rrbracket\ e \in \llbracket\delta\rrbracket^D$

$(\to E)$: Let $e \models \Gamma, \Delta$. We need to show that $\llbracket MN\rrbracket\ e \in \llbracket\kappa\to\rho\rrbracket^D$. By definition of interpretation of terms, $\llbracket MN\rrbracket\ e$ is such that, for any $k \in C$, $\llbracket MN\rrbracket\ ek = \llbracket M\rrbracket\ e\langle\llbracket N\rrbracket\ e, k\rangle$. So we need to show, by definition of interpretation of types (Def. 4.12) that, for any $k'' \in \llbracket\kappa\rrbracket^C$, $\llbracket M\rrbracket\ e\langle\llbracket N\rrbracket e, k''\rangle \in \llbracket\rho\rrbracket^R$.

By induction, we have that $\llbracket M\rrbracket e \in \llbracket\delta\times\kappa\to\rho\rrbracket$, i.e., for any $k = \langle d, k'\rangle \in \llbracket\delta\rrbracket^D \times \llbracket\kappa\rrbracket^C = \llbracket\delta\times\kappa\rrbracket^C$, $\llbracket M\rrbracket\ e\langle d, k'\rangle \in \llbracket\rho\rrbracket^R$. This implies our thesis since, by induction, we have also $\llbracket N\rrbracket\ e \in \llbracket\delta\rrbracket^D$.

$(\to I)$: Let $e \models \Gamma, \Delta$. We need to show that $\llbracket\lambda x.M\rrbracket\ e \in \llbracket\delta\times\kappa\to\rho\rrbracket^D$. In order to do that, let $k = \langle d, k'\rangle \in \llbracket\delta\times\kappa\rrbracket^C = \llbracket\delta\rrbracket^D \times \llbracket\kappa\rrbracket^C$. We need to prove that $\llbracket\lambda x.M\rrbracket\ ek \in \llbracket\rho\rrbracket^R$, that is, by definition of interpretation of terms, we need to prove that $\llbracket M\rrbracket e[x := d]k' \in \llbracket\rho\rrbracket^R$.

It is immediate to check that $e[x := d] \models \Gamma, x{:}\delta, \Delta$. Then by induction we obtain $\llbracket M\rrbracket e[x := d] \in \llbracket\kappa\to\rho\rrbracket^D$. What we need is now straightforward by definition of $\llbracket\kappa\to\rho\rrbracket^D$.

$(\bot)$: Let $e \models \Gamma, \alpha{:}\kappa, \Delta$. We need to show that $\llbracket[\alpha]M\rrbracket\ e \in \llbracket\delta\times\kappa\rrbracket^C$, i.e., by definition of interpretation of terms, that $\langle\llbracket M\rrbracket^D e, e(\alpha), \in\rangle\llbracket\delta\times\kappa\rrbracket^C$. So, by definition of interpretation of types we need to show that $\llbracket M\rrbracket^D e \in \llbracket\delta\rrbracket^D$ and $e(\alpha) \in \llbracket\kappa\rrbracket^C$. The former can be easily obtained by induction, whereas the latter derives straightforwardly from $e \models \Gamma, \alpha{:}\kappa, \Delta$.

(μ): Let $e \models \Gamma, \Delta$. We need to show that $\llbracket\mu\alpha.M\rrbracket\ e \in \llbracket\kappa\to\rho\rrbracket^D$. By definition of interpretation of types this means that we need to show that, for any $k \in \llbracket\kappa\rrbracket^C$, $\llbracket\mu\alpha.M\rrbracket\ ek \in \llbracket\rho\rrbracket^R$. Then, let $k \in \llbracket\kappa\rrbracket^C$. It is easy to check that $e[\alpha := k] \models \Gamma, \alpha{:}\kappa\Delta$. By induction we obtain $\llbracket M\rrbracket\ e[\alpha := k] \in \llbracket(\kappa'\to\rho)\times\kappa'\rrbracket^C$, that is $\pi_1(\llbracket M\rrbracket\ e[\alpha := k]) \in \llbracket\kappa'\to\rho\rrbracket^D$ and $\pi_2(\llbracket M\rrbracket\ e[\alpha := k]) \in \llbracket\kappa'\rrbracket^C$. We then obtain that
$\pi_1(\llbracket M\rrbracket\ e[\alpha := k])(\pi_2(\llbracket M\rrbracket\ e[\alpha := k])) \in \llbracket\rho\rrbracket^R$. We then get what we needed, that is $\llbracket\mu\alpha.M\rrbracket\ ek \in \llbracket\rho\rrbracket^R$, since, by definition of interpretation of terms $\llbracket\mu\alpha.M\rrbracket\ ek = \pi_1(\llbracket M\rrbracket\ e[\alpha := k])(\pi_2(\llbracket M\rrbracket\ e[\alpha := k]))$

$(\wedge I)$: Easy by induction on the interpretation of an intersection type.

(ω): Immediate by definition of interpretation of ω.

$(\leq)$: Easy, by induction and Theorem 4.13 . ■

Lemma A.5 Let $A = D, C$. $\llbracket\sigma\rrbracket^{\mathcal{F}^A} = \{f \in \mathcal{F}^A \mid \sigma \in f\}$

Proof: Easy. ■

Lemma 5.11 *Let* $A = D, C$. *Given a environment* $e \in (Var \to \mathcal{F}^D) + (Name \to \mathcal{F}^C)$, *we have*

$$\llbracket M\rrbracket^{\mathcal{F}^A} e = \{\sigma \in \mathcal{L}_A \mid \exists \Gamma, \Delta [e \models \Gamma, \Delta \ \& \ \Gamma \vdash M : \sigma \mid \Delta]\}$$

Proof: The proof is by induction on the structure of terms. In the following we shall use the notions of filter application and filter pairing as induced by the maps of Definition 4.6.

$(M \equiv x)$: $\llbracket x\rrbracket^{\mathcal{F}^D} e = e(x)$. We have to show that $e(x) = \{\delta \in \mathcal{L}_D \mid \exists \Gamma, \Delta [e \models \Gamma, \Delta \ \& \ \Gamma \vdash x : \delta \mid \Delta]\}$. Easy, by the Generation Lem. 5.3 and the fact that the elements of $\mathcal{F}^D$ are upward closed.

