Comparing Call-by-Name and Call-by-Value Reduction Strategies in Calculi for Classical Logic

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Abstract

We define call-by-name and call-by-value reduction strategies for the calculi s$\lambda\mu$ (symmetric $\lambda\mu$), $\overline{\lambda}\mu\tilde{\mu}$, and $\mathcal{X}_{\text{IS}}$ ($\mathcal{X}$ with implicit substitution). We establish a strong relation between these notions through defining a single interpretation from s$\lambda\mu$ to $\overline{\lambda}\mu\tilde{\mu}$ that respects normal reduction, as well as the call-by-name and call-by-value strategies in s$\lambda\mu$ within the their counterpart in $\overline{\lambda}\mu\tilde{\mu}$. We also define a single mapping from $\overline{\lambda}\mu\tilde{\mu}$ to $\mathcal{X}_{\text{IS}}$, and show that this also respects all three notions. We conclude with studying the natural encoding of $\mathcal{X}_{\text{IS}}$ into $\overline{\lambda}\mu\tilde{\mu}$, and show that only full reduction is respected, but that reduction steps are needed to model substitution, so the CBN and CBV strategies cannot be respected. This result underlines that $\mathcal{X}_{\text{IS}}$ and $\overline{\lambda}\mu\tilde{\mu}$ are similar, but different calculi.

keywords: classical logic, call by name, call by value, interpretations

Introduction

The λ-calculus [11, 9] has long served as a foundation for functional programming languages through its Call-by-Name (CBN) and Call-by-Value (CBV) subsystems. The former models a 'lazy' notion of reduction, in which computation of an argument is delayed until used within a function; the latter employs 'eager' evaluation, which always reduces an argument before being given to a function. Although both are subsystems of the same calculus, their semantics can vary greatly. For example, the Krivine machines (KAM) [21] for CBN need only have operations for pushing and popping arguments from the stack, whereas a CBV KAM also requires a notion of stack-frames [24] in which the function is frozen onto the stack whilst its argument is evaluated.[1] Such semantics give an operational meaning to functional languages, and the differences in the semantics can explain the difference between the behaviour of said languages.

(Classical) Logic and Computation: The λ-calculus also serves as a proof-term syntax for (an implicative fragment of) intuitionistic logic. This link between computation and logic is known as the Curry-Howard correspondence, where a proof of a proposition A corresponds to a program that will compute an answer of type A. This supports calling intuitionistic logic 'constructive', as a proof corresponds to a program that will compute an answer.

Classical logic is known to be non-constructive. It was thus thought that only intuitionistic logic enjoyed a computational counterpart, but Griffin [16] discovered an extension of the Curry-Howard correspondence [20] by typing control operators, which allow for manipulation of the current program continuation. In particular, they presented a typed λ-calculus extended with Felleisen's $\mathcal{C}$-operator of [14], for which Griffin gave the type of double-negation

[1] This feature is also present in $\overline{\lambda}\mu\tilde{\mu}$'s rule λ; see Def. 2.2.

elimination, $\neg\neg A \to A$ (or better: $((A \to \bot) \to \bot) \to A$). In this calculus, a continuation variable $k : \neg A$ represents a continuation expecting a term of type A. The logical correspondence means a notion of 'backtracking' realises classical propositions such as double-negation elimination and Pierce's law. Naturally, this led to the exploration of many 'classical calculi' – those with control operators – of which we highlight a few.

One of the earliest was Parigot's [27] $\lambda\mu$, which extends the λ-calculus with the μ-operator and a separate set of μ-variables used to denote continuations. $\lambda\mu$ allows for terms of the form $\mu\alpha.[\beta]M$, which represents a switch from the current context, which μ labels by α, to the context β. With this operator, $\lambda\mu$ implements minimal classical logic [1] through the Curry-Howard principle. A key difference compared with λC is that $\lambda\mu$ has a separate domain of variables for dealing with continuations, the μ-variables, that are (implicitly) typed with negated types, enabling it to express the 'proof by contradiction' proof step. This allows for Parigot's alternate interpretation of continuation variables as representing the 'alternative conclusions' of a sequent, rather than negated propositions.

A next significant step was the calculus $\overline{\lambda}\mu\tilde{\mu}$ [18, 12], a proof-term syntax for Gentzen's [15] sequent calculus LK, which expanded on $\lambda\mu$'s alternative conclusions. Curien and Herbelin observed the term syntax needed to reflect the symmetry of the sequents – in particular the ability to manipulate and combine assumptions. Continuations were thus made first-class as an explicit part of the syntax, where $\langle t|e\rangle$ denotes the running of term t in the continuation e. Where terms encode the proof tree of a formula on the right of a sequent, co-terms (or continuations) encode that of a formula on the left (see Def. 2.3); computation in $\overline{\lambda}\mu\tilde{\mu}$ then induces a cut-elimination in the corresponding proof tree by letting the term on the left interact with the continuation on the right. The link with cut-elimination also marks a step away from confluence.

The type system of $\overline{\lambda}\mu\tilde{\mu}$ is subtly different to LK, due to the management of the *focus* of a sequent, which states if it is concerned with a formula on the left or right of the sequent. Indeed there a two versions of the axiom rule (for focusing on the left or right), and the typing rules for μ and $\tilde{\mu}$ only shift the focus of the sequent from a neutral position: logically they perform a no-op. The shift in focus is necessary as the formula under focus is not explicitly named (assigned to a variable) by the proof-term. It would seem a true proof-term syntax for LK could be achieved by explicitly naming all formulae.

This led to the calculus $\mathcal{X}$ [7], which also came from the investigation of the computational aspects of cut-elimination [31, 22]. Whereas terms in $\overline{\lambda}\mu\tilde{\mu}$ represent the derivation of a particular formula, terms (or nets) in $\mathcal{X}$ represent the entire sequent, such that all formulae in the sequent are named by free variables (or plugs and sockets); this means that no particular formula is given precedence. $\mathcal{X}$ turns out not to just be a strong representation of LK, but is also provides a fine-grained view of computation, including the implementation of explicit substitutions. In fact, $\mathcal{X}$ represents LK rather too strongly, in that it allows for arbitrary cut-elimination, which is not strongly normalising, but of course restrictions can be made to counteract this, such as CBN and CBV.

Sub-reduction versus strategies: Although each of λ, $\lambda\mu$, $\overline{\lambda}\mu\tilde{\mu}$ and $\mathcal{X}$ have notions of CBN and CBV reduction, these can be expressed in different ways, making obtained results obscure. One method is to specify what is considered to be a valid reducible expression (redex) for λ, respectively $(\lambda x.M)N$ for CBN, and $(\lambda x.M)V$ for CBV, where V is a *value*, *i.e.* either a variable or an abstraction. CBN allows for the contraction of a redex irrespective of the shape of the argument N, whereas CBV forces the evaluation of the argument (to a value) before allowing the redex to contract. For the λ-calculus, CBV effectively becomes a sub-reduction system of CBN: all CBV redexes are CBN redexes, whereas the term $(\lambda x.x)\,(yy)$ is a CBN redex, but not one for CBV. Another method is to also specify uniquely which out of a possible multitude of redexes can be contracted, by limiting the evaluation contexts; CBN and CBV are then considered *reduction strategies*, deterministic sub-reduction systems.

For Parigot's [27] $\lambda\mu$-calculus, the situation is slightly different. Parigot's original presentation of $\lambda\mu$ was CBN, as is that of the λ-calculus. Surprisingly, the CBV $\lambda\mu$, denoted $\lambda\mu\mathrm{v}$, requires extending the reduction relation of $\lambda\mu$ with an additional rule allowing for a μ-abstraction to pull in a term on the left of an application [26], combined with the usual restrictions of arguments as values. Adding that rule to standard (CBN) $\lambda\mu$ would break confluence (this was already noted by Parigot [27]), a property that $\lambda\mu$ is set up to satisfy. But this problem disappears once the redexes are restricted to those with value for arguments, and the extra rule can be added safely. So for $\lambda\mu$, CBV is *not* a sub-reduction system of CBN; in fact, they are both sub-reduction systems for $\mathrm{s}\lambda\mu$, David and Nour's *symmetric* $\lambda\mu$ [13], which is the non-confluent system obtained from $\lambda\mu$ by adding that extra rule. For these two notions it is also possible to define CBN and CBV *strategies*, by limiting the evaluation contexts, and these have been studied extensively.

For Curien and Herbelin, Herbelin's [12, 18] $\overline{\lambda}\mu\tilde{\mu}$-calculus, the situation is again different. Reduction in $\overline{\lambda}\mu\tilde{\mu}$ is not confluent; it has a critical pair in the term $\langle \mu\alpha.c_1 | \tilde{\mu}x.c_2 \rangle$, that can be contracted in both directions, to both $c_1\{\tilde{\mu}x.c_2/\alpha\}$ and $c_2\{\mu\alpha.c_1/x\}$ (see Def. 2.2), with possibly different results. Herbelin and Curien [12] define CBN reduction by not allowing the first contraction, and CBV by not allowing the second; they do not consider strategies for either. Herbelin and Curien show that their interpretation of $\lambda\mu$ into $\overline{\lambda}\mu\tilde{\mu}$ (see Def. 5.3) respects reduction in the CBN and CBV sub-systems, but does not show that for reduction strategies; similar results are obtained in [29], but for an extended notion of $\lambda\mu$.

A similar situation exists for $\mathcal{X}$ [5], a term calculus for the implicative fragment of Gentzen's [15] sequent calculus LK. This also has a critical pair in $P\widehat{\alpha} \dagger \widehat{x}Q$ that in certain circumstances (see Def. 3.3) can reduce to both $P\widehat{\alpha} \not\dagger \widehat{x}Q$ and $P\widehat{\alpha} \dagger\!\!\!\setminus \widehat{x}Q$, terms that (can) run to different results, and CBN and CBV reduction can be defined by blocking one or the other. [5] defines an interpretation of $\lambda\mu$ and $\overline{\lambda}\mu\tilde{\mu}$ into $\mathcal{X}$, and shows that reduction is respected; it deals with CBN and CBV reduction only for the interpretations of the λ-calculus and $\lambda\mathbf{x}$ into $\mathcal{X}$.

Notions of CBN and CBV sub-reduction system for each of the calculi $\lambda\mu$, $\overline{\lambda}\mu\tilde{\mu}$, and $\mathcal{X}$ are thus obtained as restrictions on their respective notions of reduction, that differ strongly in origin and character, but are shown to correspond through various interpretations. This now naturally leads to the following questions: what if reduction *strategies* are considered? what then, if any, is the relation between these different notions? are these the same and correct restrictions? More broadly, can one be sure that the notions of CBN and CBV strategies are compatible between the calculi?

We will argue in Rem. 5.4 that it is not possible to show that Herbelin's interpretations respects the CBN and CBV reduction strategies of $\mathrm{s}\lambda\mu$; in Def. 5.5 we will give an interpretation with which such results *can* be shown. In this paper, we mainly focus on CBN and CBV *reduction strategies*, for all calculi we consider. Although these are defined in very different ways, we will see that there exists a single, natural interpretation from $\mathrm{s}\lambda\mu$ to $\overline{\lambda}\mu\tilde{\mu}$ that preserves both CBN and CBV strategies. To achieve a similar result when comparing $\overline{\lambda}\mu\tilde{\mu}$ and $\mathcal{X}$, the situation is slightly more complex. The reason for this is that $\mathcal{X}$ is a calculus defined without substitution; reduction is defined by explicitly moving a term through the syntactic structure of another in small steps, much like explicit substitution is defined for Bloo and Rose's $\lambda\mathbf{x}$ [10]. Defining a reduction strategy for a calculus with explicit substitutions is a different story altogether: for the λ-calculus, it would only propagate towards the head-variable, not the other variables, as done, for example, in [6]. Is it therefore not straightforward to compare CBN and CBV strategies for $\overline{\lambda}\mu\tilde{\mu}$ and $\mathcal{X}$; to remedy this, here we will define $\mathcal{X}_{\mathrm{IS}}$, a version of $\mathcal{X}$ that uses (implicit) substitution, for which we will follow Summers [30] and define CBN and CBV strategies, and establish the relation between $\overline{\lambda}\mu\tilde{\mu}$ and $\mathcal{X}_{\text{IS}}$.

Subsystems of Classical Calculi: Previous work has shown that each of the CBN and CBV subsystems agree – that is, for example, there is a mapping of by-name $\lambda\mu$ into by-name $\overline{\lambda}\mu\tilde{\mu}$, and $\lambda\mu$v into by-value $\overline{\lambda}\mu\tilde{\mu}$ [12] – but the mappings are *per evaluation discipline*, and thus do not answer if the restrictions on reduction are analogous. For example, in $\lambda\mu$v we have $x(\mu\alpha.C) \to \mu\gamma.C[x{\cdot}\gamma/\alpha]$. However, the CBN translation of Curien and Herbelin [12], which gives $\mu\gamma.\langle x \,|\, \mu\alpha.c{\cdot}\gamma\rangle$, is not equal to the translation of $\mu\gamma.C[x{\cdot}\gamma/\alpha]$.

The question we address in this paper is: is there is a single mapping from s$\lambda\mu$ into $\overline{\lambda}\mu\tilde{\mu}$, and a single mapping from $\overline{\lambda}\mu\tilde{\mu}$ into $\mathcal{X}$, such that equality is preserved after restricting the constituent systems through CBN and CBV reduction strategies. Concretely, if M and N are a s$\lambda\mu$-terms such that $M \to N$, then we wish not just for $[\![M]\!] =_{\overline{\lambda}} [\![N]\!]$, but also if $M \to_{\text{v}} N$ then $[\![M]\!] =^{\text{V}}_{\overline{\lambda}} [\![N]\!]$, and if $M \to_{\text{N}} N$ then $[\![M]\!] =^{\text{N}}_{\overline{\lambda}} [\![N]\!]$ (where the latter two equations mean the two terms are equal through paths using only by-value and by-name reductions, respectively; we will see that it is impossible to show these results with respect to reduction, but have to consider equality).

This paper presents two such novel mappings, satisfying a subtly stronger property, that a single reduction $M \to N$ in the source calculus induces at least one reduction in the target. More specifically, for the $\lambda\mu$ to $\overline{\lambda}\mu\tilde{\mu}$ translation, there is a t such that $[\![M]\!] \to^{+}_{\overline{\lambda}} t$ and $[\![N]\!] \to^{*}_{\overline{\lambda}} t$, and similarly for $\to^{\text{V}}_{\overline{\lambda}}$ and $\to^{\text{N}}_{\overline{\lambda}}$. The key in the mapping from $\lambda\mu$ to $\overline{\lambda}\mu\tilde{\mu}$ is that it allows for an extra possible reduction which allows the head of an application to capture its context in CBV.

Overview: The first four sections introduce the three calculi of concern: s$\lambda\mu$, $\overline{\lambda}\mu\tilde{\mu}$, and $\mathcal{X}_{\text{IS}}$. The first mapping, $[\![\cdot]\!]^{\text{S}}$, from s$\lambda\mu$ to $\overline{\lambda}\mu\tilde{\mu}$ is given in Sect. 5. Sect. 7 defines the second mapping, $[\![\cdot]\!]^{\overline{\lambda}}$, from $\overline{\lambda}\mu\tilde{\mu}$ into $\mathcal{X}$. Both the mappings $[\![\cdot]\!]^{\overline{\lambda}}$ and $[\![\cdot]\!]^{\overline{\lambda}}$ respect equality, assignable types, and call-by-name and call-by-value equality, and furthermore respect that a reduction in the source calculus gives rise to at least one reduction in the target. In Sect. 8 we will study the natural encoding of $\lceil\cdot\rfloor^{\mathcal{X}}$ of $\mathcal{X}$'s terms into $\overline{\lambda}\mu\tilde{\mu}$. Optimising that encoding slightly by avoiding to create too many cuts, we will show that $[\![\cdot]\!]^{\overline{\lambda}}$ is the right-inverse of $\lceil\cdot\rfloor^{\mathcal{X}}$, but that $\lceil\cdot\rfloor^{\mathcal{X}}$ is only $[\![\cdot]\!]^{\overline{\lambda}}$'s left-inverse up to extensionality. We will show that $\lceil\cdot\rfloor^{\mathcal{X}}$ respects reduction in $\mathcal{X}_{\text{IS}}$ by equality in $\overline{\lambda}\mu\tilde{\mu}$: to simulate $\mathcal{X}_{\text{IS}}$'s substitution in $\overline{\lambda}\mu\tilde{\mu}$ under the encoding, in the image rules (μ) and $(\tilde{\mu})$ are needed, so this encoding cannot respect the two strategies.

1 The symmetric $\lambda\mu$-calculus

The variant of $\lambda\mu$ considered in this paper extends that defined by Parigot in [27] by including the 'left-μ' reduction rule (discussed at the end of Section 3.2 in [27]); this ends up incorporating the CBV-variant as defined by Ong and Stewart [26]. The resulting calculus corresponds to David and Nour's *symmetric* $\lambda\mu$ (s$\lambda\mu$) [13], and has a non-confluent reduction system, a property shared by the two other calculi we wish to compare with.

An example of an approach for representing classical proofs, Parigot's $\lambda\mu$-calculus [27] is a natural deduction system in which there is one main conclusion that is being manipulated and possibly several alternative ones. It is a terms-as-proofs representation of a classical logic with focus. The formulas for this system are:

$$A,B \ ::= \ \varphi \mid A \to B$$

and a context Γ is a set of formulas, and the inference rules are defined through:

$$(Ax): \frac{}{\Gamma, A \vdash A \mid \Delta} \qquad (\to I): \frac{\Gamma, A \vdash B \mid \Delta}{\Gamma \vdash A \to B \mid \Delta} \qquad (\to E): \frac{\Gamma \vdash A \to B \mid \Delta \quad \Gamma \vdash A \mid \Delta}{\Gamma \vdash B \mid \Delta}$$

$$(Act): \frac{\Gamma \vdash \bot \mid A,\Delta}{\Gamma \vdash A \mid \Delta} \qquad (Pass): \frac{\Gamma \vdash A \mid A,\Delta}{\Gamma \vdash \bot \mid A,\Delta}$$

The intention of this system is to express classical logic, and for this it encapsulates the 'Proof by Contradiction' inference rule, (Pbc). The formulas in Δ are seen as negated, any statement $\Gamma \vdash_{\text{F}} A \mid \Delta$ can be seen as $\Gamma, \neg\Delta \vdash_{\text{NI}} A$ (where $\neg\Delta$ lists the negated versions of all types in Δ). With that view, the rules (Act) and $(Pass)$ corresponds to allowing the following variants of rules (Pbc) and $(\neg E)$

$$\frac{\Gamma,\neg\Delta,\neg A \vdash \bot}{\Gamma,\neg\Delta \vdash A}\,(Pbc) \qquad \frac{\overline{\Gamma,\neg\Delta,\neg A \vdash \neg A}\,(Ax) \quad \Gamma,\neg\Delta,\neg A \vdash A}{\Gamma,\neg\Delta,\neg A \vdash \bot}\,(\neg E)$$

Definition 1.1 (Syntax of s$\lambda\mu$) The terms we consider for s$\lambda\mu$ are those of $\lambda\mu$ [28], defined by the grammar:

$$\begin{array}{rcll} M,N & ::= & V \mid MN \mid \mu\alpha.\mathrm{C} & (\textit{terms}) \\ V & ::= & x \mid \lambda x.M & (\textit{values}) \\ \mathrm{C} & ::= & [\beta]M & (\textit{named terms, commands}) \end{array}$$

Recognising both λ and μ as binders, the notion of free and bound names and variables is defined as usual, and we accept Barendregt's convention to keep free and bound names and variables distinct, using (silent) α-conversion whenever necessary.

We write $x \in M$ ($\alpha \in M$) if x (α) occurs in M, either free of bound, and call a term *closed* if it has no free names or variables. We will treat the pseudo-terms of the shape $[\alpha]M$ as terms for reasons of brevity, whenever convenient.

As with Implicative Intuitionistic Logic, the reduction rules for the terms that represent the proofs correspond to proof contractions, but in $\vdash_{\text{F}}$. The reduction rules for the λ-calculus are the *logical* reductions, *i.e.* they deal with the removal of a introduction-elimination pair for a type construct; in addition to these, Parigot expresses also the *structural* rules that change the focus of a proof, where elimination essentially deals with negation and takes place for a type constructor that appears in one of the alternative conclusions (the Greek variable is the name given to a subterm). Parigot therefore needs to express that the focus of the derivation (proof) changes (see the rules in Def. 1.5), and this is achieved by extending the syntax with two new constructs $[\alpha]M$ and $\mu\alpha.M$[2] that act as witness to *passivation* and *activation* of $\vdash_{\text{F}}$, which together move the focus of the derivation, and together are called a *context switch*.

In $\lambda\mu$, reduction of terms is expressed via implicit substitution, and as usual, $M\{N/x\}$ stands for the (instantaneous) substitution of all occurrences of x in M by N. Two kinds of structural substitution are defined: the first is the standard one, defined by Parigot [27], where $M\{N{\cdot}\gamma/\alpha\}$ stands for the term obtained from M in which every command of the form $[\alpha]P$ is replaced by $[\gamma]PN$ (here γ is a fresh name). The second originates from CBV reduction, defined by Ong and Stewart [26], where $\{N{\cdot}\gamma/\alpha\}M$ stands for the term obtained from M in which every $[\alpha]P$ is replaced by $[\gamma]NP$.

They are formally defined by:

Definition 1.2 (Structural substitution) *Right-structural substitution*, $M\{N{\cdot}\gamma/\alpha\}$, and *left-structural substitution*, $\{N{\cdot}\gamma/\alpha\}M$, are defined inductively over pseudo terms by:

[2] Notice that these constructs are *pseudo* terms in that they always occur together in terms.

$$
\begin{array}{rclrcl}
x\{N\cdot\gamma/\alpha\} & \triangleq & x & \{N\cdot\gamma/\alpha\}x & \triangleq & x \\
(\lambda x.M)\{N\cdot\gamma/\alpha\} & \triangleq & \lambda x.(M\{N\cdot\gamma/\alpha\}) & \{N\cdot\gamma/\alpha\}(\lambda x.M) & \triangleq & \lambda x.(\{N\cdot\gamma/\alpha\}M) \\
(PQ)\{N\cdot\gamma/\alpha\} & \triangleq & (P\{N\cdot\gamma/\alpha\})\,(Q\{N\cdot\gamma/\alpha\}) & \{N\cdot\gamma/\alpha\}(PQ) & \triangleq & (\{N\cdot\gamma/\alpha\}P)\,(\{N\cdot\gamma/\alpha\}Q) \\
[\alpha]M\{N\cdot\gamma/\alpha\} & \triangleq & [\gamma](M\{N\cdot\gamma/\alpha\})N & \{N\cdot\gamma/\alpha\}[\alpha]M & \triangleq & [\gamma]N\,(\{N\cdot\gamma/\alpha\}M) \\
[\beta]M\{N\cdot\gamma/\alpha\} & \triangleq & [\beta](M\{N\cdot\gamma/\alpha\}) \quad (\beta \neq \alpha) & \{N\cdot\gamma/\alpha\}[\beta]M & \triangleq & [\beta]\{N\cdot\gamma/\alpha\}M \quad (\beta \neq \alpha) \\
(\mu\delta.\mathrm{C})\{N\cdot\gamma/\alpha\} & \triangleq & \mu\delta.(\mathrm{C}\{N\cdot\gamma/\alpha\}) & \{N\cdot\gamma/\alpha\}\mu\delta.\mathrm{C} & \triangleq & \mu\delta.\{N\cdot\gamma/\alpha\}\mathrm{C}
\end{array}
$$

[27] only defines the first variant of these notions of structural substitutions (so does not use the prefix 'right'); the two notions are defined together, albeit rather informally, using a notion of contexts in [26].

We have the following notions of reduction on s$\lambda\mu$. For the second and third, call by name and value, different variants exists in the literature; we follow the definitions of [1].

Definition 1.3 (s$\lambda\mu$ REDUCTION) *i*) The reduction rules of s$\lambda\mu$ are:

$$
\begin{array}{rrcll}
\textit{logical } (\beta): & (\lambda x.M)N & \to & M\{N/x\} & \\
\textit{right-structural } (\mu_{\mathrm{R}}): & (\mu\alpha.\mathrm{C})N & \to & \mu\gamma.\mathrm{C}\{N\cdot\gamma/\alpha\} & (\gamma\textit{ fresh}) \\
\textit{left-structural } (\mu_{\mathrm{L}}): & M\,(\mu\alpha.\mathrm{C}) & \to & \mu\gamma.\{M\cdot\gamma/\alpha\}\mathrm{C} & (\gamma\textit{ fresh}) \\
\textit{renaming } (\rho): & [\beta]\mu\gamma.\mathrm{C} & \to & \mathrm{C}\{\beta/\gamma\} & \\
\textit{erasing } (\theta): & \mu\alpha.[\alpha]M & \to & M & (\alpha \notin M)
\end{array}
$$

ii) We write $\rightarrow^{\mathrm{s}*}_{\lambda\mu}$ for the reflexive, transitive, and compatible closure of these steps.

Notice that for μ_{R} reduction, we have:

$$
\begin{array}{rcll}
(\mu\alpha.[\beta]M)N & \to & \mu\gamma.[\beta]M\{N\cdot\gamma/\alpha\} \quad (\beta \neq \alpha) & \text{and} \\
(\mu\alpha.[\alpha]M)N & \to & \mu\gamma.[\gamma](M\{N\cdot\gamma/\alpha\})N &
\end{array}
$$

and that for μ_{L} reduction, we have:

$$
\begin{array}{rcll}
N\,(\mu\alpha.[\beta]M) & \to & \mu\gamma.[\beta]\{N\cdot\gamma/\alpha\}M \quad (\beta \neq \alpha) & \text{and} \\
N\,(\mu\alpha.[\alpha]M) & \to & \mu\gamma.[\gamma]N\,(\{N\cdot\gamma/\alpha\}M) &
\end{array}
$$

Observe that s$\lambda\mu$ has two critical pairs given by the terms

$$
\begin{array}{c}
(\mu\alpha.[\beta]M)\,(\mu\gamma.[\delta]N) \text{ and} \\
(\lambda x.M)\,(\mu\gamma.[\delta]N)
\end{array}
$$

The first reduces to both $\mu\sigma.[\beta]M\{\mu\gamma.[\delta]N\cdot\sigma/\alpha\}$ and $\mu\tau.[\delta]\{\mu\alpha.[\beta]M\cdot\tau/\gamma\}N$ (where we assume all names are distinct), not necessarily with the same result, and similarly for the second, thus reduction in s$\lambda\mu$ is not confluent.

Historically, removing the critical pairs has led to the definition of CBN and CBV sub-reduction systems. For CBN, the standard restriction is to simply remove the rule (μ_{L}), which then yields Parigot's original $\lambda\mu$ calculus. For CBV, many different approaches exist: one is to eliminate the critical pairs by limiting the applicability of rules (μ_{L}) and (β) through allowing the contraction only in case the operand is a value V (*i.e.* a variable, or an abstraction):

$$
\begin{array}{rcll}
(\lambda x.M)V & \to & M\{V/x\} & \\
V\,(\mu\alpha.\mathrm{C}) & \to & \mu\gamma.\{V\cdot\gamma/\alpha\}\mathrm{C} & (\gamma\textit{ fresh})
\end{array}
$$

Notice that then the term $(\mu\alpha.[\beta]M)\,(\mu\gamma.[\delta]N)$ can only be a (μ_{R})-redex, and $(\lambda x.M)\,(\mu\gamma.[\delta]N)$ can only be a (μ_{L})-redex. To make this work, it is crucial that a μ-abstraction is not considered a value, although one could argue that it can be seen as a meaningful term.

This suggests allowing the contraction of *any* redex only when the argument is a value and changing rule (μ_{R}) as well:

$$
(\mu\alpha.\mathrm{C})V \;\to\; \mu\gamma.\mathrm{C}\{V\cdot\gamma/\alpha\} \quad (\gamma\textit{ fresh})
$$

as for example done by Rocheteau [29], but not every CBV reduction for $\lambda\mu$ is defined this way.

Definition 1.4 (CBN AND CBV REDUCTION STRATEGIES FOR s$\lambda\mu$)

i) The CBN *evaluation strategy* $\rightarrow^{\text{SN}}_{\lambda\mu}$ is defined by removing rule (μ_{L}), and limiting the contextual rules to:

$$P \rightarrow Q \;\Rightarrow\; \begin{cases} PM & \rightarrow\; QM \\ \mu\alpha.[\beta]P & \rightarrow\; \mu\alpha.[\beta]Q \end{cases}$$

ii) The CBV *evaluation strategy* $\rightarrow^{\text{SV}}_{\lambda\mu}$ is defined by restricting rules β and μ_{L} to:

$$\begin{array}{rrcl} (\beta_{\text{v}}): & (\lambda x.M)V & \rightarrow & M\{V/x\} \\ (\mu_{\text{LV}}): & V\,(\mu\alpha.\text{C}) & \rightarrow & \mu\gamma.\{V\cdot\gamma/\alpha\}\text{C} \quad (\gamma \textit{ fresh}) \end{array}$$

and limiting the contextual rules to:

$$P \rightarrow Q \;\Rightarrow\; \begin{cases} PM & \rightarrow\; QM \\ VP & \rightarrow\; VQ \\ \mu\alpha.[\beta]P & \rightarrow\; \mu\alpha.[\beta]Q \end{cases}$$

Notice that rule μ_{L} is not a part of the CBN strategy.

Both $\rightarrow^{\text{SN}}_{\lambda\mu}$ and $\rightarrow^{\text{SV}}_{\lambda\mu}$ are *reduction strategies* in that they pick exactly one s$\lambda\mu$-redex to contract; notice that a term may be in either CBN or CBV-normal form (*i.e.* reduction has stopped), but need not be in normal form for $\rightarrow^{\text{s}}_{\beta\mu}$. From this point onwards, we will use CBN-reduction for reduction using the CBN-reduction strategy, and likewise for CBV.

Observe that, other than in [8], we *do* consider the simplification rule θ; as argued in that paper, there it cannot be represented semantically, but creates no problems here for our interpretations.

Type assignment for s$\lambda\mu$ is defined below; since terms of s$\lambda\mu$ are the terms of $\lambda\mu$, type assignment is defined in exactly the same way. Judgements are of the shape $\Gamma \vdash_{\lambda\mu} M : A \mid \Delta$, where Δ consists of pairs of Greek characters (the *names*) and types; the left-hand context Γ, as for the λ-calculus, contains pairs of Roman characters and types, and represents the types of the free term variables of M. There is a *main*, or *active*, conclusion, labelled by the term M, and the *alternative* conclusions are labelled by names α, β, *etc* in the co-context.

Definition 1.5 (TYPING FOR $\lambda\mu$ [28]) *i*) Let φ range over a countable (infinite) set of type-variables. The set of types is defined by the grammar:

$$A, B \;::=\; \varphi \mid A \rightarrow B$$

ii) A *context* (of term variables) Γ is a partial mapping from term variables to types, denoted as a finite set of *statements* $x{:}A$, such that the *subjects* of the statements (x) are distinct. We write Γ_1, Γ_2 for the *compatible* union of Γ_1 and Γ_2 (if $x{:}A_1 \in \Gamma_1$ and $x{:}A_2 \in \Gamma_2$, then $A_1 = A_2$), and write $\Gamma, x{:}A$ for $\Gamma, \{x{:}A\}$, $x \notin \Gamma$ if there exists no A such that $x{:}A \in \Gamma$, and $\Gamma \backslash x$ for $\Gamma \backslash \{x{:}A\}$.

iii) A *context of names* Δ (or *co-context*) is a partial mapping from *names* to types, denoted as a finite set of *statements* $\alpha{:}A$, such that the *subjects* of the statements (α) are distinct. Notions Δ_1, Δ_2, as well as $\Delta, \alpha{:}A$ and $\alpha \notin \Delta$ are defined as for Γ.

iv) A *judgement* is an expression of the shape $\Gamma \vdash_{\lambda\mu} M : A \mid \Delta$; we extend the notion of free and bound variables and names to judgements $\Gamma \vdash_{\lambda\mu} M : A \mid \Delta$ and consider the term variables appearing in Γ and the names occurring in Δ as binding the free occurrences in M.

v) The type assignment rules for $\lambda\mu$ are:

$$(Ax): \frac{}{\Gamma, x{:}A \vdash x : A \mid \Delta} \qquad (\rightarrow I): \frac{\Gamma, x{:}A \vdash M : B \mid \Delta}{\Gamma \vdash \lambda x.M : A \rightarrow B \mid \Delta} \qquad (\rightarrow E): \frac{\Gamma \vdash M : A \rightarrow B \mid \Delta \quad \Gamma \vdash N : A \mid \Delta}{\Gamma \vdash MN : B \mid \Delta}$$

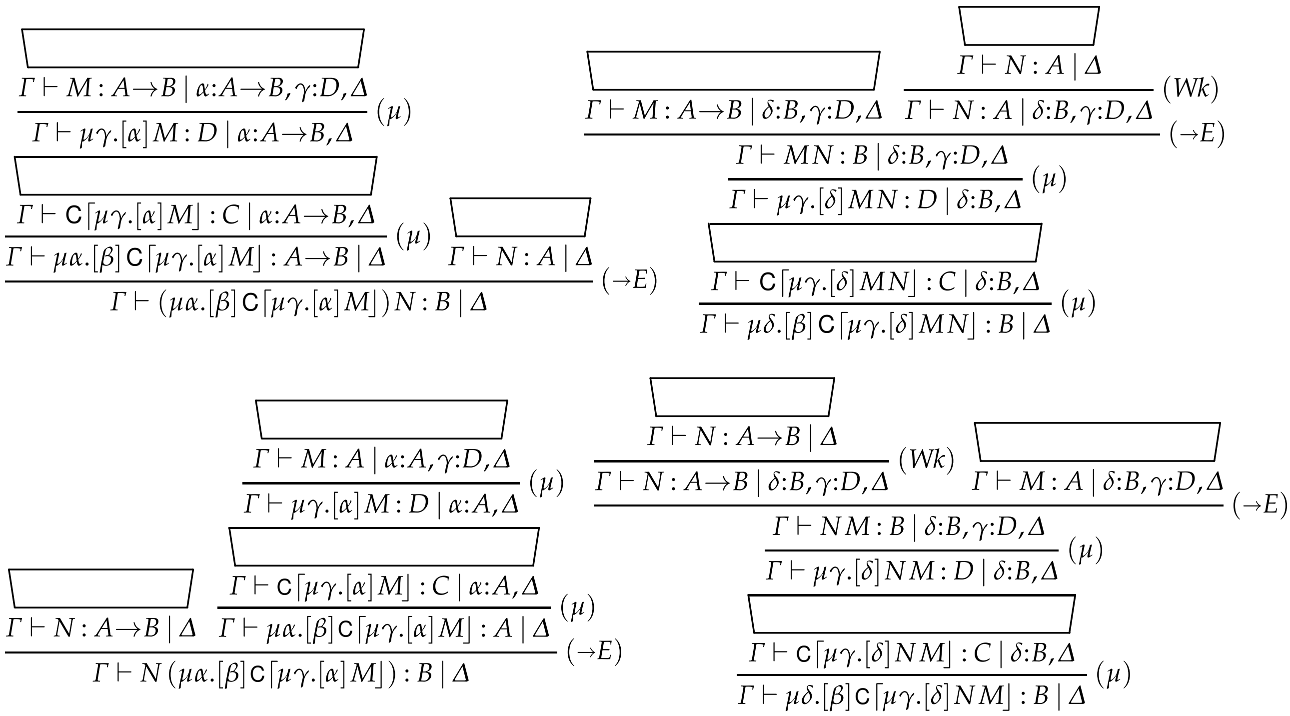

Figure 1: An illustration of structural reduction in $\lambda\mu$.

$$(\mu): \quad \frac{\Gamma \vdash M : B \mid \alpha{:}A, \beta{:}B, \Delta}{\Gamma \vdash \mu\alpha.[\beta]M : A \mid \beta{:}B, \Delta} \qquad \frac{\Gamma \vdash M : A \mid \alpha{:}A, \Delta}{\Gamma \vdash \mu\alpha.[\alpha]M : A \mid \Delta}$$

We will write $\Gamma \vdash_{\lambda\mu} M : A \mid \Delta$ for judgements derivable in this system.

We can think of $[\alpha]M$ as storing the type of M amongst the alternative conclusions by giving it the name α.

Notice that, if we erase all term information from the inference rules, we get the rules from $\vdash_{\text{F}}$, except for the variants of rule (μ); these we can infer,

$$\dfrac{\dfrac{\Gamma \vdash B \mid A, B, \Delta}{\Gamma \vdash \bot \mid A, B, \Delta}\,(Pass)}{\Gamma \vdash A \mid B, \Delta}\,(Act) \qquad \dfrac{\dfrac{\Gamma \vdash A \mid A, \Delta}{\Gamma \vdash \bot \mid A, \Delta}\,(Pass)}{\Gamma \vdash A \mid \Delta}\,(Act)$$

so they are derivable.

Fig. 1 shows type assignments for the reduction steps

$$\begin{array}{rl} (\mu_{\text{R}}): & (\mu\alpha.[\beta]\,\mathsf{C}\lceil \mu\gamma.[\alpha]M \rfloor)\,N \;\rightarrow^{s}_{\beta\mu}\; \mu\delta.[\beta]\,\mathsf{C}\lceil \mu\gamma.[\delta]MN \rfloor \\ (\mu_{\text{L}}): & N\,(\mu\alpha.[\beta]\,\mathsf{C}\lceil \mu\gamma.[\alpha]M \rfloor) \;\rightarrow^{s}_{\beta\mu}\; \mu\delta.[\beta]\,\mathsf{C}\lceil \mu\gamma.[\delta]NM \rfloor \end{array}$$

(where $\beta{:}C \in \Delta$ and α does not occur in M).

The following result is standard and of use in the proofs below.

Lemma 1.6 (Weakening and thinning for $\vdash_{\lambda\mu}$) *The following rules for* weakening *and* thinning *are admissible for* $\vdash_{\lambda\mu}$*:*

$$(Wk): \frac{\Gamma \vdash M : A \mid \Delta}{\Gamma' \vdash M : A \mid \Delta'}\ (\Gamma \subseteq \Gamma', \Delta \subseteq \Delta') \qquad (Th): \frac{\Gamma \vdash M : A \mid \Delta}{\Gamma' \vdash M : A \mid \Delta'}\ \begin{array}{l}(\Gamma' = \{\, x{:}B \in \Gamma \mid x \in fv(M) \,\}, \\ \ \Delta' = \{\, \alpha{:}B \in \Delta \mid \alpha \in fn(M) \,\})\end{array}$$

Proof: Standard. □

We will now show that type assignment is closed under reduction. This result might itself be as expected, and is presented here mostly for completeness.

First we show results for the three notions of term substitution.

Lemma 1.7 (Substitution lemma) *i) If $\Gamma, x{:}B \vdash_{\lambda\mu} M : A \mid \Delta$ and $\Gamma \vdash_{\lambda\mu} L : B \mid \Delta$, then $\Gamma \vdash_{\lambda\mu} M\{L/x\} : A \mid \Delta$.*

ii) If $\Gamma \vdash_{\lambda\mu} M : A \mid \alpha{:}B\to C, \Delta$ and $\Gamma \vdash_{\lambda\mu} L : B \mid \Delta$, then $\Gamma \vdash_{\lambda\mu} M\{L\cdot\gamma/\alpha\} : A \mid \gamma{:}C, \Delta$.

iii) If $\Gamma \vdash_{\lambda\mu} L : B\to C \mid \Delta$ and $\Gamma \vdash_{\lambda\mu} M : A \mid \alpha{:}B, \Delta$, then $\Gamma \vdash_{\lambda\mu} \{L\cdot\gamma/\alpha\}M : A \mid \gamma{:}C, \Delta$.