$(M \equiv \lambda x.N)$: Let $\langle d, k'\rangle \in \mathcal{F}^C$.

$$\begin{array}{lll}
\llbracket\lambda x.N\rrbracket^{\mathcal{F}^D}\ e\ \langle d, k'\rangle & = & \\
\llbracket N\rrbracket^{\mathcal{F}^D}\ e[x := d]\ k' & = & (IH) \\
\{\delta \in \mathcal{L}_D \mid \exists \Gamma, \Delta, \delta' [e[x := d] \models \Gamma, \Delta \ \& \ \Gamma \vdash N : \delta \mid \Delta]\} \cdot k' & = & \\
\{\kappa' \to \rho \in \mathcal{L}_D \mid \exists \Gamma, \Delta [e[x := d] \models \Gamma, \Delta \ \& \ \Gamma \vdash N : \kappa' \to \rho \mid \Delta]\} \cdot k' & = & \\
\qquad \text{(since, by Lem. A.5, } [x : d] \models x{:}\delta' \Leftrightarrow \delta' \in d) & & \\
\{\rho \in \mathcal{L}_D \mid \exists \Gamma, \Delta [e[x := d] \models \Gamma, \Delta \ \& \ \Gamma \vdash N : \kappa' \to \rho \mid \Delta \ \& \ \kappa' \in k']\} & = & \\
\{\rho \in \mathcal{L}_D \mid \exists \Gamma, \Delta [e[x := d] \models \Gamma, x{:}\delta', \Delta \ \& & & \\
\qquad \Gamma, x{:}\delta' \vdash N : \delta' \times \kappa' \to \rho \mid \Delta \ \& \ \kappa' \in k' \ \& \ \delta' \in d]\} & = & (5.3) \\
\{\delta' \times \kappa, \to \rho \in \mathcal{L}_D \mid \exists \Gamma, \Delta [e[x := d] \models \Gamma, x{:}\delta', \Delta \ \& & & \\
\qquad \Gamma, x{:}\delta' \vdash N : \delta \mid \Delta]\} \cdot \langle d, k'\rangle & = & \\
\{\delta \in \mathcal{L}_D \mid \exists \Gamma, \Delta [e \models \Gamma, \Delta \ \& \ \Gamma \vdash \lambda x.N : \delta \mid \Delta]\} \cdot \langle d, k'\rangle & &
\end{array}$$

$(M \equiv LH)$: Let $k \in \mathcal{F}^C$. By induction,

$$\begin{array}{lll}
\llbracket L\rrbracket\, e & = & \{\delta \in \mathcal{L}_D \mid \exists \Gamma, \Delta [e \models \Gamma, \Delta \ \& \ \Gamma \vdash L : \delta \mid \Delta]\} \quad \text{and} \\
\llbracket H\rrbracket\, e & = & \{\delta \in \mathcal{L}_D \mid \exists \Gamma, \Delta [e \models \Gamma, \Delta \ \& \ \Gamma \vdash H : \delta \mid \Delta]\}
\end{array}$$

Then

$$\begin{array}{lll}
\llbracket LH\rrbracket^{\mathcal{F}^D}\ e\, k & = & \\
\llbracket L\rrbracket^{\mathcal{F}^D}\ e\ \langle \llbracket H\rrbracket^{\mathcal{F}^D}\ e, k\rangle & = & \\
(\llbracket L\rrbracket^{\mathcal{F}^D}\ e) \cdot \langle \{\delta \in \mathcal{L}_D \mid \exists \Gamma, \Delta [e \models \Gamma, \Delta \ \& \ \Gamma \vdash H : \delta \mid \Delta]\}, k\rangle & = & \\
(\llbracket L\rrbracket^{\mathcal{F}^D}\ e) \cdot \{\delta \times \kappa \in \mathcal{L}_C \mid \exists \Gamma, \Delta [e \models \Gamma, \Delta, \Gamma \vdash H : \delta \mid \Delta \ \& \ \kappa \in k]\} & = & (5.3) \\
\{\rho \in \mathcal{L}_R \mid \exists \Gamma, \Delta [e \models \Gamma, \Delta, \Gamma \vdash L : \delta \times \kappa \to \rho | \Gamma \vdash H : \delta \mid \Delta \ \& \ \kappa \in k]\} & = & \\
\{\kappa \to \rho \in \mathcal{L}_D \mid \exists \Gamma, \Delta [e \models \Gamma, \Delta \ \& \ \Gamma \vdash LH : \kappa \to \delta \mid \Delta]\} \cdot k & &
\end{array}$$

$(M \equiv \mu\alpha.[\beta]N)$: Let $k \in \mathcal{F}^C$. By induction,

$$\begin{array}{lll}
\llbracket Q\rrbracket\ e[\alpha := k] & = & \\
\{\delta \times \kappa \in \mathcal{L}_C \mid \exists \Gamma, \Delta [e[\alpha := k] \models \Gamma, \alpha{:}\kappa, \Delta \ \& \ \Gamma \vdash Q : \delta \times \kappa \mid \Delta]\} & = & \langle d, k'\rangle
\end{array}$$