Proof: *i*) By induction on the definition of term substitution.

ii) By induction on the definition of right-structural substitution.

($\mu\delta.[\alpha]N\{L\cdot\gamma/\alpha\} \triangleq \mu\delta.[\gamma](N\{L\cdot\gamma/\alpha\}L)$): Then by rule ($\mu$) $\Gamma \vdash_{\lambda\mu} N : B\to C \mid \delta{:}A, \alpha{:}B\to C, \Delta$, and by induction $\Gamma \vdash_{\lambda\mu} N\{L\cdot\gamma/\alpha\} : B\to C \mid \delta{:}A, \gamma{:}C, \Delta$. Since α, δ and γ all do not occur (free) in N, we can construct

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma \vdash N\{L\cdot\gamma/\alpha\} : B\to C \mid \delta{:}A,\gamma{:}C,\Delta} \qquad \dfrac{\dfrac{\vdots}{\Gamma \vdash L : B \mid \Delta}}{\Gamma \vdash L : B \mid \delta{:}A,\gamma{:}C,\Delta}\,(Wk)}{\Gamma \vdash (N\{L\cdot\gamma/\alpha\})L : C \mid \delta{:}A,\gamma{:}C,\Delta}\,(\to E)}{\Gamma \vdash \mu\delta.[\gamma](N\{L\cdot\gamma/\alpha\})L : A \mid \gamma{:}C,\Delta}\,(\mu)$$

($(\mu\delta.[\beta]N)\{L\cdot\gamma/\alpha\} \triangleq \mu\delta.[\beta](N\{L\cdot\gamma/\alpha\})$ $(\beta \neq \alpha)$): Then by rule (μ) there exists D such that $\Delta = \beta{:}D, \Delta'$, and $\Gamma \vdash_{\lambda\mu} N : D \mid \delta{:}A, \beta{:}D, \alpha{:}B\to C, \Delta'$, and by induction $\Gamma \vdash_{\lambda\mu} N\{L\cdot\gamma/\alpha\} : D \mid \delta{:}A, \beta{:}D, \gamma{:}C, \Delta'$. But then, by rule ($\mu$), also $\mu\delta.[\beta]N\{L\cdot\gamma/\alpha\} : A : \Gamma \vdash_{\lambda\mu} \beta{:}D, \gamma{:}C, \Delta'$.

iii) By induction on the definition of left-structural substitution.

($\{L\cdot\gamma/\alpha\}\mu\delta.[\alpha]N \triangleq \mu\delta.[\gamma]L\,(\{L\cdot\gamma/\alpha\}N)$): Then by rule ($\mu$) $\Gamma \vdash_{\lambda\mu} N : B \mid \delta{:}A, \alpha{:}B, \Delta$, and by induction $\Gamma \vdash_{\lambda\mu} \{L\cdot\gamma/\alpha\}N : B \mid \delta{:}A, \gamma{:}C, \Delta$. Since δ and γ do not occur (free) in L, we can construct

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Gamma \vdash L : B\to C \mid \Delta}}{\Gamma \vdash L : B\to C \mid \delta{:}A,\gamma{:}C,\Delta}\,(Wk) \qquad \dfrac{\vdots}{\Gamma \vdash \{L\cdot\gamma/\alpha\}N : B \mid \delta{:}A,\gamma{:}C,\Delta}}{\Gamma \vdash L\{L\cdot\gamma/\alpha\}N : C \mid \delta{:}A,\gamma{:}C,\Delta}\,(\to E)}{\Gamma \vdash \mu\delta.[\gamma]L\{L\cdot\gamma/\alpha\}N : A \mid \gamma{:}C,\Delta}\,(\mu)$$

($\{L\cdot\gamma/\alpha\}(\mu\delta.[\beta]N) \triangleq \mu\delta.[\beta](\{L\cdot\gamma/\alpha\}N)$ $(\beta \neq \alpha)$): Then by rule (μ) there exists D such that $\beta{:}D, \Delta' = \Delta$, and $\Gamma \vdash_{\lambda\mu} N : D \mid \delta{:}A, \alpha{:}B, \beta{:}D, \Delta'$. Then by induction we have $\Gamma \vdash_{\lambda\mu} \{L\cdot\gamma/\alpha\}N : D \mid \delta{:}A, \gamma{:}C, \beta{:}D, \Delta'$. But then, by rule ($\mu$), also $\Gamma \vdash_{\lambda\mu} \mu\delta.[\beta]\{L\cdot\gamma/\alpha\}N : A \mid \gamma{:}C, \beta{:}D, \Delta'$. □

Theorem 1.8 (Soundness) *If $\Gamma \vdash_{\lambda\mu} M : A \mid \Delta$, and $M \to^{\text{SN}}_{\lambda\mu} N$, then $\Gamma \vdash_{\lambda\mu} N : A \mid \Delta$.*

Proof: By induction on the definition of $\to^{\text{SN}}_{\lambda\mu}$.

($(\lambda x.M)N \to^{\text{SN}}_{\lambda\mu} M\{N/x\}$): The derivation for $\Gamma \vdash_{\lambda\mu} (\lambda x.M)N : A \mid \Delta$ is shaped like

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma, x{:}B \vdash M : A \mid \Delta}}{\Gamma \vdash \lambda x.M : B\to A \mid \Delta}\,(\to I) \qquad \dfrac{\vdots}{\Gamma \vdash N : B \mid \Delta}}{\Gamma \vdash (\lambda x.M)N : A \mid \Delta}\,(\to E)$$

Then, by Lem. 1.7, we have $\Gamma \vdash_{\lambda\mu} M\{N/x\} : A \mid \Delta$.

($(\mu\alpha.[\alpha]M)N \to^{\text{SN}}_{\lambda\mu} \mu\gamma.[\gamma](M\{N\cdot\gamma/\alpha\})N$): The derivation for $(\mu\alpha.[\alpha]M)N$ is shaped like

$$\dfrac{\dfrac{\Gamma \vdash M : B\to A \mid \alpha{:}B\to A,\Delta}{\Gamma \vdash \mu\alpha.[\alpha]M : B\to A \mid \Delta}\,(\mu) \qquad \Gamma\vdash N:B\mid\Delta}{\Gamma \vdash (\mu\alpha.[\alpha]M)\,N : A\mid \Delta}\,(\to E)$$

Then by Lem. 1.7, we have $\Gamma \vdash_{\lambda\mu} M\{N{\cdot}\gamma/\alpha\} : B\to A \mid \gamma{:}A,\Delta$. Since γ is fresh, by weakening also $\Gamma \vdash_{\lambda\mu} N : B \mid \gamma{:}A,\Delta$, and we can construct

$$\dfrac{\dfrac{\Gamma \vdash M\{N{\cdot}\gamma/\alpha\} : B\to A \mid \gamma{:}A,\Delta \qquad \Gamma\vdash N : B\mid \gamma{:}A,\Delta}{\Gamma\vdash M\{N{\cdot}\gamma/\alpha\}N : A \mid \gamma{:}A,\Delta}\,(\to E)}{\Gamma\vdash \mu\gamma.[\gamma](M\{N{\cdot}\gamma/\alpha\})N : A\mid\Delta}\,(\mu)$$

($(\mu\alpha.[\delta]M)\,N \to^{\mathrm{SN}}_{\lambda\mu} \mu\gamma.[\delta]M\{N{\cdot}\gamma/\alpha\}$, *with* $\alpha\neq\delta$): The derivation for $(\mu\alpha.[\delta]M)\,N$ is shaped like

$$\dfrac{\dfrac{\Gamma \vdash M : C \mid \alpha{:}B\to A,\delta{:}C,\Delta'}{\Gamma \vdash \mu\alpha.[\delta]M : B\to A \mid \delta{:}C,\Delta'}\,(\mu) \qquad \Gamma\vdash N:B\mid\delta{:}C,\Delta'}{\Gamma \vdash (\mu\alpha.[\delta]M)\,N : A\mid \delta{:}C,\Delta'}\,(\to E)$$

with $\Delta=\delta{:}C,\Delta'$. Then by Lem. 1.7, we have $\Gamma\vdash_{\lambda\mu} M\{N{\cdot}\gamma/\alpha\} : C \mid \gamma{:}A,\delta{:}C,\Delta'$, and we can construct

$$\dfrac{\Gamma\vdash M\{N{\cdot}\gamma/\alpha\} : C\mid \gamma{:}A,\delta{:}C,\Delta'}{\Gamma\vdash \mu\gamma.[\delta]M\{N{\cdot}\gamma/\alpha\} : A\mid \delta{:}C,\Delta'}\,(\mu)$$

($M\,(\mu\alpha.[\alpha]N) \to^{\mathrm{SN}}_{\lambda\mu} \mu\gamma.[\gamma]M\,(\{M{\cdot}\gamma/\alpha\}N)$): The derivation for $M\,(\mu\alpha.[\alpha]N)$ is shaped like

$$\dfrac{\Gamma\vdash M : B\to A\mid\Delta \qquad \dfrac{\Gamma\vdash N : B\mid \alpha{:}B,\Delta}{\Gamma\vdash \mu\alpha.[\alpha]N : B\mid\Delta}\,(\mu)}{\Gamma\vdash M\,(\mu\alpha.[\alpha]N) : A\mid\Delta}\,(\to E)$$

Then by Lem. 1.7, we have $\Gamma\vdash_{\lambda\mu} \{M{\cdot}\gamma/\alpha\}N : B\mid\gamma{:}A,\Delta$, and we can construct

$$\dfrac{\dfrac{\dfrac{\Gamma\vdash M : B\to A\mid\Delta}{\Gamma\vdash M : B\to A\mid \gamma{:}A,\Delta}\,(Wk) \qquad \Gamma\vdash\{M{\cdot}\gamma/\alpha\}N : B\mid\gamma{:}A,\Delta}{\Gamma\vdash M\,(\{M{\cdot}\gamma/\alpha\}N) : A\mid\gamma{:}A,\Delta}\,(\to E)}{\Gamma\vdash\mu\gamma.[\gamma]M\,(\{M{\cdot}\gamma/\alpha\}N) : A\mid\Delta}\,(\mu)$$

($M\,(\mu\alpha.[\delta]N) \to^{\mathrm{SN}}_{\lambda\mu} \mu\gamma.[\delta]\{M{\cdot}\gamma/\alpha\}N$, *with* $\alpha\neq\delta$): The derivation for $M\,(\mu\alpha.[\delta]N)$ is shaped like

$$\dfrac{\Gamma\vdash M : B\to A\mid\delta{:}C,\Delta' \qquad \dfrac{\Gamma\vdash N : C\mid\alpha{:}B,\delta{:}C,\Delta'}{\Gamma\vdash\mu\alpha.[\delta]N : B\mid\delta{:}C,\Delta'}\,(\mu)}{M\,(\mu\alpha.[\delta]N) : A : \Gamma\vdash\delta{:}C,\Delta'}\,(\to E)$$

with $\Delta=\delta{:}C,\Delta'$. Then by Lem. 1.7, we have $\Gamma\vdash_{\lambda\mu}\{M{\cdot}\gamma/\alpha\}N : C\mid\gamma{:}A,\delta{:}C,\Delta'$, and by rule (μ) we have $\Gamma\vdash_{\lambda\mu}\mu\gamma.[\delta]\{M{\cdot}\gamma/\alpha\}N : A\mid\delta{:}C,\Delta'$.

($\mu\alpha.[\beta]\mu\gamma.[\delta]M \to^{\mathrm{SN}}_{\lambda\mu} \mu\alpha.([\delta]M)\{\beta/\gamma\}$): The derivation for $(\mu\alpha.[\delta]M)\,N$ is shaped like

$$\dfrac{\dfrac{\Gamma \vdash M : D \mid \alpha{:}A,\beta{:}B,\gamma{:}B,\delta{:}D,\Delta'}{\Gamma \vdash \mu\gamma.[\delta]M : B \mid \alpha{:}A,\beta{:}B,\delta{:}D,\Delta'}\,(\mu)}{\mu\alpha.[\beta]\mu\gamma.[\delta]M : A : \Gamma \vdash \beta{:}B,\delta{:}D,\Delta'}\,(\mu)$$

So in particular, replacing all occurrences of γ by β, we obtain a derivation for $\Gamma \vdash_{\lambda\mu} M\{\beta/\gamma\} : D \mid \alpha{:}A,\beta{:}B,\delta{:}D,\Delta'$. Now either:

$(\delta \neq \gamma)$: Then we can construct:

$$\dfrac{\Gamma \vdash M\{\beta/\gamma\} : D \mid \alpha{:}A,\beta{:}B,\delta{:}D,\Delta'}{\Gamma \vdash \mu\alpha.[\delta]M\{\beta/\gamma\} : A \mid \beta{:}B,\delta{:}D,\Delta'}\,(\mu)$$

$(\delta = \gamma)$: Then $D = B$ as well, and we can construct:

$$\dfrac{\Gamma \vdash M\{\beta/\gamma\} : B \mid \alpha{:}A,\beta{:}B,\Delta'}{\Gamma \vdash \mu\alpha.[\beta]M\{\beta/\gamma\} : A \mid \beta{:}B,\Delta'}\,(\mu)$$

$(\mu\alpha.[\alpha]M \to M,\ \textit{with}\ a \notin M)$: The derivation for $\mu\alpha.[\alpha]M$ is shaped like

$$\dfrac{\Gamma \vdash M : A \mid \alpha{:}A,\Delta}{\Gamma \vdash \mu\alpha.[\alpha]M : A \mid \Delta}\,(\mu)$$

Since $a \notin M$, by thinning we get $\Gamma \vdash_{\lambda\mu} M : A \mid \Delta$.

The contextual rules follow by induction. □

This result of course also holds for CBV and CBN-reduction as a simple corollary.

2 The calculus $\overline{\lambda}\mu\tilde{\mu}$

This section will give a short summary of Curien and Herbelin's calculus $\overline{\lambda}\mu\tilde{\mu}$, as first presented in [12]. In its typed version, $\overline{\lambda}\mu\tilde{\mu}$ is a proof-term syntax for a classical sequent calculus that treats a logic with focus, and can be seen as an extension of Parigot's $\lambda\mu$ and a variant of Gentzen's LK, restricted to implication, by adding activation and deactivation rules.

$$(cut) : \frac{\Gamma \vdash A \mid \Delta \quad \Gamma \mid A \vdash \Delta}{\Gamma \vdash \Delta} \qquad (Act\text{-}R) : \frac{c : \Gamma \vdash A,\Delta}{\Gamma \vdash A \mid \Delta} \qquad (Act\text{-}L) : \frac{c : \Gamma, A \vdash \Delta}{\Gamma \mid A \vdash \Delta}$$

$$(Ax_R) : \overline{\Gamma, A \vdash A \mid \Delta} \qquad (Ax_L) : \overline{\Gamma \mid A \vdash A,\Delta} \qquad (\to R) : \frac{\Gamma, A \vdash B \mid \Delta}{\Gamma \vdash A \to B \mid \Delta} \qquad (\to L) : \frac{\Gamma \vdash A \mid \Delta \quad \Gamma \mid B \vdash \Delta}{\Gamma \mid A \to B \vdash \Delta}$$

As in $\lambda\mu$, for $\overline{\lambda}\mu\tilde{\mu}$ there are two sets of variables: x,y,z, *etc.*, label the types of the hypotheses and α,β,γ, *etc.*, label the types of the conclusions. The syntax of $\overline{\lambda}\mu\tilde{\mu}$ has three different categories: commands, terms, and environments. Commands c form the computational units in $\overline{\lambda}\mu\tilde{\mu}$ and are composed of a pair $\langle t \,|\, e\rangle$ of a term t and its environment e.

Reduction in $\overline{\lambda}\mu\tilde{\mu}$ is dual, in that both parameter call and environment call are represented: parameter call through the environment $\tilde{\mu}x.c$ that can pull the corresponding term in to the places marked by x, and environment call through the term $\mu\alpha.c$ that places the corresponding environment in the places marked by α.

Definition 2.1 (Commands, Terms, and Contexts [12]) Let $x,y,z,\ldots$ range over an infinite, countable set of *term variables* and $\alpha,\beta,\gamma,\ldots$ range over an infinite countable set of *environment variables* (or *names*).

There are three categories of expressions in $\overline{\lambda}\mu\tilde{\mu}$, defined by:

$$\begin{array}{rcll} c & ::= & \langle t\,|\,e\rangle & (\textit{commands}) \\ t & ::= & x \mid \lambda x.t \mid \mu\beta.c & (\textit{terms}) \\ e & ::= & \alpha \mid t{\cdot}e \mid \tilde{\mu}x.c & (\textit{environments}) \end{array}$$

Here λ, μ, and $\tilde{\mu}$ are binders, and the notion of free or bound term and environment variables is defined as usual.

The environment $t{\cdot}e$ can be thought of as $e[[\,]\,t]$, and the environment $t_1{\cdot}(\cdots(t_n{\cdot}\alpha)\cdots)$ (we can omit these brackets and write $t_1{\cdot}\cdots{\cdot}t_n{\cdot}\alpha$) as a *stack*; $\mu\alpha.c$ is inherited from $\lambda\mu$, as is $\langle t|\alpha\rangle$ which corresponds to $\lambda\mu$'s *naming* construct $[\alpha]t$, giving name α to the implicit output name of t; the construct $\tilde{\mu}x.c$ can be thought of as $\mathsf{let}\ x = [\,]\ \mathsf{in}\ c$, so is an environment that can pull in a term.

Notice that each environment is a sequence of terms, ending either with a name or with an environment of the shape $\tilde{\mu}x.c$:

$$e \ ::= \begin{cases} t_1 \cdot \cdots \cdot t_n{\cdot}\alpha \\ t_1 \cdot \cdots \cdot t_n{\cdot}\tilde{\mu}x.c \end{cases}$$

Commands can be computed (thus eliminating the cut in the corresponding proof):

Definition 2.2 (Reduction in $\overline{\lambda}\mu\tilde{\mu}$ [12, 19]) Let $c\{e/\beta\}$ stand for the implicit substitution of the free occurrences of the environment variable β by the environment e, and $c\{t/x\}$ for that of x by the term t. The reduction rules are defined by:

$$\begin{array}{llcl@{\qquad}llcll} & \multicolumn{3}{c}{\textit{logical rules}} & & \multicolumn{4}{c}{\textit{extensional rules}} \\ (\lambda): & \langle\lambda x.t_1|t_2{\cdot}e\rangle & \to & \langle t_2\,|\,\tilde{\mu}x.\langle t_1|e\rangle\rangle & (\eta): & \lambda x.\mu\beta.\langle t|x{\cdot}\beta\rangle & \to & t & (x,\beta \notin fv(t)) \\ (\mu): & \langle\mu\beta.c|e\rangle & \to & c\{e/\beta\} & (\eta\mu): & \mu\alpha.\langle t|\alpha\rangle & \to & t & (\alpha \notin fv(t)) \\ (\tilde{\mu}): & \langle t|\tilde{\mu}x.c\rangle & \to & c\{t/x\} & (\eta\tilde{\mu}): & \tilde{\mu}x.\langle x|e\rangle & \to & e & (x \notin fv(e)) \end{array}$$

contextual rules

$$t \to t' \Rightarrow \begin{cases} \langle t|e\rangle \to \langle t'|e\rangle \\ \lambda x.t \to \lambda x.t' \\ t{\cdot}e \to t'{\cdot}e \end{cases} \qquad e \to e' \Rightarrow \begin{cases} \langle t|e\rangle \to \langle t|e'\rangle \\ t{\cdot}e \to t{\cdot}e' \end{cases} \qquad c \to c' \Rightarrow \begin{cases} \mu\beta.c \to \mu\beta.c' \\ \tilde{\mu}x.c \to \tilde{\mu}x.c' \end{cases}$$

We use $\to_{\overline{\lambda}}$ for this notion of reduction and $=_{\overline{\lambda}}$ for the induced equality.

We say that the reductions $\langle\mu\beta.c|e\rangle \to_{\overline{\lambda}} c\{e/\beta\}$ and $\langle t|\tilde{\mu}x.c\rangle \to_{\overline{\lambda}} c\{t/x\}$ take place over β, respectively x, and write $c \to_{\overline{\lambda}} (n)\ c'$ when the reduction step takes place over n, mainly to help the reader.

The rules (λ), (μ), and $(\tilde{\mu})$ reduce commands to commands, rules (η) and $(\eta\mu)$ reduce a term to a term, and rule $(\eta\tilde{\mu})$ reduces an environment to an environment. Apart from Thm. 5.8, the extensional rules play no role in this paper. Not all commands can be reduced: *e.g.* $\langle x|\alpha\rangle$, $\langle\lambda x.t|\alpha\rangle$ and $\langle x|t{\cdot}e\rangle$ are irreducible; this is one of the differences between LK and $\overline{\lambda}\mu\tilde{\mu}$.

Although $\overline{\lambda}\mu\tilde{\mu}$ has abstraction, it does not have application, as that corresponds to an *elimination rule*, which are not part of LK. In fact, abstraction's counterpart is that of *environment construction* $t{\cdot}e$, where a term with a hole is built, offering the operand t and the continuation e. The main operators are μ and $\tilde{\mu}$ abstraction, which, in a sense correspond to (delayed) substitution (parameter call) and to environment call.

$\overline{\lambda}\mu\tilde{\mu}$ has both *explicit* and *implicit* variables: the implicit variables are for example in $t{\cdot}e$, where the hole $\cdot$ (which acts as input) does not have an identity, and in $\lambda x.t$ where the environment (output) is anonymous. We can make these variables explicit by *naming*, respectively, $\tilde{\mu}y.\langle y|t{\cdot}e\rangle$ and $\mu\alpha.\langle\lambda x.t|\alpha\rangle$; when y (α) is fresh, these terms are η redexes, but, in general, the implicit variable can be made to correspond to one that already occurs.

Herbelin's $\overline{\lambda}\mu\tilde{\mu}$-calculus expresses elegantly the duality of LK's left- and right introduction in a very symmetric syntax. However, this duality notwithstanding, $\overline{\lambda}\mu\tilde{\mu}$ does not fully represent LK. The LK proof

$$\dfrac{\dfrac{\dfrac{\square}{\Gamma,A \vdash B,\Delta}}{\Gamma \vdash A{\rightarrow}B,\Delta}\,(\rightarrow R) \qquad \dfrac{\dfrac{\square}{\Gamma \vdash A,\Delta} \quad \dfrac{\square}{\Gamma,B \vdash \Delta}}{\Gamma,A{\rightarrow}B \vdash \Delta}\,(\rightarrow L)}{\Gamma \vdash \Delta}\,(cut)$$

reduces to both

$$\dfrac{\dfrac{\square}{\Gamma \vdash A,\Delta} \qquad \dfrac{\dfrac{\square}{\Gamma,A \vdash B,\Delta} \quad \dfrac{\dfrac{\square}{\Gamma,B \vdash \Delta}}{\Gamma,A,B \vdash \Delta}\,(Wk)}{\Gamma,A \vdash \Delta}\,(cut)}{\Gamma \vdash \Delta}\,(cut) \quad \text{and} \quad \dfrac{\dfrac{\dfrac{\dfrac{\square}{\Gamma \vdash A,\Delta}}{\Gamma \vdash A,B,\Delta}\,(Wk) \quad \dfrac{\square}{\Gamma,A \vdash B,\Delta}}{\Gamma \vdash B,\Delta}\,(cut) \qquad \dfrac{\square}{\Gamma,B \vdash \Delta}}{\Gamma \vdash \Delta}\,(cut)$$

The first result is represented in the normal reduction system of $\overline{\lambda}\mu\tilde{\mu}$ through rule (λ), but the second is not, whereas both are represented in $\mathcal{X}$, by the two right-hand sides of rule (*exp-imp*) (see Def. 3.3). This implies of course that there does not exist a full reduction-preserving interpretation of $\mathcal{X}$ into $\overline{\lambda}\mu\tilde{\mu}$. One solution would be to add the second alternative to $\overline{\lambda}\mu\tilde{\mu}$ as well, which would result in the reduction rule

$$\langle \lambda y.t \,|\, t'{\cdot}e\rangle \;\rightarrow\; \begin{cases} \langle t' \,|\, \tilde{\mu}y.\langle t|e\rangle\rangle & \textsc{cbv} \\ \langle \mu\gamma.\langle t' \,|\, \tilde{\mu}y.\langle t|\gamma\rangle\rangle \,|\, e\rangle \;\; (\gamma \textit{ fresh}) & \textsc{cbn} \end{cases}$$

Since γ is fresh, we have

$$\langle \mu\gamma.\langle t' \,|\, \tilde{\mu}y.\langle t|\gamma\rangle\rangle \,|\, e\rangle \;\rightarrow_{\overline{\lambda}}(\gamma)\; \langle t' \,|\, \tilde{\mu}y.\langle t|e\rangle\rangle$$

We will do that in Def. 8.5.

Adding this alternative would extend the expressivity of $\overline{\lambda}\mu\tilde{\mu}$, since now we would have also the reduction:

$$\begin{array}{rcl} \langle \lambda y.t \,|\, t'{\cdot}\tilde{\mu}z.c\rangle & \rightarrow & \langle \mu\gamma.\langle t' \,|\, \tilde{\mu}y.\langle t|\gamma\rangle\rangle \,|\, \tilde{\mu}z.c\rangle \\ & \rightarrow & c\{\mu\gamma.\langle t' \,|\, \tilde{\mu}y.\langle t|\gamma\rangle\rangle/z\} \end{array}$$

As to the encoding of CBV-reduction, little is gained by adding this rule; the positioning of sub-terms in $\langle \mu\beta.\langle t' \,|\, \tilde{\mu}z.\langle t|\beta\rangle\rangle \,|\, e\rangle$ and $\langle t' \,|\, \tilde{\mu}z.\langle t|e\rangle\rangle$ is very similar.

(Implicative) Typing for $\overline{\lambda}\mu\tilde{\mu}$ is defined by:

Definition 2.3 (Typing for $\overline{\lambda}\mu\tilde{\mu}$ [12]) Using the notion of types, and contexts of variables and names of Definition 1.5, type assignment for $\overline{\lambda}\mu\tilde{\mu}$ is defined via the rules:

$$(cut):\; \dfrac{\Gamma \vdash_{\overline{\lambda}} t : A \mid \Delta \quad \Gamma \mid e : A \vdash_{\overline{\lambda}} \Delta}{\langle t|e\rangle : \Gamma \vdash_{\overline{\lambda}} \Delta}$$

$$(Ax_R):\; \overline{\Gamma,x{:}A \vdash_{\overline{\lambda}} x : A \mid \Delta} \qquad (Ax_L):\; \overline{\Gamma \mid \alpha : A \vdash_{\overline{\lambda}} \alpha{:}A,\Delta}$$

$$(\rightarrow R):\; \dfrac{\Gamma,x{:}A \vdash_{\overline{\lambda}} t : B \mid \Delta}{\Gamma \vdash_{\overline{\lambda}} \lambda x.t : A{\rightarrow}B \mid \Delta} \qquad (\rightarrow L):\; \dfrac{\Gamma \vdash_{\overline{\lambda}} t : A \mid \Delta \quad \Gamma \mid e : B \vdash_{\overline{\lambda}} \Delta}{\Gamma \mid t{\cdot}e : A{\rightarrow}B \vdash_{\overline{\lambda}} \Delta}$$

$$(\mu):\; \dfrac{c : \Gamma \vdash_{\overline{\lambda}} \alpha{:}A,\Delta}{\Gamma \vdash_{\overline{\lambda}} \mu\alpha.c : A \mid \Delta} \qquad (\tilde{\mu}):\; \dfrac{c : \Gamma,x{:}A \vdash_{\overline{\lambda}} \Delta}{\Gamma \mid \tilde{\mu}x.c : A \vdash_{\overline{\lambda}} \Delta}$$

We write $c : \Gamma \vdash_{\overline{\lambda}} \Delta$, $\Gamma \vdash_{\overline{\lambda}} t : A \mid \Delta$, and $\Gamma \mid e : A \vdash_{\overline{\lambda}} \Delta$ if there exists a derivation built using these rules that has this judgement in the bottom line.

We will now show soundness, *i.e.* that type assignment is respected by reduction; first we show a substitution lemma.

Lemma 2.4 *i) If* $c : \Gamma, x{:}A \vdash_{\overline{\lambda}} \Delta$*, and* $\Gamma \vdash_{\overline{\lambda}} t : A \mid \Delta$*, then* $c\{t/x\} : \Gamma \vdash_{\overline{\lambda}} \Delta$*.*
ii) If $c : \Gamma \vdash_{\overline{\lambda}} \beta{:}A, \Delta$*, and* $\Gamma \mid e : A \vdash_{\overline{\lambda}} \Delta$*, then* $c\{e/\beta\} : \Gamma \vdash_{\overline{\lambda}} \Delta$*.*

Proof: Straightforward by induction. □

We can now show that assignable types are preserved under reduction.

Theorem 2.5 (Subject reduction) *i) If* $c : \Gamma \vdash_{\overline{\lambda}} \Delta$*, and* $c \to_{\overline{\lambda}} c'$*, then* $c' : \Gamma \vdash_{\overline{\lambda}} \Delta$*.*
ii) If $\Gamma \vdash_{\overline{\lambda}} t : A \mid \Delta$*, and* $t \to_{\overline{\lambda}} t'$*, then* $\Gamma \vdash_{\overline{\lambda}} t' : A \mid \Delta$*.*
iii) If $\Gamma \mid e : A \vdash_{\overline{\lambda}} \Delta$*, and* $e \to_{\overline{\lambda}} e'$*, then* $\Gamma \mid e' : A \vdash_{\overline{\lambda}} \Delta$*.*

Proof: Simultaneous by induction on the definition of $\to^*_{\overline{\lambda}}$; we will only show the base cases.

($\langle \lambda x.t_1 | t_2{\cdot}e \rangle \to \langle t_2 \mid \tilde{\mu}x.\langle t_1 | e \rangle \rangle$): If $\langle \lambda x.t_1 | t_2{\cdot}e \rangle : \Gamma \vdash_{\overline{\lambda}} \Delta$, then the derivation is shaped like on the left; regrouping the sub-derivations, we can construct the one on the right.

$$\dfrac{\dfrac{\Gamma, x{:}A \vdash_{\overline{\lambda}} t_1 : B \mid \Delta}{\Gamma \vdash_{\overline{\lambda}} \lambda x.t_1 : A{\to}B \mid \Delta} \qquad \dfrac{\Gamma \vdash_{\overline{\lambda}} t_2 : A \mid \Delta \quad \Gamma \mid e : B \vdash_{\overline{\lambda}} \Delta}{\Gamma \mid t_2{\cdot}e : A{\to}B \vdash_{\overline{\lambda}} \Delta}}{\langle \lambda x.t_1 | t_2{\cdot}e \rangle : \Gamma \vdash_{\overline{\lambda}} \Delta} \qquad \dfrac{\Gamma \vdash_{\overline{\lambda}} t_2 : A \mid \Delta \qquad \dfrac{\dfrac{\Gamma, x{:}A \vdash_{\overline{\lambda}} t_1 : B \mid \Delta \quad \dfrac{\Gamma \mid e : B \vdash_{\overline{\lambda}} \Delta}{\Gamma, x{:}A \mid e : B \vdash_{\overline{\lambda}} \Delta}\,(Wk)}{\langle t_1 | e \rangle : \Gamma, x{:}A \vdash_{\overline{\lambda}} \Delta}}{\Gamma \mid \tilde{\mu}x.\langle t_1 | e \rangle : A \vdash_{\overline{\lambda}} \Delta}}{\langle t_2 \mid \tilde{\mu}x.\langle t_1 | e \rangle \rangle : \Gamma \vdash_{\overline{\lambda}} \Delta}$$

($\langle \mu\beta.c | e \rangle \to c\{e/\beta\}$): If $\langle \mu\beta.c | e \rangle : \Gamma \vdash_{\overline{\lambda}} \Delta$, then the derivation is shaped like

$$\dfrac{\dfrac{c : \Gamma \vdash_{\overline{\lambda}} \beta{:}A, \Delta}{\Gamma \vdash_{\overline{\lambda}} \mu\beta.c : A \mid \Delta} \quad \Gamma \mid e : A \vdash_{\overline{\lambda}} \Delta}{\langle \mu\beta.c | e \rangle : \Gamma \vdash_{\overline{\lambda}} \Delta}$$

By Lemma 2.4 we get $c\{e/\beta\} : \Gamma \vdash_{\overline{\lambda}} \Delta$.

($\langle t | \tilde{\mu}x.c \rangle \to c\{t/x\}$): If $\langle t | \tilde{\mu}x.c \rangle : \Gamma \vdash_{\overline{\lambda}} \Delta$, then the derivation is shaped like

$$\dfrac{\Gamma \vdash_{\overline{\lambda}} t : A \mid \Delta \quad \dfrac{c : \Gamma, x{:}A \vdash_{\overline{\lambda}} \Delta}{\Gamma \mid \tilde{\mu}x.c : A \vdash_{\overline{\lambda}} \Delta}}{\langle t | \tilde{\mu}x.c \rangle : \Gamma \vdash_{\overline{\lambda}} \Delta}$$

By Lemma 2.4 we get $c\{t/x\} : \Gamma \vdash_{\overline{\lambda}} \Delta$.

($\lambda x.\mu\beta.\langle t | x{\cdot}\beta \rangle \to t,\ x,\beta \notin fv(t)$): If $\Gamma \vdash_{\overline{\lambda}} \lambda x.\mu\beta.\langle t | x{\cdot}\beta \rangle : A \mid \Delta$, then the derivation is shaped like

$$\dfrac{\dfrac{\dfrac{\Gamma, x{:}A \vdash_{\overline{\lambda}} t : A{\to}B \mid \beta{:}B, \Delta \qquad \dfrac{\overline{\Gamma, x{:}A \vdash_{\overline{\lambda}} x : A \mid \beta{:}B, \Delta} \quad \overline{\Gamma, x{:}A \mid \beta : B \vdash_{\overline{\lambda}} \beta{:}B, \Delta}}{\Gamma, x{:}A \mid x{\cdot}\beta : A{\to}B \vdash_{\overline{\lambda}} \beta{:}B, \Delta}}{\langle t | x{\cdot}\beta \rangle : \Gamma, x{:}A \vdash_{\overline{\lambda}} \beta{:}B, \Delta}}{\Gamma, x{:}A \mid \mu\beta.\langle t | x{\cdot}\beta \rangle : B \vdash_{\overline{\lambda}} \Delta}}{\Gamma \vdash_{\overline{\lambda}} \lambda x.\mu\beta.\langle t | x{\cdot}\beta \rangle : A{\to}B \mid \Delta}$$

From $\Gamma, x{:}A \vdash_{\overline{\lambda}} t : A{\to}B \mid \beta{:}B, \Delta$ and $x, \beta \notin fv(t)$, by Thinning we get $\Gamma \vdash_{\overline{\lambda}} t : A{\to}B \mid \Delta$.

($\mu\alpha.\langle t | \alpha \rangle \to t,\ \alpha \notin fv(t)$): If $\Gamma \vdash_{\overline{\lambda}} \mu\alpha.\langle t | \alpha \rangle : A \mid \Delta$, then the derivation is shaped like

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma \vdash_{\overline{\lambda}} t : A \mid \alpha{:}A,\Delta} \qquad \dfrac{}{\Gamma \mid \alpha : A \vdash_{\overline{\lambda}} \alpha{:}A,\Delta}}{\langle t|\alpha\rangle : \Gamma \vdash_{\overline{\lambda}} \alpha{:}A,\Delta}}{\Gamma \vdash_{\overline{\lambda}} \mu\alpha.\langle t|\alpha\rangle : A \mid \Delta}$$

From $\Gamma \vdash_{\overline{\lambda}} t : A \mid \alpha{:}A,\Delta$ and $\alpha \notin fv(t)$, by Thinning we get $\Gamma \vdash_{\overline{\lambda}} t : A \mid \Delta$.

$(\tilde{\mu}x.\langle x|e\rangle \to e,\ x \notin fv(e))$: If $\Gamma \vdash_{\overline{\lambda}} \mu\alpha.\langle t|\alpha\rangle : A \mid \Delta$, then the derivation is shaped like

$$\dfrac{\dfrac{\dfrac{}{\Gamma,x{:}A \vdash_{\overline{\lambda}} x : A \mid \Delta} \qquad \dfrac{\vdots}{\Gamma,x{:}A \mid e : A \vdash_{\overline{\lambda}} \Delta}}{\langle x|e\rangle : \Gamma,x{:}A \vdash_{\overline{\lambda}} \Delta}}{\Gamma \mid \tilde{\mu}x.\langle x|e\rangle : A \vdash_{\overline{\lambda}} \Delta}$$

From $\Gamma,x{:}A \mid e : A \vdash_{\overline{\lambda}} \Delta$ and $x \notin fv(e)$, by Thinning we get $\Gamma \mid e : A \vdash_{\overline{\lambda}} \Delta$. □

We can extend this last result also for the second alternative to rule (λ), since we can derive:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Gamma \vdash_{\overline{\lambda}} t' : A \mid \Delta} \qquad \dfrac{\dfrac{\dfrac{\vdots}{\Gamma,x{:}A \vdash_{\overline{\lambda}} t : B \mid \Delta} \quad \dfrac{}{\Gamma,y{:}A \vdash_{\overline{\lambda}} \gamma : B \mid \gamma{:}B,\Delta}}{\langle t|\gamma\rangle : \Gamma,y{:}A \vdash_{\overline{\lambda}} \gamma{:}B,\Delta}}{\Gamma \mid \tilde{\mu}y.\langle t|\gamma\rangle : A \vdash_{\overline{\lambda}} \gamma{:}B,\Delta}}{\langle t' \mid \tilde{\mu}y.\langle t|\gamma\rangle\rangle : \Gamma \vdash_{\overline{\lambda}} \gamma{:}B,\Delta}}{\Gamma \vdash_{\overline{\lambda}} \mu\gamma.\langle t' \mid \tilde{\mu}y.\langle t|\gamma\rangle\rangle : A \mid \Delta} \qquad \dfrac{\vdots}{\Gamma \mid e : B \vdash_{\overline{\lambda}} \Delta}}{\langle \mu\gamma.\langle t' \mid \tilde{\mu}y.\langle t|\gamma\rangle\rangle \mid e\rangle : \Gamma \vdash_{\overline{\lambda}} \Delta}$$

$\overline{\lambda}\mu\tilde{\mu}$ has a critical pair in the command $\langle \mu\alpha.c_1 | \tilde{\mu}x.c_2\rangle$, which reduces to both $c_1\{\tilde{\mu}x.c_2/\alpha\}$ and $c_2\{\mu\alpha.c_1/x\}$; since *cut*-elimination of the classical sequent calculus is not confluent, neither is reduction in $\overline{\lambda}\mu\tilde{\mu}$. For example, in LK the proof (where (W) is the admissible weakening rule)

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma \vdash \Delta}}{\Gamma \vdash A,\Delta}\,(W) \qquad \dfrac{\dfrac{\mathcal{D}_2}{\Gamma \vdash \Delta}}{\Gamma,A \vdash \Delta}\,(W)}{\Gamma \vdash \Delta}\,(cut)$$

reduces to both $\mathcal{D}_1$ and $\mathcal{D}_2$, different proofs, albeit for the same sequence; likewise, in $\vdash_{\overline{\lambda}}$ we can derive (where α does not appear in c_1, and x does not appear in c_2):

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{c_1 : \Gamma \vdash_{\overline{\lambda}} \Delta}}{c_1 : \Gamma \vdash_{\overline{\lambda}} \alpha{:}A,\Delta}\,(W)}{\Gamma \vdash_{\overline{\lambda}} \mu\alpha.c_1 : A \mid \Delta}\,(\mu) \qquad \dfrac{\dfrac{\dfrac{\vdots}{c_2 : \Gamma \vdash_{\overline{\lambda}} \Delta}}{c_2 : \Gamma,x{:}A \vdash_{\overline{\lambda}} \Delta}\,(W)}{\Gamma \mid \tilde{\mu}x.c_2 : A \vdash_{\overline{\lambda}} \Delta}\,(\tilde{\mu})}{\langle \mu\alpha.c_1 | \tilde{\mu}x.c_2\rangle : \Gamma \vdash_{\overline{\lambda}} \Delta}\,(cut)$$

and $\langle \mu\alpha.c_1 | \tilde{\mu}x.c_2\rangle$ reduces to both c_1 and c_2: witnesses to the same sequent, but not necessarily the same proof.