Then

$$\begin{array}{lll}
(\llbracket \mu\alpha.Q\rrbracket^{\mathcal{F}^D}\ e\, k) & = & \\
d(k') & = & \\
\{\rho \in \mathcal{L}_R \mid \exists \Gamma, \Delta [e[\alpha := k] \models \Gamma, \alpha{:}\kappa, \Delta \ \& \ \Gamma \vdash Q : (\kappa' \to \rho) \times \kappa' \mid \alpha{:}\kappa, \Delta]\} & = & \\
\{\rho \in \mathcal{L}_R \mid \exists \Gamma, \Delta [e \models \Gamma, \Delta \ \& \ \Gamma \vdash Q : (\kappa' \to \rho) \times \kappa' \mid \alpha{:}\kappa, \Delta \ \& \ \kappa \in k]\} & = & \\
\qquad \text{(since, by Lem. A.5, } [\alpha{:}\kappa] \models \alpha{:}\kappa \Leftrightarrow \kappa \in k) & & \\
\{\kappa \to \rho \in \mathcal{L}_D \mid \exists \Gamma, \Delta [e \models \Gamma, \Delta \ \& \ \Gamma \vdash Q : (\kappa' \to \rho) \times \kappa' \mid \alpha{:}\kappa, \Delta]\} \cdot k & = & (5.3) \\
\{\delta \in \mathcal{L}_D \mid \exists \Gamma, \Delta [e \models \Gamma, \Delta \ \& \ \Gamma \vdash \mu\alpha.[\beta]N : \delta \mid \Delta]\} \cdot k. & &
\end{array}$$

$(M \equiv [\alpha]N)$:

$$\begin{array}{ll}
\llbracket [\alpha]N \rrbracket^{\mathcal{F}^C} e & = \\
\langle \llbracket N \rrbracket^{\mathcal{F}^D} e, e\,\alpha \rangle & = \; (IH) \\
\langle \{ \delta \in \mathcal{L}_D \mid \exists \Gamma,\Delta [e \models \Gamma,\Delta \;\&\; \Gamma \vdash N:\delta,\Delta] \}, e\,\alpha \rangle & = \\
\qquad\qquad (\text{since, by Lem. A.5, } e \models \alpha{:}\kappa' \iff \kappa' \in e\,\alpha) & \\
\{ \delta{\times}\kappa' \in \mathcal{L}_C \mid \exists \Gamma,\Delta [e \models \Gamma,\alpha{:}\kappa',\Delta \;\&\; \Gamma \vdash N:\delta \mid \alpha{:}\kappa',\Delta] \} & = \; (5.3) \\
\{ \kappa \in \mathcal{L}_C \mid \exists \Gamma,\Delta [e \models \Gamma,\Delta \;\&\; \Gamma \vdash [\alpha]N:\delta \mid \Delta] \} &
\end{array}$$

Theorem 5.12 (COMPLETENESS) $\Gamma \models M:\delta \mid \Delta \;\Rightarrow\; \Gamma \vdash M:\delta \mid \Delta$.

Proof: Let $\Gamma \models M:\delta \mid \Delta$. We begin by defining a particular environment $e_{\Gamma,\Delta}$ as follows

$$e_{\Gamma,\Delta}(x) \;=\; \begin{cases} \uparrow\delta & \text{if } x{:}\delta \in \Gamma \\ \uparrow\omega & \text{otherwise} \end{cases} \qquad\qquad e_{\Gamma,\Delta}(\alpha) \;=\; \begin{cases} \uparrow\kappa & \text{if } \alpha{:}\kappa \in \Delta \\ \uparrow\omega & \text{otherwise} \end{cases}$$

By construction, $e_{\Gamma,\Delta} \models \Gamma,\Delta$. Now,

$$\begin{array}{ll}
\Gamma \models M:\delta \mid \Delta & \Rightarrow \\
\llbracket M \rrbracket e_{\Gamma,\Delta} \in \llbracket \delta \rrbracket & \Rightarrow \; (A.5) \\
\delta \in \llbracket M \rrbracket & \Rightarrow \; (5.11) \\
\exists \Gamma',\Delta' [e_{\Gamma,\Delta} \models \Gamma',\Delta' \;\&\; \Gamma' \vdash M:\delta \mid \Delta'] &
\end{array}$$

If $e_{\Gamma,\Delta} \models \Gamma',\Delta'$ then, for any $x{:}\delta' \in \Gamma'$ and $k \in \kappa' \in \Delta'$, we have that $\delta' \in e_{\Gamma,\Delta}(x)$ and $\kappa' \in e_{\Gamma,\Delta}(x)$; therefore there exist δ'' and κ'' such that $x{:}\delta'' \in \Gamma$ and $k \in \kappa'' \in \Delta$, where $\delta'' \leq \delta'$ and $\kappa'' \leq \kappa'$. Since we have that $\Gamma' \vdash M:\delta \mid \Delta'$, it is possible to use A.6 in order to obtain what we needed, namely that $\Gamma \vdash M:\delta \mid \Delta$. ■

Lemma A.6 (THINNING AND WEAKENING) *i) If $\Gamma \vdash M:\delta \mid \Delta$, then $\Gamma' \vdash M:\delta \mid \Delta'$, where $\Gamma' = \{ x{:}\delta \in \Gamma \mid x \in fv(M) \}$, and $\Delta = \{ \alpha{:}\kappa \in \Delta \mid \alpha \in fn(M) \}$.*
ii) If $\Gamma \vdash M:\delta \mid \Delta$, then $\Gamma' \vdash M:\delta \mid \Delta'$, where $\Gamma' \leq \Gamma$ and $\Delta' \leq \Delta$.

Proof: Since the system is defined using an expressive $\leq$-relation, the proof is relatively easy. ■

Notice that the weakening result is surprising: we allow Δ to become *more* specific, not less; this underlines again that the intersection operator, whose occurrence in Δ has a clear motivation from the domains we constructed above, is *not* a logical operator.

In the next proofs, we will normally assume that (wlog) $\delta = \kappa \rightarrow \rho$; we can do this, because the case $\delta_1 \wedge \delta_2$ is dealt with by splitting up the two cases, and for $\delta = \omega$ the proof becomes trivial.

Lemma 5.4 (TERM SUBSTITUTION LEMMA) *$\Gamma \vdash M[L/x]:\sigma \mid \Delta$ iff there exists δ' such that $\Gamma, x{:}\delta' \vdash M:\sigma \mid \Delta$ and $\Gamma \vdash L:\delta \mid \Delta$.*

Proof: $(M \equiv x)$: $(\Rightarrow)$: If $\Gamma \vdash x[L/x]:\delta$, then $\Gamma, x{:}\delta \vdash x:\delta$ and $\Gamma \vdash L:\delta$.