On the other hand, the term $\mu\gamma.\langle \lambda x.t \mid \mu\alpha.c{\cdot}\gamma\rangle$ is *not* a $\overline{\lambda}\mu\tilde{\mu}$ critical pair, whereas its s$\lambda\mu$-counterpart $(\lambda x.M)\,(\mu\alpha.\mathrm{C})$ (see Sect. 5) *is* a s$\lambda\mu$ critical pair. We will come back to this at the end of Sect. 5.

The $\overline{\lambda}\mu\tilde{\mu}$-calculus expresses the duality of LK's left and right introduction in a very symmetric syntax. But the duality goes beyond that: for instance, the symmetry of the reduction rules displays syntactically the duality between the CBV and CBN evaluations (see also [33]).

In [12] the CBV sub-reduction is not defined as a strategy but is obtained by forbidding a $\tilde{\mu}$-reduction when the command is also a μ-redex, whereas the CBN sub-reduction forbids a

μ-reduction when the redex is also a $\tilde{\mu}$-redex; there is no other restriction defined in [12, 19] in terms of not permitting certain contextual rules in the definition of CBV and CBN. Since we want CBN and CBV to be reduction *strategies* in the sense that each term has at most one contractable cut, we will define those here.

Definition 2.6 (CBN AND CBV REDUCTION STRATEGIES FOR $\overline{\lambda}\mu\tilde{\mu}$) *i*) *Values* V are defined by $V ::= x \mid \lambda x.t$, and *stacks*[3] S are defined by $S ::= \alpha \mid t{\cdot}S$.

ii) The CBN-reduction strategy $\rightarrow_{\lambda}^{\mathrm{N}}$ is defined by limiting rule (μ) and restricting the contextual rules:

$$\begin{array}{rrcll} (\lambda): & \langle \lambda x.t_1 | t_2{\cdot}e \rangle & \rightarrow & \langle t_2 \,|\, \tilde{\mu}x.\langle t_1 | e \rangle\rangle & \\ (\mu_{\mathrm{N}}): & \langle \mu\beta.c | S \rangle & \rightarrow & c\{S/\beta\} & t \rightarrow t' \;\Rightarrow\; \langle t|e\rangle \rightarrow \langle t'|e\rangle \\ (\tilde{\mu}): & \langle t | \tilde{\mu}x.c \rangle & \rightarrow & c\{t/x\} & c \rightarrow c' \;\Rightarrow\; \mu\beta.c \rightarrow \mu\beta.c' \\ (\eta\mu): & \mu\alpha.\langle t|\alpha\rangle & \rightarrow & t \quad (\alpha \notin fv(t)) & \end{array}$$

iii) The CBV-reduction strategy $\rightarrow_{\lambda}^{\mathrm{V}}$ is defined by limiting rule $(\tilde{\mu})$ and restricting the contextual rules:

$$\begin{array}{rrcll} (\lambda): & \langle \lambda x.t_1 | t_2{\cdot}e \rangle & \rightarrow & \langle t_2 \,|\, \tilde{\mu}x.\langle t_1 | e \rangle\rangle & \\ (\mu): & \langle \mu\beta.c | e \rangle & \rightarrow & c\{e/\beta\} & t \rightarrow t' \;\Rightarrow\; \langle t|e\rangle \rightarrow \langle t'|e\rangle \\ (\tilde{\mu}_{\mathrm{v}}): & \langle V | \tilde{\mu}x.c \rangle & \rightarrow & c\{V/x\} & c \rightarrow c' \;\Rightarrow\; \mu\beta.c \rightarrow \mu\beta.c' \\ (\eta\mu): & \mu\alpha.\langle t|\alpha\rangle & \rightarrow & t \quad (\alpha \notin fv(t)) & \end{array}$$

Both notions only reduce terms or commands, never environments.

Of course Thm. 2.5 holds for the CBN and CBV strategies as well.

3 The calculus $\mathcal{X}$

In this section we will give the definition of the $\mathcal{X}$-calculus which has been proven to be a fine-grained implementation model for various well-known calculi [7, 5], like the λ-calculus, $\lambda\mathbf{x}$, $\lambda\mu$, and $\overline{\lambda}\mu\tilde{\mu}$. The calculus $\mathcal{X}$ is inspired by the sequent calculus LK, introduced by Gentzen in [15]; the fragment of LK we will consider has only implication, and no structural rules.

LK is a logical system in which the rules only introduce connectives (but on both sides of a sequent), in contrast to natural deduction which uses introduction and elimination rules. The only way to eliminate a connective is to eliminate the whole formula in which it appears, via an application of the (*cut*)-rule. Gentzen's calculus for classical logic LK allows sequents of the form $A_1,\ldots,A_n \vdash B_1,\ldots,B_m$, where $A_1,\ldots,A_n$ is to be understood as $A_1 \wedge \ldots \wedge A_n$ and $B_1,\ldots,B_m$ is to be understood as $B_1 \vee \ldots \vee B_m$. Thus, LK appears as a very symmetrical system.

The variant of the sequent calculus we consider offers an extremely natural presentation of the classical propositional calculus with implication, and is a variant of system LK. It has four rules: *axiom*, *right introduction* of the arrow, *left introduction* and *cut*.

$$(ax): \frac{}{\Gamma, A \vdash A, \Delta} \qquad (\Rightarrow L): \frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \Rightarrow B \vdash \Delta} \qquad (\Rightarrow R): \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta} \qquad (cut): \frac{\Gamma \vdash A, \Delta \quad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta}$$

The elimination of rule (*cut*) plays a major role in LK, since for proof theoreticians, cut-free proofs enjoy nice properties. Proof reductions by cut-elimination have been proposed by Gentzen; those reductions become the fundamental principle of computation in $\mathcal{X}$.

$\mathcal{X}$ features two separate categories of 'connectors', *plugs* and *sockets*, that act as input and output channels, respectively.

[3] In [19], stacks are called *linear evaluation contexts*.

Definition 3.1 (SYNTAX FOR $\mathcal{X}$ [5]) The terms (or *nets*) of the $\mathcal{X}$-calculus are defined by the following syntax, where x,y range over the infinite set of *term variables* (also called *sockets*), and α,β over the infinite set of *context variables* (also called *plugs*); the common name for both is *connector*.

$$\begin{array}{rcccccccc} P,Q & ::= & \langle x{\cdot}\alpha\rangle & \mid & \widehat{y}P\widehat{\beta}\cdot\alpha & \mid & P\widehat{\beta}\,[y]\,\widehat{x}Q & \mid & P\widehat{\alpha}\dagger\widehat{x}Q \\ & & \textit{capsule} & & \textit{export} & & \textit{import} & & \textit{cut} \end{array}$$

The syntax is extended with two *flagged* or *active* cuts:

$$P,Q \;::=\; \ldots \mid P\widehat{\alpha}\not\dagger\widehat{x}Q \mid P\widehat{\alpha}\,\rotatebox[origin=c]{180}{$\not\dagger$}\,\widehat{x}Q$$

Terms constructed without these flagged cuts are called *pure*.

We borrow the terminology from $\overline{\lambda}\mu\tilde{\mu}$, and call P in $P\widehat{\alpha}\dagger\widehat{x}Q$ or $P\widehat{\beta}\,[y]\,\widehat{x}Q$ a term, and Q a context.

The $\hat{\cdot}$ symbolises that the connector underneath is bound in the adjacent term. The notion of bound and free connector is defined as usual, and we will identify terms that only differ in the names of bound connectors, as usual.

Definition 3.2 ([5]) The *bound connectors* in a term are defined through:

$$\begin{array}{lcllcl} bs(\langle x{\cdot}\alpha\rangle) & = & \emptyset & bp(\langle x{\cdot}\alpha\rangle) & = & \emptyset \\ bs(\widehat{x}P\widehat{\alpha}\cdot\beta) & = & bs(P)\cup\{x\} & bp(\widehat{x}P\widehat{\alpha}\cdot\beta) & = & bp(P)\cup\{\alpha\} \\ bs(P\widehat{\alpha}\,[y]\,\widehat{x}Q) & = & bs(P)\cup bs(Q)\cup\{x\} & bp(P\widehat{\alpha}\,[y]\,\widehat{x}Q) & = & bp(P)\cup\{\alpha\}\cup bp(Q) \\ bs(P\widehat{\alpha}\dagger\widehat{x}Q) & = & bs(P)\cup bs(Q)\cup\{x\} & bp(P\widehat{\alpha}\dagger\widehat{x}Q) & = & bp(P)\cup\{\alpha\}\cup bp(Q) \end{array}$$

A connector occurring in P which is not bound is called *free*, and we write $x\in fs(P)$ and $\alpha\in fp(P)$. We will write $x\in fs(P,Q)$ for $x\in fs(P)\wedge x\in fs(Q)$, etc.

We adopt Barendregt's convention in that free and bound connectors of terms will be different.

The calculus, defined by the reduction rules below, explains in detail how cuts are propagated through terms to be eventually evaluated at the level of *capsules*, where renaming takes place. Reduction is defined by specifying both the interaction between well-connected basic syntactic structures, and how to deal with propagating active nodes to points in the term where they can interact.

It is important to know when a connector is introduced, *i.e.* is connectable, *i.e.* is exposed and unique; this will play a crucial role in the reduction rules.

Definition 3.3 (REDUCTION ON $\mathcal{X}$ [5]) (*Introduction*):

(P *introduces* x): Either $P=Q\widehat{\beta}\,[x]\,\widehat{y}R$ with $x\notin fs(Q,R)$, or $P=\langle x{\cdot}\alpha\rangle$.

(P *introduces* α): Either $P=\widehat{x}Q\widehat{\beta}\cdot\alpha$ and $\alpha\notin fp(Q)$, or $P=\langle x{\cdot}\alpha\rangle$.

(*Logical rules*): Let α and x be introduced in, respectively, the left and right-hand side of the main cuts below.

$$\begin{array}{rrcl} (cap): & \langle y{\cdot}\alpha\rangle\widehat{\alpha}\dagger\widehat{x}\langle x{\cdot}\beta\rangle & \to & \langle y{\cdot}\beta\rangle \\ (exp): & (\widehat{y}P\widehat{\beta}\cdot\alpha)\widehat{\alpha}\dagger\widehat{x}\langle x{\cdot}\gamma\rangle & \to & \widehat{y}P\widehat{\beta}\cdot\gamma \\ (imp): & \langle y{\cdot}\alpha\rangle\widehat{\alpha}\dagger\widehat{x}(Q\widehat{\beta}\,[x]\,\widehat{z}R) & \to & Q\widehat{\beta}\,[y]\,\widehat{z}R \\ (exp\text{-}imp): & (\widehat{y}P\widehat{\beta}\cdot\alpha)\widehat{\alpha}\dagger\widehat{x}(Q\widehat{\gamma}\,[x]\,\widehat{z}R) & \to & \begin{cases} Q\widehat{\gamma}\dagger\widehat{y}\,(P\widehat{\beta}\dagger\widehat{z}R) \\ (Q\widehat{\gamma}\dagger\widehat{y}P)\,\widehat{\beta}\dagger\widehat{z}R \end{cases} \end{array}$$

(*Activation*): We define two cut-activation rules.

$$\begin{array}{rlcll} (a\not\dagger): & P\widehat{\alpha}\dagger\widehat{x}Q & \to & P\widehat{\alpha}\not\dagger\widehat{x}Q & (P \textit{ does not introduce } \alpha) \\ (\rotatebox[origin=c]{180}{$\not\dagger$}a): & P\widehat{\alpha}\dagger\widehat{x}Q & \to & P\widehat{\alpha}\,\rotatebox[origin=c]{180}{$\not\dagger$}\,\widehat{x}Q & (Q \textit{ does not introduce } x) \end{array}$$

(*Propagation rules*): Left propagation rules:

$$
\begin{array}{rrcll}
(d\not\nearrow): & \langle y{\cdot}\alpha\rangle\,\widehat{\alpha}\not\nearrow\widehat{x}P & \to & \langle y{\cdot}\alpha\rangle\,\widehat{\alpha}\dagger\widehat{x}P & \\
(cap\not\nearrow): & \langle y{\cdot}\beta\rangle\,\widehat{\alpha}\not\nearrow\widehat{x}P & \to & \langle y{\cdot}\beta\rangle & (\beta\neq\alpha)\\
(exp\text{-}out\not\nearrow): & (\widehat{y}Q\widehat{\beta}\cdot\alpha)\,\widehat{\alpha}\not\nearrow\widehat{x}P & \to & (\widehat{y}\,(Q\widehat{\alpha}\not\nearrow\widehat{x}P)\widehat{\beta}\cdot\gamma)\,\widehat{\gamma}\dagger\widehat{x}P, & (\gamma\ fresh)\\
(exp\text{-}in\not\nearrow): & (\widehat{y}Q\widehat{\beta}\cdot\gamma)\,\widehat{\alpha}\not\nearrow\widehat{x}P & \to & \widehat{y}\,(Q\widehat{\alpha}\not\nearrow\widehat{x}P)\widehat{\beta}\cdot\gamma & (\gamma\neq\alpha)\\
(imp\not\nearrow): & (Q\widehat{\beta}\,[z]\,\widehat{y}R)\,\widehat{\alpha}\not\nearrow\widehat{x}P & \to & (Q\widehat{\alpha}\not\nearrow\widehat{x}P)\,\widehat{\beta}\,[z]\,\widehat{y}\,(R\widehat{\alpha}\not\nearrow\widehat{x}P) & \\
(cut\not\nearrow): & (Q\widehat{\beta}\dagger\widehat{y}R)\,\widehat{\alpha}\not\nearrow\widehat{x}P & \to & (Q\widehat{\alpha}\not\nearrow\widehat{x}P)\,\widehat{\beta}\dagger\widehat{y}\,(R\widehat{\alpha}\not\nearrow\widehat{x}P) &
\end{array}
$$

Right propagation rules:

$$
\begin{array}{rrcll}
(\nwarrow\!\!\!\!\!\setminus d): & P\widehat{\alpha}\nwarrow\!\!\!\!\!\setminus\widehat{x}\,\langle x{\cdot}\beta\rangle & \to & P\widehat{\alpha}\dagger\widehat{x}\,\langle x{\cdot}\beta\rangle & \\
(\nwarrow\!\!\!\!\!\setminus cap): & P\widehat{\alpha}\nwarrow\!\!\!\!\!\setminus\widehat{x}\,\langle y{\cdot}\beta\rangle & \to & \langle y{\cdot}\beta\rangle & (y\neq x)\\
(\nwarrow\!\!\!\!\!\setminus exp): & P\widehat{\alpha}\nwarrow\!\!\!\!\!\setminus\widehat{x}\,(\widehat{y}Q\widehat{\beta}\cdot\gamma) & \to & \widehat{y}\,(P\widehat{\alpha}\nwarrow\!\!\!\!\!\setminus\widehat{x}Q)\widehat{\beta}\cdot\gamma & \\
(\nwarrow\!\!\!\!\!\setminus imp\text{-}out): & P\widehat{\alpha}\nwarrow\!\!\!\!\!\setminus\widehat{x}\,(Q\widehat{\beta}\,[x]\,\widehat{y}R) & \to & P\widehat{\alpha}\dagger\widehat{z}\,((P\widehat{\alpha}\nwarrow\!\!\!\!\!\setminus\widehat{x}Q)\,\widehat{\beta}\,[z]\,\widehat{y}\,(P\widehat{\alpha}\nwarrow\!\!\!\!\!\setminus\widehat{x}R)) & (z\ fresh)\\
(\nwarrow\!\!\!\!\!\setminus imp\text{-}in): & P\widehat{\alpha}\nwarrow\!\!\!\!\!\setminus\widehat{x}\,(Q\widehat{\beta}\,[z]\,\widehat{y}R) & \to & (P\widehat{\alpha}\nwarrow\!\!\!\!\!\setminus\widehat{x}Q)\,\widehat{\beta}\,[z]\,\widehat{y}\,(P\widehat{\alpha}\nwarrow\!\!\!\!\!\setminus\widehat{x}R) & (z\neq x)\\
(\nwarrow\!\!\!\!\!\setminus cut): & P\widehat{\alpha}\nwarrow\!\!\!\!\!\setminus\widehat{x}\,(Q\widehat{\beta}\dagger\widehat{y}R) & \to & (P\widehat{\alpha}\nwarrow\!\!\!\!\!\setminus\widehat{x}Q)\,\widehat{\beta}\dagger\widehat{y}\,(P\widehat{\alpha}\nwarrow\!\!\!\!\!\setminus\widehat{x}R) &
\end{array}
$$

(*Reduction*): We write $\to_x$ for the compatible closure of the above logical, propagation and activation rules, and use $\to_x^*$ for the reflexive, transitive reduction relation generated by $\to_x$.

The first three logical rules above specify a renaming procedure, whereas the fourth rule specifies the basic computational step: it links the *export* of a function, available on the plug α, to an adjacent *import* via the *socket* x. The effect of the reduction will be that the exported function is placed in-between the two sub-terms of the *import*, acting as interface. Notice that two cuts are created in the result, that can be grouped in two ways; these alternatives do not necessarily share all normal forms (reduction is non-confluent, so normal forms are not unique). And in fact, this rule presents a *critical pair*.

Notice that, by the activation rules, in case both α is not introduced in P and x is not introduced in Q, activation can take place in both directions, so the cut $P\widehat{\alpha}\dagger\widehat{x}Q$ forms another critical pair, which is a second source of non-confluence. The activation rules define how to reduce a cut when one of its sub-terms does not introduce a connector mentioned in the cut. This will involve moving the cut inwards, towards a position where the connector *is* introduced. In case both connectors are not introduced this search can start in either direction, indicated by the tilting of the dagger, via the *activation* of the cut.

The (full) reduction relation $\to_x$ is not confluent: assuming α does not occur in P and x does not occur in Q, then $P\widehat{\alpha}\dagger\widehat{x}Q$ reduces to both P and Q.

As observed in [5], although activated cuts cannot 'cross', it can be mimicked, which can lead to non-termination for typeable nets.