$(\Leftarrow)$: If $\Gamma, x{:}\delta' \vdash x:\delta \mid \Delta$, then $\delta' \leq \delta$ by Lem. 5.3. From $\Gamma \vdash L:\delta' \mid \Delta$ and rule $(\leq)$, we have $\Gamma \vdash L:\delta \mid \Delta$, so also $\Gamma \vdash x[L/x]:\delta \mid \Delta$.

$(M \equiv y \neq x)$: $(\Rightarrow)$: By Thinning, since $y[L/x] \equiv y$, and $x \notin fv(y)$.

$(\Leftarrow)$: $\Gamma \vdash y[L/x]:\delta \mid \Delta \Rightarrow \Gamma \vdash y:\delta \mid \Delta$. Take $\delta' = \omega$; by Weakening, $\Gamma, x{:}\omega \vdash y:\delta \mid \Delta$.

$(M \equiv \lambda y.N):\quad \exists \delta' [\, \Gamma, x{:}\delta' \vdash \lambda y.N : \delta''{\times}\kappa{\to}\rho \mid \Delta \;\&\; \Gamma \vdash L : \delta' \mid \Delta \,] \iff (\to I, 5.3)$
$\quad \exists \delta' [\, \Gamma, x{:}\delta', y{:}\delta'' \vdash N : \kappa{\to}\rho \mid \Delta \;\&\&\; \Gamma \vdash L : \delta' \mid \Delta \,] \iff (IH)$
$\quad \Gamma, y{:}\delta'' \vdash N[L/x] : \kappa{\to}\rho \mid \Delta \iff (\to I, 5.3)$
$\quad \Gamma \vdash \lambda y.(N[L/x]) : \delta''{\times}\kappa{\to}\rho \mid \Delta \triangleq \Gamma \vdash (\lambda y.N)[L/x] : \delta''{\times}\kappa{\to}\rho \mid \Delta$

$(M = M_1M_2):$ Notice that $(M_1M_2)[L/x] = P[L/x]Q[L/x]$.

$(\Rightarrow):$ Then, by Lem. 5.3, there exists δ'',κ,ρ such that $\Gamma \vdash M_1[L/x] : \delta''{\times}\kappa{\to}\rho \mid \Delta$, $\delta = \kappa{\to}\rho$, and $\Gamma \vdash M_2[L/x] : \delta'' \mid \Delta$. Then by induction, there are δ_1,δ_2 such that:

* $\Gamma, x{:}\delta_1 \vdash M_1 : \delta''{\times}\kappa{\to}\rho \mid \Delta$ and $\Gamma \vdash L : \delta_1 \mid \Delta$, as well as
* $\Gamma, x{:}\delta_2 \vdash M_2 : \delta'' \mid \Delta$ and $\Gamma \vdash L : \delta_2 \mid \Delta$.

Take $\delta' = \delta_1 \wedge \delta_2$; then by weakening and $(\to E)$, we get $\Gamma, x{:}\delta' \vdash M_1M_2 : \delta \mid \Delta$; notice that $\Gamma \vdash L : \delta' \mid \Delta$ by $(\wedge I)$.

$(\Leftarrow):$ If $\Gamma, x{:}\delta' \vdash M_1M_2 : \delta \mid \Delta$, then, by Lem. 5.3 there exists δ'',κ,ρ such that $\delta = \delta''{\times}\kappa{\to}\pi$, $\Gamma, x{:}\delta' \vdash M_1 : \kappa{\to}\rho \mid \Delta$ and $\Gamma, x{:}\delta' \vdash M_2 : \delta'' \mid \Delta$. Then, by induction, we have $\Gamma \vdash M_1[L/x] : \delta''{\times}\kappa{\to}\pi \mid \Delta$ and $\Gamma \vdash M_2[L/x] : \delta'' \mid \Delta$; the result follows by $(\to E)$.

$(M \equiv \mu\alpha.Q):\quad \exists \delta' [\, \Gamma, x{:}\delta' \vdash \mu\alpha.Q : \delta \mid \Delta \;\&\; \Gamma \vdash L : \delta' \mid \Delta \,] \iff (\mu, 5.3)$
$\quad \exists \delta',\kappa,\kappa',\rho [\, \Gamma, x{:}\delta' \vdash Q : \kappa'{\to}\rho{\times}\kappa' \mid \alpha{:}\kappa, \Delta \;\&\; \delta = \kappa{\to}\rho \;\&\; \Gamma \vdash L : \delta' \mid \Delta \,] \iff (IH)$
$\quad \Gamma, x{:}C \vdash Q[L/x] : \kappa'{\to}\rho{\times}\kappa' \mid \alpha{:}\kappa, \Delta \iff (\mu, 5.3)$
$\quad \Gamma \vdash \mu\alpha.Q[L/x] : \kappa{\to}\rho \mid \Delta \triangleq \Gamma \vdash (\mu\alpha.Q)[L/x] : \delta \mid \Delta$

$(M \equiv [\alpha]N):\quad \exists \delta' [\, \Gamma, x{:}\delta' \vdash [\alpha]N : \kappa \mid \Delta \;\&\; \Gamma \vdash L : \delta' \mid \Delta \,] \iff (\times, 5.3)$
$\quad \exists \delta',\delta'',\kappa' [\, \Gamma, x{:}\delta' \vdash N : \delta'' \mid \Delta \;\&\; \kappa = \delta''{\times}\kappa' \;\&\; \alpha{:}\kappa' \in \Delta \;\&\; \Gamma \vdash L : \delta' \mid \Delta \,] \iff (IH)$
$\quad \exists \delta'',\kappa'' [\, \Gamma \vdash N[L/x] : \delta'' \mid \Delta \;\&\; \kappa = \delta''{\times}\kappa' \;\&\; \alpha{:}\kappa' \in \Delta \,] \iff (\times, 5.3)$
$\quad \Gamma \vdash [\alpha]N[L/x] : \kappa \mid \Delta \triangleq \Gamma \vdash ([\alpha]N)[L/x] : \kappa \mid \Delta$ ■

Lemma 5.6 (Structural substitution lemma) *$\Gamma \vdash M[\alpha \Leftarrow L] : \sigma \mid \alpha{:}\kappa, \Delta$ if and only if there exists δ' such that $\Gamma \vdash L : \delta' \mid \Delta$, and $\Gamma \vdash M : \sigma \mid \alpha{:}\delta'{\times}\kappa, \Delta$.*

Proof: We only show the interesting cases.