Example 3.4 Assume $x\notin fs(Q), \beta\notin fp(P)$, and P,Q both pure, then:

$$
\begin{array}{llll}
P\widehat{\alpha}\dagger\widehat{x}\,(\langle x{\cdot}\beta\rangle\,\widehat{\beta}\dagger\widehat{z}Q) & \to(\nwarrow\!\!\!\!\!\setminus a) & P\widehat{\alpha}\nwarrow\!\!\!\!\!\setminus\widehat{x}\,(\langle x{\cdot}\beta\rangle\,\widehat{\beta}\dagger\widehat{z}Q) & \to(\nwarrow\!\!\!\!\!\setminus cut)\\
(P\widehat{\alpha}\nwarrow\!\!\!\!\!\setminus\widehat{x}\,\langle x{\cdot}\beta\rangle)\,\widehat{\beta}\dagger\widehat{z}\,(P\widehat{\alpha}\nwarrow\!\!\!\!\!\setminus\widehat{x}Q) & \to(\nwarrow\!\!\!\!\!\setminus d),(\nwarrow\!\!\!\!\!\setminus gc) & (P\widehat{\alpha}\dagger\widehat{x}\,\langle x{\cdot}\beta\rangle)\,\widehat{\beta}\dagger\widehat{z}Q & \to(a\not\nearrow)\\
(P\widehat{\alpha}\dagger\widehat{x}\,\langle x{\cdot}\beta\rangle)\,\widehat{\beta}\not\nearrow\widehat{z}Q & \to(cut\not\nearrow) & (P\widehat{\beta}\not\nearrow\widehat{z}Q)\,\widehat{\alpha}\dagger\widehat{x}\,(\langle x{\cdot}\beta\rangle\,\widehat{\beta}\not\nearrow\widehat{z}Q) & \to(d\not\nearrow),(gc\not\nearrow)\\
P\widehat{\alpha}\dagger\widehat{x}\,(\langle x{\cdot}\beta\rangle\,\widehat{\beta}\dagger\widehat{z}Q) & & &
\end{array}
$$

[5] defines CBN and CBV sub-reduction systems by limiting the activation rules, favouring one kind of activating whenever the above critical pair occurs.

Definition 3.5 ([5]) • If a cut can be activated in two ways, CBN only allows to activate it via $(\nwarrow\!\!\!\!\!\setminus a)$; this is obtained by replacing rule $(a\not\nearrow)$ by:

$$(a\not\nearrow_{\text{N}}):\ P\widehat{\alpha}\dagger\widehat{x}Q \ \to\ P\widehat{\alpha}\not\nearrow\widehat{x}Q \quad (P\ does\ not\ introduce\ \alpha\ and\ Q\ introduces\ x)$$

- CBV can only activate such a cut via $(a\nearrow)$. We can reformulate this as the reduction system obtained by replacing rule $(\nwarrow a)$ by:

$$(\nwarrow a_{\mathrm{v}}):\ P\widehat{\alpha}\dagger\widehat{x}Q \ \rightarrow\ P\widehat{\alpha}\nwarrow\widehat{x}Q \quad (P \textit{ introduces } \alpha \textit{ and } Q \textit{ does not introduce } x)$$

It is possible to define reduction on terms in $\mathcal{X}$ without using activation, allowing *cuts* to propagate over *cuts* at will; of course then it would be difficult to define CBN or CBV reduction as we do here. Urban and Bierman introduced the activated cuts to obtain a more controlled (limited) reduction, making a proof of strong normalisation for typeable terms possible [32].

We have the following results.

Lemma 3.6 (Renaming and Garbage Collection [5]) *The following rules are admissible:*

i) $P\widehat{\alpha}\nearrow\widehat{x}\langle x{\cdot}\beta\rangle \rightarrow_{x}^{*} P\{\beta/\alpha\}$, *and* $P\widehat{\alpha}\dagger\widehat{x}\langle x{\cdot}\beta\rangle \rightarrow_{x}^{*} P\{\beta/\alpha\}$.

ii) $\langle y{\cdot}\alpha\rangle\widehat{\alpha}\nwarrow\widehat{x}P \rightarrow_{x}^{*} P\{y/x\}$, *and* $\langle y{\cdot}\alpha\rangle\widehat{\alpha}\dagger\widehat{x}P \rightarrow_{x}^{*} P\{y/x\}$.

iii) $(\nearrow gc)$: $Q\widehat{\alpha}\nearrow\widehat{x}P \rightarrow_{x}^{*} Q$, $\alpha \notin fn(Q)$.

iv) $(\nwarrow gc)$: $P\widehat{\alpha}\nwarrow\widehat{x}Q \rightarrow_{x}^{*} Q$, $x \notin fv(Q)$.

The propagation rules in fact correspond to explicit substitution, as is exemplified by the fact that $\lambda\mathbf{x}$, Bloo and Rose's calculus of explicit substitution can easily be embedded in $\mathcal{X}$.

Definition 3.7 ([10]) The syntax of $\lambda\mathbf{x}$ is an extension of that of the λ-calculus:

$$M,N \ ::=\ x \mid \lambda x.M \mid MN \mid M\langle x{:=}N\rangle$$

The reduction relation is defined as the compatible closure of the following rules:

$$\begin{array}{rlcl@{\qquad}rlcl}
(\mathsf{B}): & (\lambda x.M)P & \rightarrow & M\langle x{:=}P\rangle & (\mathsf{Var}): & x\langle x{:=}P\rangle & \rightarrow & P\\
(\mathsf{App}): & (MN)\langle x{:=}P\rangle & \rightarrow & M\langle x{:=}P\rangle N\langle x{:=}P\rangle & (\mathsf{VarK}): & y\langle x{:=}P\rangle & \rightarrow & y\\
(\mathsf{Abs}): & (\lambda y.M)\langle x{:=}P\rangle & \rightarrow & \lambda y.(M\langle x{:=}P\rangle) & (\mathsf{gc}): & M\langle x{:=}P\rangle & \rightarrow & M\ \ (x \notin fv(M))
\end{array}$$

The notion of reduction $\lambda\mathbf{x}$ is obtained by deleting rule (gc), and the notion of reduction $\lambda\mathbf{x}_{\mathsf{gc}}$ is obtained by deleting rule (VarK). The rule (gc) is called 'garbage collection', as it removes useless substitutions.

Definition 3.8 (Interpretation of λ, $\lambda\mathbf{x}$, and $\lambda\mu$ in $\mathcal{X}$ [5]) $[\![\cdot]\!]_{\alpha}^{\lambda}$,[4] the interpretation of λ terms into $\mathcal{X}$, is defined as follows:

$$\begin{array}{rcl}
[\![x]\!]_{\alpha}^{\lambda} & \triangleq & \langle x{\cdot}\alpha\rangle\\
[\![\lambda x.M]\!]_{\alpha}^{\lambda} & \triangleq & \widehat{x}\,[\![M]\!]_{\beta}^{\lambda}\,\widehat{\beta}\cdot\alpha\\
[\![MN]\!]_{\alpha}^{\lambda} & \triangleq & [\![M]\!]_{\gamma}^{\lambda}\,\widehat{\gamma}\dagger\widehat{x}\,([\![N]\!]_{\beta}^{\lambda}\,\widehat{\beta}\,[x]\,\widehat{y}\,\langle y{\cdot}\alpha\rangle)
\end{array}$$

This can be extended to $[\![\cdot]\!]_{\alpha}^{\lambda x}$, that maps $\lambda\mathbf{x}$-terms to $\mathcal{X}$, by adding:

$$[\![M\langle x{:=}N\rangle]\!]_{\alpha}^{\lambda x} \ \triangleq\ [\![N]\!]_{\beta}^{\lambda x}\,\widehat{\beta}\nwarrow\widehat{x}\,[\![M]\!]_{\alpha}^{\lambda x}$$

The interpretation of $\lambda\mu$-term to $\mathcal{X}$, $[\![\cdot]\!]_{\alpha}^{\lambda\mu}$, is obtained by adding

$$[\![\mu\delta.[\gamma]M]\!]_{\alpha}^{\lambda\mu} \ \triangleq\ [\![M]\!]_{\gamma}^{\lambda\mu}\,\widehat{\delta}\dagger\widehat{x}\,\langle x{\cdot}\alpha\rangle$$

to $[\![\cdot]\!]_{\alpha}^{\lambda}$.

Notice that the case for application directly represents how the natural deduction rule Modes Ponens $(\rightarrow E)$ gets represented in LK.

---

[4] We should remark that the notation introduced here might be misleading. Indexing of terms, as in t_1 and t_2, is normally done to use the same identifier for two different items. This is not the case for $[\![x]\!]_{\alpha}^{\lambda}$ and $[\![x]\!]_{\beta}^{\lambda}$, which yield $\langle x{\cdot}\alpha\rangle$ and $\langle x{\cdot}\beta\rangle$. Perhaps a notation like $[\![M]\!](\alpha)$ would have been better, since that correctly suggests that α is a parameter to the interpretation of M; unfortunately, this notation becomes rather unreadable, especially in the proofs that follow.

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma \vdash A \Rightarrow B}}{\Gamma \vdash A \Rightarrow B, B}\,(W) \qquad \dfrac{\dfrac{\dfrac{\vdots}{\Gamma \vdash A}}{\Gamma \vdash A, B}\,(W) \quad \dfrac{}{\Gamma, B \vdash B}\,(Ax)}{\Gamma, A \Rightarrow B \vdash B}\,(\rightarrow L)}{\Gamma \vdash B}\,(cut)$$

(cf [15]). [5] shows that the interpretations of Def. 3.8 respect reduction.

Type assignment for $\mathcal{X}$ is defined as follows:

Definition 3.9 (Typing for $\mathcal{X}$ [7]) Using the notion of types, and contexts of variables and names of Definition 1.5, type assignment for $\mathcal{X}$ is defined through:

i) *Type judgements* are expressed via the ternary relation $P \mathrel{:\cdot} \Gamma \vdash_{\mathcal{X}} \Delta$, where Γ is a context of *sockets* and Δ is a context of *plugs*, and P is a term. We say that P is the *witness* of this judgement.

ii) *Context assignment for* $\mathcal{X}$ is defined by the following rules:

$$(cap): \dfrac{}{\langle y{\cdot}\beta\rangle \mathrel{:\cdot} \Gamma, y{:}A \vdash \beta{:}A, \Delta} \qquad (imp): \dfrac{P \mathrel{:\cdot} \Gamma \vdash \alpha{:}A, \Delta \quad Q \mathrel{:\cdot} \Gamma, x{:}A \vdash \Delta}{P\widehat{\alpha}\,[y]\,\widehat{x}Q \mathrel{:\cdot} \Gamma, y{:}A{\rightarrow}B \vdash \Delta}$$

$$(exp): \dfrac{P \mathrel{:\cdot} \Gamma, x{:}A \vdash \alpha{:}B, \Delta}{\widehat{x}P\widehat{\alpha}\cdot\beta \mathrel{:\cdot} \Gamma \vdash \beta{:}A{\rightarrow}B, \Delta} \qquad (cut): \dfrac{P \mathrel{:\cdot} \Gamma \vdash \alpha{:}A, \Delta \quad Q \mathrel{:\cdot} \Gamma, x{:}A \vdash \Delta}{P\widehat{\alpha} \dagger \widehat{x}Q \mathrel{:\cdot} \Gamma \vdash \Delta}$$

We write $P \mathrel{:\cdot} \Gamma \vdash_{\mathcal{X}} \Delta$ if there exists a derivation that has this judgement in the bottom line.

Notice that Γ and Δ carry the types of the free connectors in P, as unordered sets. There is no notion of type for P itself, instead the derivable statement shows how P is connectable.

The soundness result of simple type assignment with respect to reduction is stated as usual:

Theorem 3.10 (Witness reduction [5]) *If* $P \mathrel{:\cdot} \Gamma \vdash_{\mathcal{X}} \Delta$, *and* $P \rightarrow_{\mathcal{X}} Q$, *then* $Q \mathrel{:\cdot} \Gamma \vdash_{\mathcal{X}} \Delta$.

4 $\mathcal{X}$ with implicit substitution

In [5] it is argued that $\mathcal{X}$ is a calculus with explicit substitution, which makes it suitable to encode calculi like the λ-calculus, $\lambda\mathbf{x}$, $\lambda\mu$ and $\overline{\lambda}\mu\tilde{\mu}$, as shown in that paper; these results are shown with respect to full reduction. Since in this paper we look to model similar results for restrictions of those calculi to cbn or cbv strategies, this explicit character of $\mathcal{X}$ poses a problem.

When modelling $\overline{\lambda}\mu\tilde{\mu}$'s reduction rule (μ), a property like

$$[\![\langle \mu\beta.c \,|\, e\rangle]\!]^{\overline{\lambda}} \rightarrow^{*}_{\mathcal{X}} [\![c\{e/\beta\}]\!]^{\overline{\lambda}}$$

needs justification, for we would need to show that substitution is preserved under the translation function $[\![\cdot]\!]^{\overline{\lambda}}$. Since substitution is not part of the definition of $\mathcal{X}$, the only thing that is possible to show, as is done in [5], is that the interpretation of the implicit substitution of $\overline{\lambda}\mu\tilde{\mu}$ gets *executed* through reduction in $\mathcal{X}$, mainly through the propagation rules. This implies that, in a proof for this result, to fully achieve $[\![c\{e/\beta\}]\!]^{\overline{\lambda}}$, these steps need to be executed *in full*, irrespective of the restriction in evaluation contexts that the cbn or cbv-reduction strategy might impose, thus forcing us, in practice, to allow for reduction, at least of the propagation rules, to take place everywhere.

To avoid this problem altogether, here we choose to work with $\mathcal{X}_{\text{IS}}$, a variant of $\mathcal{X}$ defined by Summers [30] that has a notion of implicit substitution. In [30], Summers defines $\mathcal{X}^{i}$ as a variant or $\mathcal{X}$ (adding also negation), that replaces the propagation rules by a substitution-like operation $P\alpha \leftrightarrow x$ and $\alpha \leftrightarrow xQ$; the benefit of this change is that activated and unactivated cuts cannot interfere, and makes it more clear that the intention of reduction is that activated cuts run to completion. We agree with Summers that separating the rules that express propagation of activated cuts from the logical reduction rules gives a notion of reduction that is easier to deal with; it should be noted, however, that treating activated cuts as implicit substitution, rather than explicit substitution, as was the case in [5], restricts $\mathcal{X}$'s notion of reduction and possible reducts.

As we will see below, when defining CBN and CBV reduction strategies on $\mathcal{X}$, it becomes natural to separate the propagation steps from the logical steps. In particular, to show the preservations results in Sect. 7 for our encoding of $\overline{\lambda}\mu\tilde{\mu}$ into $\mathcal{X}$, we need to be able to simulate the implicit substitution of $\overline{\lambda}\mu\tilde{\mu}$; this is non-problematic for full reduction, as shown in [5], but when modelling the CBN and CBV strategies, in the proofs of Lem. 7.3 and 7.4 we would be forced to propagate the active cuts all through the terms, even where not permitted under CBN or CBV strategies for $\mathcal{X}$, so would not be able to show full simulation.

No longer considering active cuts as reducible terms, but rather expressing propagation of active cuts through substitution, avoids that problem. We therefore change our definition to that of $\mathcal{X}_{\text{IS}}$, $\mathcal{X}$ with implicit substitutions; notice that part of the justification of this definition lies in Lem. 3.6.

Definition 4.1 (SUBSTITUTION ON $\mathcal{X}_{\text{IS}}$) The terms of $\mathcal{X}_{\text{IS}}$ are those of $\mathcal{X}$:

$$P,Q ::= \langle x{\cdot}\alpha\rangle \mid \widehat{y}P\widehat{\beta}\cdot\alpha \mid P\widehat{\beta}\,[y]\,\widehat{x}Q \mid P\widehat{\alpha}\dagger\widehat{x}Q$$

Right substitution on $\mathcal{X}_{\text{IS}}$ is defined through:

$$\begin{array}{rrcll}
(d_{\text{R}}): & \langle y{\cdot}\alpha\rangle\{\alpha\not\dagger\widehat{x}P\} & = & P\{y/x\} & \\
(gc_{\text{R}}): & Q\{\alpha\not\dagger\widehat{x}P\} & = & Q & (\alpha\notin fn(Q)) \\
(exp\text{-}out_{\text{R}}): & (\widehat{y}Q\widehat{\beta}\cdot\alpha)\{\alpha\not\dagger\widehat{x}P\} & = & (\widehat{y}\,(Q\{\alpha\not\dagger\widehat{x}P\})\widehat{\beta}\cdot\gamma)\,\widehat{\gamma}\dagger\widehat{x}P & (\gamma\ fresh) \\
(exp\text{-}in_{\text{R}}): & (\widehat{y}Q\widehat{\beta}\cdot\gamma)\{\alpha\not\dagger\widehat{x}P\} & = & \widehat{y}\,(Q\{\alpha\not\dagger\widehat{x}P\})\widehat{\beta}\cdot\gamma & (\gamma\neq\alpha) \\
(imp_{\text{R}}): & (Q\widehat{\beta}\,[z]\,\widehat{y}R)\{\alpha\not\dagger\widehat{x}P\} & = & (Q\{\alpha\not\dagger\widehat{x}P\})\,\widehat{\beta}\,[z]\,\widehat{y}\,(R\{\alpha\not\dagger\widehat{x}P\}) & \\
(cut_{\text{R}}): & (Q\widehat{\beta}\dagger\widehat{y}R)\{\alpha\not\dagger\widehat{x}P\} & = & (Q\{\alpha\not\dagger\widehat{x}P\})\,\widehat{\beta}\dagger\widehat{y}\,(R\{\alpha\not\dagger\widehat{x}P\}) &
\end{array}$$

and *left substitution* on $\mathcal{X}_{\text{IS}}$ through:

$$\begin{array}{rlcll}
(d_{\text{L}}): & \{P\widehat{\alpha}\dagger\!\!\!\backslash\, x\}\langle x{\cdot}\beta\rangle & = & P\{\beta/\alpha\} & \\
(gc_{\text{L}}): & \{P\widehat{\alpha}\dagger\!\!\!\backslash\, x\}Q & = & Q & (x\notin fv(Q)) \\
(exp_{\text{L}}): & \{P\widehat{\alpha}\dagger\!\!\!\backslash\, x\}(\widehat{y}Q\widehat{\beta}\cdot\gamma) & = & \widehat{y}\,(\{P\widehat{\alpha}\dagger\!\!\!\backslash\, x\}Q)\widehat{\beta}\cdot\gamma & \\
(imp\text{-}out_{\text{L}}): & \{P\widehat{\alpha}\dagger\!\!\!\backslash\, x\}(Q\widehat{\beta}\,[x]\,\widehat{y}R) & = & P\widehat{\alpha}\dagger\widehat{z}\,((\{P\widehat{\alpha}\dagger\!\!\!\backslash\, x\}Q)\,\widehat{\beta}\,[z]\,\widehat{y}\,(\{P\widehat{\alpha}\dagger\!\!\!\backslash\, x\}R)) & z\ fresh) \\
(imp\text{-}in_{\text{L}}): & \{P\widehat{\alpha}\dagger\!\!\!\backslash\, x\}(Q\widehat{\beta}\,[z]\,\widehat{y}R) & = & (\{P\widehat{\alpha}\dagger\!\!\!\backslash\, x\}Q)\,\widehat{\beta}\,[z]\,\widehat{y}\,(\{P\widehat{\alpha}\dagger\!\!\!\backslash\, x\}R) & (z\neq x) \\
(cut_{\text{L}}): & \{P\widehat{\alpha}\dagger\!\!\!\backslash\, x\}(Q\widehat{\beta}\dagger\widehat{y}R) & = & (\{P\widehat{\alpha}\dagger\!\!\!\backslash\, x\}Q)\,\widehat{\beta}\dagger\widehat{y}\,(\{P\widehat{\alpha}\dagger\!\!\!\backslash\, x\}R) &
\end{array}$$

Notice that these now no longer are reduction rules, but define a notion of implicit substitution that percolates through the terms; the only reduction rules now are the logical rules (*cap*), (*exp*), (*imp*), and (*cut*) (as in Def 3.3), as well as the two rules that start the substitution. However, rules ($exp\text{-}out_{\text{R}}$) and ($exp_{\text{L}}$) introduce new cuts.