$(M = [\alpha]N):\quad \Gamma \vdash ([\alpha]N)[\alpha \Leftarrow L] : \kappa_1 \mid \alpha{:}\kappa, \Delta \triangleq$
$\quad \Gamma \vdash [\alpha]N[\alpha \Leftarrow L]L : \kappa_1 \mid \alpha{:}\kappa, \Delta \iff (\times, 5.3)$
$\quad \exists \delta_1 [\, \Gamma \vdash N[\alpha \Leftarrow L]L : \delta_1 \mid \alpha{:}\kappa, \Delta \;\&\; \kappa_1 = \delta_1{\times}\kappa \,] \iff (\to E, 5.3)$
$\quad \exists \delta_2,\kappa_2 [\, \Gamma \vdash N[\alpha \Leftarrow L] : \delta_2{\times}\kappa_2{\to}\rho \mid \alpha{:}\kappa, \Delta \;\&$
$\quad\quad \Gamma \vdash L : \delta_2 \mid \Delta \;\&\; \kappa_1 = (\kappa_2{\to}\rho){\times}\kappa \,] \iff (IH)$
$\quad \exists \delta,\delta_2,\kappa_2 [\, \Gamma \vdash N : \delta_2{\times}\kappa_2{\to}\rho \mid \alpha{:}\delta{\times}\kappa, \Delta \;\&\; \Gamma \vdash L : \delta \mid \Delta \;\&$
$\quad\quad \Gamma \vdash L : \delta_2 \mid \Delta \;\&\; \kappa_1 = (\kappa_2{\to}\rho){\times}\kappa \,] \iff (\wedge)$
$\quad \exists \delta,\delta_2,\kappa_2 [\, \Gamma \vdash N : \delta_2{\times}\kappa_2{\to}\rho \mid \alpha{:}\delta{\times}\kappa, \Delta \;\&\; \Gamma \vdash L : \delta \wedge \delta_2 \mid \Delta \;\&$
$\quad\quad \&\; \kappa_1 = (\kappa_2{\to}\rho){\times}\kappa \,] \iff (\leq, W)$
$\quad \exists \delta,\delta_2,\kappa_2 [\, \Gamma \vdash N : \delta \wedge \delta_2{\times}\kappa_2{\to}\rho \mid \alpha{:}\delta \wedge \delta_2{\times}\kappa, \Delta \;\&\; \Gamma \vdash L : \delta \wedge \delta_2 \mid \Delta \;\&$
$\quad\quad \&\; \kappa_1 = (\kappa_2{\to}\rho){\times}\kappa \,] \iff (\times, 5.3)$
$\quad \exists \delta' [\, \Gamma \vdash [\alpha]N : \kappa_1 \mid \alpha{:}\delta'{\times}\kappa, \Delta \;\&\; \Gamma \vdash L : \delta' \mid \Delta \,]$

$(M = [\beta]N \textit{ with } \alpha \neq \beta):\quad \Gamma \vdash ([\beta]N)[\alpha \Leftarrow L] : \kappa_1 \mid \alpha{:}\kappa, \Delta \triangleq$
$\quad \Gamma \vdash [\beta]N[\alpha \Leftarrow L] : \kappa_1 \mid \alpha{:}\kappa, \Delta \iff (\times, 5.3)$
$\quad \exists \delta,\kappa_2 [\, \Gamma \vdash N[\alpha \Leftarrow L] : \delta \mid \alpha{:}\kappa, \Delta \;\&\; \kappa_1 = \delta{\times}\kappa_2 \;\&\; \beta{:}\kappa_2 \in \Delta \,] \iff (IH)$
$\quad \exists \delta',\delta,\kappa_2 [\, \Gamma \vdash N : \delta \mid \alpha{:}\delta'{\times}\kappa, \Delta \;\&\; \Gamma \vdash L : \delta' \mid \Delta \;\&$
$\quad\quad \kappa_1 = \delta{\times}\kappa_2 \;\&\; \beta{:}\kappa_2 \in \Delta \,] \iff (\times, 5.3)$
$\quad \exists \delta' [\, \Gamma \vdash [\beta]N : \kappa_1 \mid \alpha{:}\delta'{\times}\kappa, \Delta \;\&\; \Gamma \vdash L : \delta' \mid \Delta \,]$

$(M = x):\quad (\Rightarrow):$ Assume there exists δ' such that $\Gamma \vdash L : \delta' \mid \Delta$, and $\Gamma \vdash x : \delta \mid \alpha{:}\delta'{\times}\kappa, \Delta$;

since $\alpha \notin fn\,(x)$, by thinning and weakening also $\Gamma \vdash x:\delta \mid \alpha{:}\kappa,\Delta$.

$(\Leftarrow)$: Let $\Gamma \vdash x:\delta \mid \alpha{:}\kappa,\Delta$; take $\delta' = \omega$, then by rule (ω), $\Gamma \vdash L:\omega \mid \Delta$, and by thinning and weakening $\Gamma \vdash x:\delta \mid \alpha{:}\omega{\times}\kappa,\Delta$.

$(M = \lambda x.N)$: By induction.

$(M = M_1M_2)$: Then $M_1M_2[\alpha \Leftarrow L] = P\,[\alpha \Leftarrow L]\ Q\,[\alpha \Leftarrow L]$.

$(\Rightarrow)$: By Lem. 5.3, $\sigma = \kappa'{\rightarrow}\rho$ and there exists δ'' such that $\Gamma \vdash M_2\,[\alpha \Leftarrow L]:\delta'' \mid \gamma{:}\kappa,\Delta$ and $\Gamma \vdash M_1\,[\alpha \Leftarrow L]:\delta''{\times}\kappa'{\rightarrow}\rho \mid \alpha{:}\kappa,\Delta$. Then by induction, there are δ_1 and δ_2 such that

* $\Gamma \vdash M_1:\delta''{\times}\kappa'{\rightarrow}\rho \mid \alpha{:}\delta_1{\times}\kappa,\Delta$ and $\Gamma \vdash L:\delta_1 \mid \Delta$, as well as
* $\Gamma \vdash M_2:\delta'' \mid \alpha{:}\delta_2{\times}\kappa,\Delta$ and $\Gamma \vdash L:\delta_2 \mid \Delta$.

Then by weakening and $({\rightarrow}E)$, we get $\Gamma \vdash M_1M_2:\kappa'{\rightarrow}\rho \mid \alpha{:}(\delta_1{\times}\kappa){\wedge}(\delta_2{\times}\kappa),\Delta$; notice that
$\Gamma \vdash N:\delta_1{\wedge}\delta_2 \mid \Delta$ by $(\wedge)$, and $\Gamma \vdash M_1M_2:\kappa'{\rightarrow}\rho \mid \alpha{:}(\delta_1{\wedge}\delta_2){\times}\kappa,\Delta$ by weakening.

$(\Leftarrow)$: If $\Gamma \vdash M_1M_2:\kappa'{\rightarrow}\rho \mid \alpha{:}\kappa,\Delta$, then there exists δ such that $\Gamma \vdash M_1:\delta{\times}\kappa'{\rightarrow}\rho \mid \alpha{:}\kappa,\Delta$ and
$\Gamma \vdash M_2:\delta \mid \alpha{:}\kappa,\Delta$. Then, by induction, $\Gamma \vdash M_1[\alpha \Leftarrow L]:\kappa'{\rightarrow}\rho \mid \alpha{:}\delta'{\times}\kappa,\Delta$ and
$\Gamma \vdash M_2[\alpha \Leftarrow L]:\delta \mid \alpha{:}\delta'{\times}\kappa,\Delta$; the result follows by $({\rightarrow}E)$.

$$\begin{array}{lll}
(M = \mu\beta.Q): \quad \Gamma \vdash \mu\beta.Q:\delta \mid \alpha{:}\kappa,\Delta & \iff & (\mu,5.3) \\
\exists \kappa_1,\kappa_2\,[\,\Gamma \vdash Q:(\kappa_2{\rightarrow}\rho){\times}\kappa_2 \mid \alpha{:}\kappa,\Delta \ \&\ \delta = \kappa_1{\rightarrow}\rho \ \&\ \beta{:}\kappa_1 \in \Delta] & \iff & (IH) \\
\exists \delta',\kappa_1,\kappa_2\,[\,\Gamma \vdash Q[\alpha \Leftarrow L]:(\kappa_2{\rightarrow}\rho){\times}\kappa_2 \mid \alpha{:}\delta'{\times}\kappa,\Delta \ \& & & \\
\qquad\qquad \delta = \kappa_1{\rightarrow}\rho \ \&\ \beta{:}\kappa_1 \in \Delta] & \iff & (\mu,5.3) \\
\exists \delta'\,[\,\Gamma \vdash \mu\beta.Q[\alpha \Leftarrow L]:\delta \mid \alpha{:}\delta'{\times}\kappa,\Delta] \qquad \blacksquare & &
\end{array}$$

Theorem 5.7 (Subject expansion) *If $M \rightarrow N$, and $\Gamma \vdash N:\delta \mid \Delta$, then $\Gamma \vdash M:\delta \mid \Delta$.*

Proof: By induction on the definition of reduction, where we focus on the rules.

$((\lambda x.M)N \rightarrow M[N/x])$: If $\Gamma,x{:}B \vdash M[N/x]:\delta \mid \Delta$, then by Lem. 5.4 there exists a δ' such that $\Gamma,x{:}\delta' \vdash M:\delta \mid \Delta$ and $\Gamma \vdash N:\delta' \mid \Delta$; assume (wlog) that $\delta = \kappa{\rightarrow}\rho$, then, by applying rule $({\rightarrow}I)$ to the first result we get $\Gamma \vdash \lambda x.M:\delta'{\times}\kappa{\rightarrow}\rho \mid \Delta$ and by $({\rightarrow}E)$ we get $\Gamma \vdash (\lambda x.M)N:\delta \mid \Delta$.

$((\mu\alpha.Q)N \rightarrow \mu\alpha.Q[\alpha \Leftarrow N])$: If $\Gamma \vdash \mu\alpha.Q[\alpha \Leftarrow N]:\delta \mid \Delta$, then (wlog) $\delta = \kappa{\rightarrow}\rho$, and by Lem. 5.3, there exists κ' such that $\Gamma \vdash Q[\alpha \Leftarrow N]:(\kappa'{\rightarrow}\rho){\times}\kappa' \mid \alpha{:}\kappa,\Delta$. Then, by Lem. 5.6, there exists δ' such that $\Gamma \vdash N:\delta' \mid \Delta$, and $\Gamma \vdash Q:(\kappa'{\rightarrow}\rho){\times}\kappa' \mid \alpha{:}\delta'{\times}\kappa,\Delta$. Then, by rule (μ), $\Gamma \vdash \mu\alpha.Q:\delta'{\times}\kappa{\rightarrow}\rho \mid \Delta$, and $\Gamma \vdash (\mu\alpha.Q)N:\kappa{\rightarrow}\rho \mid \Delta$ follows by rule $({\rightarrow}E)$.