Definition 4.2 (REDUCTION ON $\mathcal{X}_{\text{IS}}$) The single step reduction steps for $\mathcal{X}_{\text{IS}}$ are defined through:

$$\begin{array}{rlcll}
(cap): & \langle y{\cdot}\alpha\rangle\widehat{\alpha}\dagger\widehat{x}\langle x{\cdot}\beta\rangle & \to & \langle y{\cdot}\beta\rangle & \\
(exp): & (\widehat{y}P\widehat{\beta}\cdot\alpha)\widehat{\alpha}\dagger\widehat{x}\langle x{\cdot}\gamma\rangle & \to & \widehat{y}P\widehat{\beta}\cdot\gamma & \\
(imp): & \langle y{\cdot}\alpha\rangle\widehat{\alpha}\dagger\widehat{x}(Q\widehat{\beta}\,[x]\,\widehat{z}R) & \to & Q\widehat{\beta}\,[y]\,\widehat{z}R & \\
(exp\text{-}imp): & (\widehat{y}P\widehat{\beta}\cdot\alpha)\widehat{\alpha}\dagger\widehat{x}(Q\widehat{\gamma}\,[x]\,\widehat{z}R) & \to & \begin{cases} Q\widehat{\gamma}\dagger\widehat{y}\,(P\widehat{\beta}\dagger\widehat{z}R) \\ (Q\widehat{\gamma}\dagger\widehat{y}P)\,\widehat{\beta}\dagger\widehat{z}R \end{cases} & \\
(sub_{\text{R}}): & P\widehat{\alpha}\dagger\widehat{x}Q & \to & P\{\alpha\not\dagger\widehat{x}Q\} & (P\ \textit{does not introduce}\ \alpha) \\
(sub_{\text{L}}): & P\widehat{\alpha}\dagger\widehat{x}Q & \to & \{P\widehat{\alpha}\dagger\!\!\!\backslash\, x\}Q & (Q\ \textit{does not introduce}\ x)
\end{array}$$

We write $P \rightarrow_{\mathcal{X}_{\text{IS}}} Q$ if P reduces to Q using one of the above logical or substitution activation rules, and use $\rightarrow^*_{\mathcal{X}_{\text{IS}}}$ for the reflexive, transitive, compatible reduction relation generated by $\rightarrow_{\mathcal{X}_{\text{IS}}}$.

Notice that, by these rules, in case both α is not introduced in P and x is not introduced in Q, activation can take place in both directions, so then the cut $P\widehat{\alpha} \dagger \widehat{x}Q$ again forms a critical pair.

As to the reduction in Exm. 3.4, this problem is now avoided:

$$\begin{array}{lll} P\widehat{\alpha} \dagger \widehat{x}\,(\langle x{\cdot}\beta\rangle\,\widehat{\beta} \dagger \widehat{z}Q) & \rightarrow (sub_{\text{L}}) & \{P\widehat{\alpha} \nwarrow x\}(\langle x{\cdot}\beta\rangle\,\widehat{\beta} \dagger \widehat{z}Q) \;=(cut_{\text{L}}) \\ (\{P\widehat{\alpha} \nwarrow x\}\langle x{\cdot}\beta\rangle)\,\widehat{\beta} \dagger \widehat{z}\,(\{P\widehat{\alpha} \nwarrow x\}Q) & \rightarrow (d_{\text{L}}, gc_{\text{L}}) & P\{\beta/\alpha\}\,\widehat{\beta} \dagger \widehat{z}Q \end{array}$$

We will now define the CBN and CBV reduction strategies for $\mathcal{X}_{\text{IS}}$.

Definition 4.3 (Call By Name reduction strategy for $\mathcal{X}_{\text{IS}}$) For $\mathcal{X}_{\text{IS}}$, the CBN-reduction strategy $\rightarrow^{\text{N}*}_{\mathcal{X}_{\text{IS}}}$ is defined by limiting $\rightarrow^*_{\mathcal{X}_{\text{IS}}}$ through:

- We replace rule (sub_{R}) with:

$$(sub^{\text{N}}_{\text{R}}):\; P\widehat{\alpha} \dagger \widehat{x}Q \;\rightarrow_{\text{N}}\; P\{\alpha \not\nearrow \widehat{x}Q\},\; (\textit{if } P \textit{ does not introduce } \alpha \textit{ and } Q \textit{ introduces } x).$$

- As in [23], we only allow one variant of (*exp-imp*):

$$(\widehat{y}P\widehat{\beta}\cdot\alpha)\,\widehat{\alpha} \dagger \widehat{x}\,(Q\widehat{\gamma}\,[x]\,\widehat{z}R) \;\rightarrow_{\text{N}}\; Q\widehat{\gamma} \dagger \widehat{y}\,(P\widehat{\beta} \dagger \widehat{z}R)$$

- The contextual rule for the CBN-evaluation strategy is defined through:

$$P \rightarrow^{\text{N}*}_{\mathcal{X}_{\text{IS}}} Q \;\Rightarrow\; P\widehat{\alpha} \dagger \widehat{x}R \rightarrow^{\text{N}*}_{\mathcal{X}_{\text{IS}}} Q\widehat{\alpha} \dagger \widehat{x}R$$

Definition 4.4 (Call By Value reduction strategy for $\mathcal{X}_{\text{IS}}$) The CBV-reduction strategy $\rightarrow^{\text{V}*}_{\mathcal{X}_{\text{IS}}}$ is defined by limiting $\rightarrow^*_{\mathcal{X}_{\text{IS}}}$ through:

- We replace rule (sub_{L}) with:

$$(sub^{\text{V}}_{\text{L}}):\; P\widehat{\alpha} \dagger \widehat{x}Q \;\rightarrow_{\text{V}}\; \{P\widehat{\alpha} \nwarrow x\}Q,\; (P \textit{ introduces } \alpha \textit{ and } Q \textit{ does not introduce } x).$$

- As for CBN, we only allow the first variant of (*exp-imp*):

$$(\widehat{y}P\widehat{\beta}\cdot\alpha)\,\widehat{\alpha} \dagger \widehat{x}\,(Q\widehat{\gamma}\,[x]\,\widehat{z}R) \;\rightarrow_{\text{V}}\; Q\widehat{\gamma} \dagger \widehat{y}\,(P\widehat{\beta} \dagger \widehat{z}R)$$

- The contextual rule for the CBV-evaluation strategy is defined through:

$$P \rightarrow^{\text{V}*}_{\mathcal{X}_{\text{IS}}} Q \;\Rightarrow\; P\widehat{\alpha} \dagger \widehat{x}R \rightarrow^{\text{V}*}_{\mathcal{X}_{\text{IS}}} Q\widehat{\alpha} \dagger \widehat{x}R$$

This way, we obtain two notions of reduction that are clearly confluent because of the absence of critical pairs. Notice that the only difference between CBN and CBV reduction lies in activation and that both strategies do not allow for reduction in contexts, nor inside substitutions.

By the way reduction in $\mathcal{X}_{\text{IS}}$ is defined, directly based on that of $\mathcal{X}$, reduction in $\mathcal{X}$ implements that of $\mathcal{X}_{\text{IS}}$, and the following result is straightforward.

Theorem 4.5 *If $P \rightarrow_{\mathcal{X}_{\text{IS}}} Q$, then $P \rightarrow^*_{\mathcal{X}} Q$.*

Since $\rightarrow^*_{\mathcal{X}}$ is more fine-grained, of course the converse does not hold.

We have the following results. These are already shown in [5], but for $\mathcal{X}$; since here the exact steps that are needed in the reduction must be known when modelling CBN or CBV, we give the proofs in detail.

Lemma 4.6 i) $P\{\alpha \not\nearrow \widehat{x}\,\langle x{\cdot}\beta\rangle\} = P\{\beta/\alpha\}$, and $P\widehat{\alpha} \dagger \widehat{x}\,\langle x{\cdot}\beta\rangle \rightarrow^*_{\mathcal{X}_{\text{IS}}} P\{\beta/\alpha\}$.

ii) $\{\langle y{\cdot}\alpha\rangle\,\widehat{\alpha} \nwarrow x\}P = P\{y/x\}$, and $\langle y{\cdot}\alpha\rangle\,\widehat{\alpha} \dagger \widehat{x}P \rightarrow^*_{\mathcal{X}_{\text{IS}}} P\{y/x\}$.

iii) (gc_{R}): $Q\{\alpha \not\nearrow \widehat{x}P\} = Q$, $\alpha \notin fn(Q)$.

iv) (gc_{L}): $\{P\widehat{\alpha} \nwarrow x\}Q = Q$, $x \notin fv(Q)$.

Proof: By induction on the structure of nets.

i) $(P = \langle y{\cdot}\alpha\rangle)$: $\langle y{\cdot}\alpha\rangle\{\alpha \not\,\widehat{x}\,\langle x{\cdot}\beta\rangle\} \;=(d_{\mathrm{R}})\; \langle y{\cdot}\beta\rangle \;\triangleq\; \langle y{\cdot}\alpha\rangle\{\beta/\alpha\}$

$(P = \langle y{\cdot}\gamma\rangle,\ \gamma \neq \alpha)$: $\langle y{\cdot}\gamma\rangle\{\alpha \not\,\widehat{x}\,\langle x{\cdot}\beta\rangle\} \;=(gc_{\mathrm{R}})\; \langle y{\cdot}\gamma\rangle \;\triangleq\; \langle y{\cdot}\gamma\rangle\{\beta/\alpha\}$

$(P = \widehat{y}Q\widehat{\gamma}\cdot\alpha)$: $(\widehat{y}Q\widehat{\gamma}\cdot\alpha)\{\alpha \not\,\widehat{x}\,\langle x{\cdot}\beta\rangle\} \;=(exp\text{-}out_{\mathrm{R}})\; \widehat{y}\,(Q\{\alpha \not\,\widehat{x}\,\langle x{\cdot}\beta\rangle\})\widehat{\gamma}\cdot\beta \;=(ih)$
$\widehat{y}\,Q\{\beta/\alpha\}\,\widehat{\gamma}\cdot\beta \;\triangleq\; (\widehat{y}Q\widehat{\gamma}\cdot\alpha)\{\beta/\alpha\}$

$(P = \widehat{y}Q\widehat{\gamma}\cdot\delta,\ \delta \neq \alpha)$: $(\widehat{y}Q\widehat{\gamma}\cdot\delta)\{\alpha \not\,\widehat{x}\,\langle x{\cdot}\beta\rangle\} \;=(exp\text{-}in_{\mathrm{R}})\; \widehat{y}\,(Q\{\alpha \not\,\widehat{x}\,\langle x{\cdot}\beta\rangle\})\widehat{\gamma}\cdot\delta \;=(ih)$
$\widehat{y}\,Q\{\beta/\alpha\}\,\widehat{\gamma}\cdot\delta \;\triangleq\; (\widehat{y}Q\widehat{\gamma}\cdot\delta)\{\beta/\alpha\}$

$(P = Q\widehat{\gamma}\,[y]\,\widehat{z}R)$: $(Q\widehat{\gamma}\,[y]\,\widehat{z}R)\{\alpha \not\,\widehat{x}\,\langle x{\cdot}\beta\rangle\} \;=(imp_{\mathrm{R}})\; (Q\{\alpha \not\,\widehat{x}\,\langle x{\cdot}\beta\rangle\})\,\widehat{\gamma}\,[y]\,\widehat{z}\,(R\{\alpha \not\,\widehat{x}\,\langle x{\cdot}\beta\rangle\})$
$=(ih)\; Q\{\beta/\alpha\}\,\widehat{\gamma}\,[y]\,\widehat{z}\,R\{\beta/\alpha\} \;\triangleq\; (Q\widehat{\gamma}\,[y]\,\widehat{z}R)\{\beta/\alpha\}$

$(P = Q\widehat{\gamma}\dagger\widehat{z}R)$: $(Q\widehat{\gamma}\dagger\widehat{z}R)\{\alpha \not\,\widehat{x}\,\langle x{\cdot}\beta\rangle\} \;=(cut_{\mathrm{R}})\; (Q\{\alpha \not\,\widehat{x}\,\langle x{\cdot}\beta\rangle\})\,\widehat{\gamma}\dagger\widehat{z}\,(R\{\alpha \not\,\widehat{x}\,\langle x{\cdot}\beta\rangle\})$
$=(ih)\; Q\{\beta/\alpha\}\,\widehat{\gamma}\dagger\widehat{z}\,R\{\beta/\alpha\} \;\triangleq\; (Q\widehat{\gamma}\dagger\widehat{z}R)\{\beta/\alpha\}$

For the second part, if α is introduced in P, the result follows by rules (*cap*) or (*exp*); otherwise $P\widehat{\alpha}\dagger\widehat{x}\,\langle x{\cdot}\beta\rangle \to_{\mathcal{X}} (sub_{\mathrm{R}})\; P\{\alpha \not\,\widehat{x}\,\langle x{\cdot}\beta\rangle\}$, and the result follows by the first part.

ii) $(P = \langle x{\cdot}\beta\rangle)$: $\{\langle y{\cdot}\alpha\rangle\,\widehat{\alpha}\nwarrow x\}\langle x{\cdot}\beta\rangle \;=(d_{\mathrm{L}})\; \langle x{\cdot}\beta\rangle \;\triangleq\; \langle y{\cdot}\beta\rangle\{y/x\}$

$(P = \langle z{\cdot}\beta\rangle,\ z \neq x)$: $\{\langle y{\cdot}\alpha\rangle\,\widehat{\alpha}\nwarrow x\}\langle z{\cdot}\beta\rangle \;=(gc_{\mathrm{L}})\; \langle z{\cdot}\beta\rangle \;\triangleq\; \langle z{\cdot}\beta\rangle\{y/x\}$

$(P = \widehat{z}Q\widehat{\gamma}\cdot\beta)$: $\{\langle y{\cdot}\alpha\rangle\,\widehat{\alpha}\nwarrow x\}(\widehat{z}Q\widehat{\gamma}\cdot\beta) \;=(exp_{\mathrm{L}})\; \widehat{z}\,(\{\langle y{\cdot}\alpha\rangle\,\widehat{\alpha}\nwarrow x\}Q)\widehat{\gamma}\cdot\beta \;=(ih)\; \widehat{z}\,Q\{y/x\}\,\widehat{\gamma}\cdot\beta$
$\triangleq\; (\widehat{z}Q\widehat{\gamma}\cdot\beta)\{y/x\}$

$(P = Q\widehat{\gamma}\,[x]\,\widehat{z}R)$: $\{\langle y{\cdot}\alpha\rangle\,\widehat{\alpha}\nwarrow x\}(Q\widehat{\gamma}\,[x]\,\widehat{z}R) \;=(imp\text{-}out_{\mathrm{L}})$
$(\{\langle y{\cdot}\alpha\rangle\,\widehat{\alpha}\nwarrow x\}Q)\,\widehat{\gamma}\,[y]\,\widehat{z}\,(\{\langle y{\cdot}\alpha\rangle\,\widehat{\alpha}\nwarrow x\}R) \;=(ih)$
$Q\{y/x\}\,\widehat{\gamma}\,[y]\,\widehat{z}\,R\{y/x\} \;\triangleq\; (Q\widehat{\gamma}\,[x]\,\widehat{z}R)\{y/x\}$

$(P = Q\widehat{\gamma}\,[v]\,\widehat{z}R,\ v \neq x)$: $\{\langle y{\cdot}\alpha\rangle\,\widehat{\alpha}\nwarrow x\}(Q\widehat{\gamma}\,[v]\,\widehat{z}R) \;=(imp\text{-}in_{\mathrm{L}})$
$(\{\langle y{\cdot}\alpha\rangle\,\widehat{\alpha}\nwarrow x\}Q)\,\widehat{\gamma}\,[v]\,\widehat{z}\,(\{\langle y{\cdot}\alpha\rangle\,\widehat{\alpha}\nwarrow x\}R) \;=(ih)$
$Q\{y/x\}\,\widehat{\gamma}\,[v]\,\widehat{z}\,R\{y/x\} \;\triangleq\; (Q\widehat{\gamma}\,[v]\,\widehat{z}R)\{y/x\}$

$(P = Q\widehat{\gamma}\dagger\widehat{z}R)$: $\{\langle y{\cdot}\alpha\rangle\,\widehat{\alpha}\nwarrow x\}(Q\widehat{\gamma}\dagger\widehat{z}R) \;=(cut_{\mathrm{L}})\; (\{\langle y{\cdot}\alpha\rangle\,\widehat{\alpha}\nwarrow x\}Q)\,\widehat{\gamma}\dagger\widehat{z}\,(\{\langle y{\cdot}\alpha\rangle\,\widehat{\alpha}\nwarrow x\}R)$
$=(ih)\; Q\{y/x\}\,\widehat{\gamma}\dagger\widehat{z}\,R\{y/x\} \;\triangleq\; (Q\widehat{\gamma}\dagger\widehat{z}R)\{y/x\}$

For the second part, if x is introduced in P, the result follows by rules (*cap*) or (*imp*); otherwise $\langle y{\cdot}\alpha\rangle\,\widehat{\alpha}\dagger\widehat{x}P \to_{\mathcal{X}} \{\langle y{\cdot}\alpha\rangle\,\widehat{\alpha}\nwarrow x\}P$, and the result follows by the first part.

iii) $(Q = \langle y{\cdot}\gamma\rangle,\ \gamma \neq \alpha)$: $\langle y{\cdot}\gamma\rangle\{\alpha \not\,\widehat{x}P\} \;=(gc_{\mathrm{R}})\; \langle y{\cdot}\gamma\rangle$

$(Q = \widehat{y}R\widehat{\gamma}\cdot\delta,\ \delta \neq \alpha)$: $(\widehat{y}R\widehat{\gamma}\cdot\delta)\{\alpha \not\,\widehat{x}P\} \;=(exp\text{-}in_{\mathrm{R}})\; \widehat{y}\,(R\{\alpha \not\,\widehat{x}P\})\widehat{\gamma}\cdot\delta \;=(ih)\; \widehat{y}R\widehat{\gamma}\cdot\delta$

$(Q = R\widehat{\gamma}\,[y]\,\widehat{z}S)$: $(R\widehat{\gamma}\,[y]\,\widehat{z}S)\{\alpha \not\,\widehat{x}P\} \;=(imp_{\mathrm{R}})\; (R\{\alpha \not\,\widehat{x}P\})\,\widehat{\gamma}\,[y]\,\widehat{z}\,(S\{\alpha \not\,\widehat{x}P\}) \;=(ih)$
$Q\widehat{\gamma}\,[y]\,\widehat{z}R$

$(P = R\widehat{\gamma}\dagger\widehat{z}S)$: $(R\widehat{\gamma}\dagger\widehat{z}R)\{\alpha \not\,\widehat{x}P\} \;=(cut_{\mathrm{R}})\; (R\{\alpha \not\,\widehat{x}P\})\,\widehat{\gamma}\dagger\widehat{z}\,(S\{\alpha \not\,\widehat{x}P\}) \;=(ih)\; R\widehat{\gamma}\dagger\widehat{z}S$

iv) $(P = \langle z{\cdot}\beta\rangle,\ z \neq x)$: $\{P\widehat{\alpha}\nwarrow x\}\langle z{\cdot}\beta\rangle \;=(gc_{\mathrm{L}})\; \langle z{\cdot}\beta\rangle$

$(P = \widehat{z}R\widehat{\gamma}\cdot\beta)$: $\{P\widehat{\alpha}\nwarrow x\}(\widehat{z}R\widehat{\gamma}\cdot\beta) \;=(exp_{\mathrm{L}})\; \widehat{z}\,(\{P\widehat{\alpha}\nwarrow x\}R)\widehat{\gamma}\cdot\beta \;=(ih)\; \widehat{z}R\widehat{\gamma}\cdot\beta$

$(P = Q\widehat{\gamma}\,[v]\,\widehat{z}R,\ v \neq x)$: $\{P\widehat{\alpha}\nwarrow x\}(Q\widehat{\gamma}\,[v]\,\widehat{z}R) \;=(imp\text{-}in_{\mathrm{L}})\; (\{P\widehat{\alpha}\nwarrow x\}Q)\,\widehat{\gamma}\,[v]\,\widehat{z}\,(\{P\widehat{\alpha}\nwarrow x\}R)$
$=(ih)\; Q\widehat{\gamma}\,[v]\,\widehat{z}R$

$(P = Q\widehat{\gamma}\dagger\widehat{z}R)$: $\{P\widehat{\alpha}\nwarrow x\}(Q\widehat{\gamma}\dagger\widehat{z}R) \;=(cut_{\mathrm{L}})\; (\{P\widehat{\alpha}\nwarrow x\}Q)\,\widehat{\gamma}\dagger\widehat{z}\,(\{P\widehat{\alpha}\nwarrow x\}R) \;=(ih)$
$Q\widehat{\gamma}\dagger\widehat{z}R$ □

Notice that substitution activation plays no role in this proof.