$(\mu\alpha[\beta]\mu\gamma.[\alpha']M \rightarrow \mu\alpha.[\alpha']M[\beta/\gamma])$: If $\Gamma \vdash \mu\alpha.[\alpha']M[\beta/\gamma]:\delta \mid \Delta$, then by Lem. 5.3 there exist $\kappa_1,\kappa_2,\kappa_3$ such that $\delta = \kappa_2{\rightarrow}\rho$, and $\Gamma \vdash M[\beta/\gamma]:\kappa_2{\rightarrow}\rho \mid \alpha{:}\kappa_1,\alpha'{:}\kappa_2,\beta{:}\kappa_3,\Delta$.

$$\dfrac{\dfrac{\Gamma \vdash M[\beta/\gamma]:\kappa_2{\rightarrow}\rho \mid \alpha{:}\kappa_1,\alpha'{:}\kappa_2,\beta{:}\kappa_3,\Delta}{\Gamma \vdash [\alpha']M[\beta/\gamma]:(\kappa_2{\rightarrow}\rho){\times}\kappa_2 \mid \alpha{:}\kappa_1,\alpha'{:}\kappa_2,\beta{:}\kappa_3,\Delta}\,(\times)}{\Gamma \vdash \mu\alpha.[\alpha']M[\beta/\gamma]:\kappa_1{\rightarrow}\rho \mid \alpha'{:}\kappa_2,\beta{:}\kappa_3,\Delta}\,(\mu)$$

Since M can contain β as well, this means that there are κ^1,κ^2 with $\kappa_3 = \kappa^1{\wedge}\kappa^2$, such that κ^1 is an intersection of the types used for the 'original' β, and κ^2 for those inserted by the substitution. Then we have $\Gamma \vdash M:\kappa \mid \kappa_2{\rightarrow}\rho \mid \alpha{:}\kappa_1,\alpha'{:}\kappa_2,\beta{:}\kappa^1,\gamma{:}\kappa^2,\Delta$ as well, and, by weakening, also
$\Gamma \vdash M:\kappa_2{\rightarrow}\rho \mid \alpha{:}\kappa_1,\alpha'{:}\kappa_2,\beta{:}\kappa_3,\gamma{:}\kappa_3,\Delta$. We can now derive:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\boxed{\qquad\qquad}}{\Gamma \vdash M : \kappa_2 \rightarrow \rho \mid \alpha{:}\kappa_1, \alpha'{:}\kappa_2, \beta{:}\kappa_3, \gamma{:}\kappa_3, \Delta}}{\Gamma \vdash [\alpha']M : (\kappa_2 \rightarrow \rho) \times \kappa_2 \mid \alpha{:}\kappa_1, \alpha'{:}\kappa_2, \beta{:}\kappa_3, \gamma{:}\kappa_3, \Delta}\,(\times)}{\Gamma \vdash \mu\gamma.[\alpha']M : \kappa_3 \rightarrow \rho \mid \alpha{:}\kappa_1, \alpha'{:}\kappa_2, \beta{:}\kappa_3, \Delta}\,(\mu)}{\Gamma \vdash [\beta]\mu\gamma.[\alpha']M : (\kappa_3 \rightarrow \rho) \times \kappa_3 \mid \alpha{:}\kappa_1, \alpha'{:}\kappa_2, \beta{:}\kappa_3, \Delta}\,(\times)}{\Gamma \vdash \mu\alpha.[\beta]\mu\gamma.[\alpha']M : \kappa_1 \rightarrow \rho \mid \alpha'{:}\kappa_2, \beta{:}\kappa_3, \Delta}\,(\mu)$$

which shows the result.

($\mu\alpha.[\alpha]M \rightarrow M$, *where* $\alpha \notin fn(M)$) : Assume $\Gamma \vdash M : \delta \mid \Delta$, such that $\alpha \notin fn(M)$, and (wlog) that $d = \kappa \rightarrow \rho$. Then, by Thinning and Weakening, we can assume that α does not occur in Δ, and we can derive $\Gamma \vdash M : \delta \mid \alpha{:}\kappa, \Delta$. Then, by rule $(\times)$, also $\Gamma \vdash [\alpha]M : (\kappa \rightarrow \rho) \times \kappa \mid \alpha{:}\kappa, \Delta$, and by rule μ, so $\Gamma \vdash \mu\alpha.[\alpha]M : \kappa \rightarrow \rho \mid \Delta$ by rule (μ). ■

Theorem 5.8 (Subject reduction) *If* $M \rightarrow N$, *and* $\Gamma \vdash M : \delta \mid \Delta$, *then* $\Gamma \vdash N : \delta \mid \Delta$

Proof: ($(\lambda x.M)N \rightarrow M[N/x]$) : Assume (wlog) that $\delta = \kappa \rightarrow \rho$. Then by Lem. 5.3 there exists δ' such that $\Gamma \vdash \lambda x.M : \delta' \times \kappa \rightarrow \rho \mid \Delta$ and also $\Gamma \vdash N : \delta' \mid \Delta$; from the first, by the same lemma, also $\Gamma, x{:}\delta' \vdash M : \kappa \rightarrow \rho \mid \Delta$. Then, by Lem. 5.4, we have $\Gamma \vdash M[N/x] : \kappa \rightarrow \rho \mid \Delta$.

($(\mu\alpha.Q)N \rightarrow \mu\alpha.Q[\alpha \Leftarrow N]$) : Assume (wlog) that $\delta = \kappa \rightarrow \rho$. Then by Lem. 5.3 there exist δ' such that $\Gamma \vdash \mu\alpha.Q : \delta' \times \kappa \rightarrow \rho \mid \Delta$ and $\Gamma \vdash N : \delta' \mid \Delta$, and, by the same lemma, from the first there exists κ' such that $\Gamma \vdash Q : (\kappa' \rightarrow \rho) \times \kappa' \mid \alpha{:}\delta' \times \kappa, \Delta$. Then, by Lem. 5.6, $\Gamma \vdash Q[\alpha \Leftarrow N] : (\kappa' \rightarrow \rho) \times \kappa' \mid \alpha{:}\kappa, \Delta$ and
$\Gamma \vdash \mu\alpha.Q[\alpha \Leftarrow N] : \kappa \rightarrow \rho \mid \Delta$ follows by rule (μ).

($\mu\alpha.[\beta]\mu\gamma.[\alpha']M \rightarrow \mu\alpha.[\alpha'](M[\beta/\gamma])$) : If $\Gamma \vdash \mu\alpha.[\alpha']\mu\gamma.[\delta]M : \delta \mid \Delta$, then by Lem. 5.3 there exist $\rho, \kappa_1, \kappa_2, \kappa_3$ such that $\Gamma \vdash M : \kappa_2 \rightarrow \rho \mid \alpha{:}\kappa_1, \alpha'{:}\kappa_2, \beta{:}\kappa_3, \gamma{:}\kappa_3, \Delta$ and $\delta = \kappa \rightarrow \rho$. Then, obviously, also
$\Gamma \vdash M[\beta/\gamma] : \kappa_2 \rightarrow \rho \mid \alpha{:}\kappa_1, \alpha'{:}\kappa_2, \beta{:}\kappa_3, \Delta$, and applying rule $(\times)$ and (μ) to this derivation gives
$\Gamma \vdash \mu\alpha.[\alpha'](M[\beta/\gamma]) : \delta \mid \alpha'{:}\kappa_2, \beta{:}\kappa_3, \Delta$.

($\mu\alpha.[\alpha]M \rightarrow M$, *where* $\alpha \notin fn(M)$) : Then by Lem. 5.3 there exists κ, κ', ρ such that (wlog) $\delta = \kappa \rightarrow \rho$, and $\Gamma \vdash [\alpha]M : (\kappa' \rightarrow \rho) \times \kappa' \mid \alpha{:}\kappa, \Delta$. Then $\kappa = \kappa'$, so we have $\Gamma \vdash M : (\kappa \rightarrow \rho) \times \kappa \mid \alpha{:}\kappa, \Delta$, so also
$\Gamma \vdash M : \kappa \rightarrow \rho \mid \alpha{:}\kappa, \Delta$; by thinning, we have also $\Gamma \vdash M : \kappa \rightarrow \rho \mid \Delta$. ■

Lemma A.7 $\overline{A} = \underline{A} \rightarrow v_a$.

Proof: ($A = \varphi$) : $\overline{\varphi} = (v_a \times \omega) \rightarrow v_a = \underline{\varphi} \rightarrow v_a$.

$$
\begin{array}{llcll}
(A = B \rightarrow C): & \overline{B \rightarrow C} & = & \overline{B} \times \kappa \rightarrow v_a \quad \textit{where } \overline{C} = \kappa \rightarrow v_a & = \ (IH) \\
 & & & (\underline{B} \rightarrow v_a) \times \kappa \rightarrow v_a \ \textit{where } \underline{C} = \kappa & = \\
 & & & (\underline{B} \rightarrow v_a) \times \underline{C} \rightarrow v_a & = \ \underline{(B \rightarrow C)} \rightarrow v_a
\end{array}
$$

Theorem 6.3 (Type preservation) *If* $\Gamma \vdash_{\mathrm{P}} M : A \mid \Delta$, *then* $\overline{\Gamma} \vdash M : \overline{A} \mid \underline{\Delta}$

Proof: (Ax) : Then $M = x$, and $x{:}A \in \Gamma$; then also $x{:}\overline{A} \in \overline{\Gamma}$, so also $\overline{\Gamma} \vdash M : \overline{A} \mid \underline{\Delta}$.

$(\bot_B)$: Then $M = [\alpha]N$, $A = \bot_B$ and $\Gamma \vdash_{\mathrm{P}} N : B \mid \Delta$ and $\alpha{:}B \in \Delta$; by induction, we have $\overline{\Gamma} \vdash N : \overline{B} \mid \underline{\alpha{:}B, \Delta}$. By Lem. A.7 and Def. 6.2, we have $\overline{\Gamma} \vdash N : \underline{B} \rightarrow v_a \mid \alpha{:}\underline{B}, \underline{\Delta}$ and can derive $\overline{\Gamma} \vdash [\alpha]N : (\underline{B} \rightarrow v_a) \times \underline{B} \mid \underline{\alpha{:}B, \Delta}$ using rule $(\times)$. Since $\overline{\bot_B} = (\underline{B} \rightarrow v_a) \times \underline{B}$, we have: $\overline{\Gamma} \vdash [\alpha]N : \underline{\bot_B} \mid \underline{\alpha{:}B, \Delta}$

(μ): Then $M = \mu\alpha.Q$ and $\Gamma \vdash_{\mathrm{P}} Q : \bot_B \mid \alpha{:}A,\Delta$; by induction, $\overline{\Gamma} \vdash Q : \overline{\bot_B} \mid \underline{\alpha{:}A,\Delta}$. Since $\underline{\alpha{:}A,\Delta} = \alpha{:}\underline{A},\underline{\Delta}$ and $\overline{\bot_B} = (\underline{B} \to v_a) \times \underline{B}$, we have $\overline{\Gamma} \vdash \mu\alpha.Q : \underline{A} \to \varphi \mid \underline{\Delta}$ by rule (μ).

$(\to I)$: Then $M = \lambda x.N$; notice that $A \neq \bot$, so $A = B \to C$, and $\Gamma, x{:}B \vdash_{\mathrm{P}} N : C \mid \Delta$; by induction, we have $\overline{\Gamma, x{:}B} \vdash N : \overline{C} \mid \underline{\Delta}$. Let $\overline{C} = \kappa \to \rho$; since $\overline{\Gamma, x{:}B} = \overline{\Gamma}, x{:}\overline{B}$, by rule $(\to I)$ we have $\overline{\Gamma} \vdash \lambda x.N : \overline{B} \times \kappa \to \rho \mid \underline{\Delta}$; notice that $\overline{B \to C} = \overline{B} \times \kappa \to \rho$.

$(\to E)$: Then $M = NL$ and there exists B such that both $\Gamma \vdash_{\mathrm{P}} N : B \to A \mid \Delta$ and $\Gamma \vdash_{\mathrm{P}} L : B \mid \Delta$; by induction, we have $\overline{\Gamma} \vdash N : \overline{B \to A} \mid \underline{\Delta}$ and $\overline{\Gamma} \vdash L : \overline{B} \mid \underline{\Delta}$. Since $\overline{B \to A} = \overline{B} \times \kappa \to \rho$, with $\overline{A} = \kappa \to \rho$, by rule $(\to E)$ we have $\overline{\Gamma} \vdash NL : \overline{A} \mid \underline{\Delta}$. ∎