Type assignment for $\mathcal{X}_{\mathrm{IS}}$ is defined as for $\mathcal{X}$, and the following soundness result of type assignment with respect to $\to_{\mathcal{X}_{\mathrm{IS}}}$ reduction is stated as usual and is easy to show.

Theorem 4.7 (Witness reduction for $\to_{\mathcal{X}_{\mathrm{IS}}}$) *If $P :\cdot\, \Gamma \vdash_{\mathcal{X}} \Delta$, and $P \to_{\mathcal{X}_{\mathrm{IS}}} Q$, then $Q :\cdot\, \Gamma \vdash_{\mathcal{X}} \Delta$.*

Proof: By Thm. 3.10 and 4.5. □

5 Embedding the $\mathrm{s}\lambda\mu$-calculus in to $\overline{\lambda}\mu\tilde{\mu}$

Essentially following [12], an interpretation $[\![\cdot]\!]$ of $\mathrm{s}\lambda\mu$ into $\overline{\lambda}\mu\tilde{\mu}$ can be defined as follows:

$$\begin{array}{rcl} [\![x]\!] & \triangleq & x \\ [\![\lambda x.M]\!] & \triangleq & \lambda x.[\![M]\!] \\ [\![MN]\!] & \triangleq & \mu\alpha.\langle [\![M]\!] \,|\, [\![N]\!]\cdot\alpha \rangle \\ [\![\mu\beta.[\gamma]M]\!] & \triangleq & \mu\beta.\langle [\![M]\!] | \gamma \rangle \end{array}$$

Using this interpretation, and the observation that

$$[\![PQR]\!] \triangleq \mu\alpha.\langle [\![PQ]\!] \,|\, [\![R]\!]\cdot\alpha \rangle \triangleq \mu\alpha.\langle \mu\beta.\langle [\![P]\!] \,|\, [\![Q]\!]\cdot\beta \rangle \,|\, [\![R]\!]\cdot\alpha \rangle \rightarrow_{\overline{\lambda}} \mu\alpha.\langle [\![P]\!] \,|\, [\![Q]\!]\cdot[\![R]\!]\cdot\alpha \rangle$$

we can point out the fundamental difference between μ-reduction in $\lambda\mu$ and $\overline{\lambda}\mu\tilde{\mu}$.

Remark 5.1 In $\lambda\mu$, as discussed above, the intention of μ-reduction is to redirect an applicative context, but it has to do that 'one term at the time'.

$$\begin{array}{rcll} (\mu\alpha.[\tau]x\,(\mu\sigma.[\alpha]M))\,PQR & \rightarrow_{\lambda\mu} & (\mu\gamma.[\tau]x\,(\mu\sigma.[\gamma]MP))\,QR & \rightarrow_{\lambda\mu} \\ & & (\mu\delta.[\tau]x\,(\mu\sigma.[\delta]MPQ))\,R & \rightarrow_{\lambda\mu} \\ & & \mu\beta.[\tau]x\,(\mu\sigma.[\beta]MPQR) & \end{array}$$

and therefore has to be recursive in nature. This is not the case for $\overline{\lambda}\mu\tilde{\mu}$, where we have the reduction

$$\begin{array}{ll} [\![(\mu\alpha.[\tau]x\,(\mu\sigma.[\alpha]M))\,PQR]\!] & \triangleq \\ \mu\beta.\langle \mu\gamma.\langle \mu\delta.\langle \mu\alpha.\langle \mu\rho.\langle x \,|\, \mu\sigma.\langle [\![M]\!] | \alpha \rangle\cdot\rho \rangle \,|\, \tau \rangle \,|\, [\![P]\!]^{\mathrm{s}}\cdot\delta \rangle \,|\, [\![Q]\!]^{\mathrm{s}}\cdot\gamma \rangle \,|\, [\![R]\!]^{\mathrm{s}}\cdot\beta \rangle & \rightarrow^{*}_{\overline{\lambda}} \\ \mu\beta.\langle \mu\alpha.\langle \mu\rho.\langle x \,|\, \mu\sigma.\langle [\![M]\!] | \alpha \rangle\cdot\rho \rangle \,|\, \tau \rangle \,|\, [\![P]\!]^{\mathrm{s}}\cdot[\![Q]\!]^{\mathrm{s}}\cdot[\![R]\!]^{\mathrm{s}}\cdot\beta \rangle & \rightarrow_{\overline{\lambda}}(\alpha) \\ \mu\beta.\langle \mu\rho.\langle x \,|\, \mu\sigma.\langle [\![M]\!] \,|\, [\![P]\!]^{\mathrm{s}}\cdot[\![Q]\!]^{\mathrm{s}}\cdot[\![R]\!]^{\mathrm{s}}\cdot\beta \rangle\cdot\rho \rangle \,|\, \tau \rangle & \triangleq \end{array}$$

where the whole environment $[\![P]\!]^{\mathrm{s}}\cdot[\![Q]\!]^{\mathrm{s}}\cdot[\![R]\!]^{\mathrm{s}}\cdot\beta$ gets pulled in in one step (notice that the first sequence of steps does not deal with the $\mu\alpha$-redex contraction, but just prepares the environment, contracting the μ-redexes that are generated by the interpretation). We can also reduce the term as follows:

$$\begin{array}{ll} [\![(\mu\alpha.[\tau]x\,(\mu\sigma.[\alpha]M))\,PQR]\!] & \triangleq \\ \mu\beta.\langle \mu\gamma.\langle \mu\delta.\langle \mu\alpha.\langle \mu\rho.\langle x \,|\, \mu\sigma.\langle [\![M]\!] | \alpha \rangle\cdot\rho \rangle \,|\, \tau \rangle \,|\, [\![P]\!]^{\mathrm{s}}\cdot\delta \rangle \,|\, [\![Q]\!]^{\mathrm{s}}\cdot\gamma \rangle \,|\, [\![R]\!]^{\mathrm{s}}\cdot\beta \rangle & \rightarrow_{\overline{\lambda}}(\alpha) \\ \mu\beta.\langle \mu\gamma.\langle \mu\delta.\langle \mu\rho.\langle x \,|\, \mu\sigma.\langle [\![M]\!] \,|\, [\![P]\!]^{\mathrm{s}}\cdot\delta \rangle\cdot\rho \rangle \,|\, \tau \rangle \,|\, [\![Q]\!]^{\mathrm{s}}\cdot\gamma \rangle \,|\, [\![R]\!]^{\mathrm{s}}\cdot\beta \rangle & \rightarrow_{\overline{\lambda}}(\delta) \\ \mu\beta.\langle \mu\gamma.\langle \mu\rho.\langle x \,|\, \mu\sigma.\langle [\![M]\!] \,|\, [\![P]\!]^{\mathrm{s}}\cdot[\![Q]\!]^{\mathrm{s}}\cdot\gamma \rangle\cdot\rho \rangle \,|\, \tau \rangle \,|\, [\![R]\!]^{\mathrm{s}}\cdot\beta \rangle & \rightarrow^{\mathrm{v}}_{\overline{\lambda}}(\gamma) \\ \mu\beta.\langle \mu\rho.\langle x \,|\, \mu\sigma.\langle [\![M]\!] \,|\, [\![P]\!]^{\mathrm{s}}\cdot[\![Q]\!]^{\mathrm{s}}\cdot[\![R]\!]^{\mathrm{s}}\cdot\beta \rangle\cdot\rho \rangle \,|\, \tau \rangle & \end{array}$$

which 'pulls in one term at the time', but does not do that using α but rather the μ-redexes generated by the interpretation for applications. In fact, in a way the μ-abstractions added by the interpretation implement the repetitive character of $\lambda\mu$'s μ-reduction; for each surrounding application, a μ-abstraction is inserted, that will be used to execute one of the recursive steps.

Example 5.2 It is worthwhile to remark that reduction in the image of $[\![\cdot]\!]$, even when restricted to the λ-calculus, is not confluent. In fact, we have both:

$$\begin{array}{ll}
[\![(\lambda z.zz)\,(pp)]\!] & \triangleq \\
\mu\alpha.\langle [\![\lambda z.zz]\!] \,|\, [\![pp]\!]{\cdot}\alpha\rangle & \triangleq \\
\mu\alpha.\langle \lambda z.(zz)^n \,|\, [\![pp]\!]{\cdot}\alpha\rangle & \rightarrow_{\overline{\lambda}}(\lambda) \\
\mu\alpha.\langle [\![pp]\!] \,|\, \tilde{\mu}z.\langle (zz)^n|\alpha\rangle\rangle & \triangleq \\
\mu\alpha.\langle \mu\beta.\langle p|p{\cdot}\beta\rangle \,|\, \tilde{\mu}z.\langle (zz)^n|\alpha\rangle\rangle & \rightarrow^{\mathrm{V}}_{\overline{\lambda}}(\mu) \\
\mu\alpha.\langle p \,|\, p{\cdot}\tilde{\mu}z.\langle (zz)^n|\alpha\rangle\rangle & \triangleq \\
\mu\alpha.\langle p \,|\, p{\cdot}\tilde{\mu}z.\langle \mu\gamma.\langle z|z{\cdot}\gamma\rangle \,|\, \alpha\rangle\rangle &
\end{array}$$

$$\begin{array}{ll}
[\![(\lambda z.zz)\,(pp)]\!] & \rightarrow_{\overline{\lambda}}(\lambda) \\
\mu\alpha.\langle [\![pp]\!] \,|\, \tilde{\mu}z.\langle (zz)^n|\alpha\rangle\rangle & \triangleq \\
\mu\alpha.\langle [\![pp]\!] \,|\, \tilde{\mu}z.\langle \mu\gamma.\langle z|z{\cdot}\gamma\rangle \,|\, \alpha\rangle\rangle & \rightarrow^{\mathrm{N}}_{\overline{\lambda}}(\tilde{\mu}) \\
\mu\alpha.\langle \mu\gamma.\langle [\![pp]\!] \,|\, [\![pp]\!]{\cdot}\gamma\rangle \,|\, \alpha\rangle & \rightarrow^{\mathrm{N}}_{\overline{\lambda}}(\mu) \\
\mu\alpha.\langle [\![pp]\!] \,|\, [\![pp]\!]{\cdot}\alpha\rangle & \triangleq \\
\mu\alpha.\langle \mu\beta.\langle p|p{\cdot}\beta\rangle \,|\, [\![pp]\!]{\cdot}\alpha\rangle & \rightarrow^{\mathrm{N}}_{\overline{\lambda}}(\mu) \\
\mu\alpha.\langle p \,|\, p{\cdot}[\![pp]\!]{\cdot}\alpha\rangle & \triangleq \\
\mu\alpha.\langle p \,|\, p{\cdot}\mu\beta.\langle p|p{\cdot}\beta\rangle{\cdot}\alpha\rangle &
\end{array}$$

This holds for all the interpretations from $\mathrm{s}\lambda\mu$ to $\overline{\lambda}\mu\tilde{\mu}$ we discuss in this paper. Notice that the right reduction is in CBN. The left is CBV; it can be extended with

$$\mu\alpha.\langle p \,|\, p{\cdot}\tilde{\mu}z.\langle \mu\gamma.\langle z|z{\cdot}\gamma\rangle \,|\, \alpha\rangle\rangle \;\rightarrow_{\overline{\lambda}}(\mu)\; \mu\alpha.\langle p \,|\, p{\cdot}\tilde{\mu}z.\langle z|z{\cdot}\alpha\rangle\rangle$$

but here reduction takes place in the environment.

Curien and Herbelin [12] define two separate encodings, one, $\cdot^n$, to model CBN and another, $\cdot^v$, to model CBV:

Definition 5.3 ([12]) The interpretations $\cdot^v$ and $\cdot^n$ of $\lambda\mu$ into $\overline{\lambda}\mu\tilde{\mu}$ are defined by:

$$\begin{array}{rcl}
x^v & \triangleq & x \\
(\lambda x.M)^v & \triangleq & \lambda x.M^v \\
(MN)^v & \triangleq & \mu\alpha.\langle N^v \,|\, \tilde{\mu}x.\langle M^v|x{\cdot}\alpha\rangle\rangle \\
(\mu\beta.\mathrm{C})^v & \triangleq & \mu\beta.\mathrm{C}^v \\
([\alpha]M)^v & \triangleq & \langle M^v|\alpha\rangle
\end{array}
\qquad
\begin{array}{rcl}
x^n & \triangleq & x \\
(\lambda x.M)^n & \triangleq & \lambda x.M^n \\
(MN)^n & \triangleq & \mu\alpha.\langle M^n|N^n{\cdot}\alpha\rangle \\
(\mu\beta.\mathrm{C})^n & \triangleq & \mu\beta.\mathrm{C}^n \\
([\alpha]M)^n & \triangleq & \langle M^n|\alpha\rangle
\end{array}$$

Note that these also deal with $\Lambda\mu$, where naming and μ-binding are separate.

Observe that $\cdot^n$ is the interpretation $[\![\cdot]\!]$ we mentioned above, and that these interpretations only differ in the case for application; remark that we have:

$$\mu\alpha.\langle t_1 \,|\, \tilde{\mu}x.\langle t_2|x{\cdot}\alpha\rangle\rangle \quad \rightarrow^{\mathrm{N}}_{\overline{\lambda}}(x)\; \mu\alpha.\langle t_2|t_1{\cdot}\alpha\rangle$$

and in effect, Herbelin only considers $\cdot^v$ in [12].

Now the problem signalled above disappears, since we have (assuming that $M^v \rightarrow^{\mathrm{V}*}_{\overline{\lambda}} N^v$):

$$(VM)^v \;\triangleq\; \mu\alpha.\langle M^v \,|\, \tilde{\mu}x.\langle V^v|x{\cdot}\alpha\rangle\rangle \;\rightarrow^{\mathrm{V}*}_{\overline{\lambda}}\; \mu\alpha.\langle N^v \,|\, \tilde{\mu}x.\langle V^v|x{\cdot}\alpha\rangle\rangle \;\triangleq\; (VN)^v$$

so we can simulate the evaluation of the argument of a redex without allowing reduction in the environment.

[12] in fact only deals with (β) and (μ_{R})-reduction, and defines CBV reduction by limiting the operands in those rules to values. It states some preservation results, but gives very few details.

We wanted to investigate if there could be an interpretation that respects both CBN and CBV strategies for $\mathrm{s}\lambda\mu$ as well. We first make the following observations:

Remark 5.4 • The interpretation $\cdot^n$ creates problems when interpreting the CBV (μ_{L})-reduction

$$N\,(\mu\beta.[\beta]M) \;\rightarrow^{s}_{\beta\mu}\; \mu\gamma.[\gamma]N\,(\{N{\cdot}\gamma/\beta\}M)$$

We would like to show that

$$\begin{array}{rl}
(N\,(\mu\beta.[\beta]M))^n & \triangleq \\
\mu\alpha.\langle N^n \,|\, \mu\beta.\langle M^n|\beta\rangle\cdot\alpha\rangle & \downarrow^{\mathrm{V}}_{\overline{\lambda}}\;(?)\;\; \mu\alpha.\langle N^n \,|\, (\{[N{\cdot}\gamma]/\beta\}M)^n\cdot\alpha\rangle \\
& {}_{\overline{\lambda}}^{\mathrm{V}}\!\leftarrow(\gamma)\;\; \mu\gamma.\langle \mu\alpha.\langle N^n \,|\, (\{N{\cdot}\gamma/\beta\}M)^n\cdot\alpha\rangle \,|\, \gamma\rangle \\
& \triangleq\;\; (\mu\gamma.[\gamma]N\,(\{N{\cdot}\gamma/\beta\}M))^n
\end{array}$$

(as will be explained below, the interpretation respects reduction through equality; not through reduction), but cannot: the term $\mu\alpha.\langle N^n \,|\, \mu\beta.\langle M^n|\beta\rangle \cdot \alpha\rangle$ is not a (μ)-redex (over β). In Sect. 6 we will discuss adding this kind of term to the redexes of $\overline{\lambda}\mu\tilde{\mu}$.

- As already observed above, $\cdot^n$ also does not deal well with the contextual reduction rules for CBV. For the rule $M \rightarrow^{\mathrm{v}}_{\lambda} N \Rightarrow VM \rightarrow^{\mathrm{v}}_{\lambda} VN$ we have:

$$(VM)^n \;\triangleq\; \mu\alpha.\langle V^n|M^n{\cdot}\alpha\rangle \;\downarrow^{\mathrm{v}}_{\lambda}(?)\; \mu\alpha.\langle V^n|N^n{\cdot}\alpha\rangle \;\triangleq\; (VN)^n$$

which asks for reduction in the environment.

- That $\cdot^v$ deals correctly with (μ_{L}) is illustrated by:

$$\begin{array}{rl} (N\,(\mu\beta.[\beta]M))^v & \triangleq \\ \mu\alpha.\langle \mu\beta.\langle M^v|\beta\rangle \,|\, \tilde{\mu}x.\langle N^v|x{\cdot}\alpha\rangle\rangle & \rightarrow_{\overline{\lambda}}(\mu)\; \mu\alpha.\langle M^v\{\tilde{\mu}x.\langle N^v|x{\cdot}\alpha\rangle/\beta\} \,|\, \tilde{\mu}x.\langle N^v|x{\cdot}\alpha\rangle\rangle \\ & {}^{*}_{\lambda}\!\leftarrow\; \mu\alpha.\langle (\{N{\cdot}\alpha/\beta\}M)^v \,|\, \tilde{\mu}x.\langle M^v|x{\cdot}\alpha\rangle\rangle \\ & \triangleq\; (M\,(\{N{\cdot}\alpha/\beta\}M))^v \end{array}$$

provided of course that we verify that $M^v\{\tilde{\mu}x.\langle N^v|x{\cdot}\alpha\rangle/\beta\} = (\{N{\cdot}\alpha/\beta\}M)^v$; we will do so in Lem. 5.13 and in Sect. 6.

- There is a problem in showing $M \rightarrow^{\mathrm{v}}_{\lambda\mu} N \Rightarrow M^v \downarrow^{\mathrm{v}}_{\lambda} N^v$ when dealing with the contextual reduction rules. The first, $M \rightarrow N \Rightarrow VM \rightarrow VN$ now follows easily, since we have:

$$(VM)^v \;\triangleq\; \mu\alpha.\langle M^v \,|\, \tilde{\mu}x.\langle V^v|x{\cdot}\alpha\rangle\rangle \;\downarrow^{\mathrm{v}}_{\lambda}(ih)\; \mu\alpha.\langle N^v \,|\, \tilde{\mu}x.\langle V^v|x{\cdot}\alpha\rangle\rangle \;\triangleq\; (VN)^v$$

benefitting from the swap between the terms, but for the second $M \rightarrow N \Rightarrow MP \rightarrow NP$ we now have:

$$(MP)^v \;\triangleq\; \mu\alpha.\langle P^v \,|\, \tilde{\mu}x.\langle M^v|x{\cdot}\alpha\rangle\rangle \;\downarrow^{\mathrm{v}}_{\lambda}(?)\; \mu\alpha.\langle P^v \,|\, \tilde{\mu}x.\langle N^v|x{\cdot}\alpha\rangle\rangle \;\triangleq\; (VN)^v$$

for which we need to allow for reduction to take place inside a $\tilde{\mu}$-term, so inside the environment.

When modelling CBN reduction under this interpretation, there is no need to reduce in the environment, since we can then contract the $(\tilde{\mu})$-redexes:

$$\begin{array}{llll} (MP)^v \triangleq & \mu\alpha.\langle P^v \,|\, \tilde{\mu}x.\langle M^v|x{\cdot}\alpha\rangle\rangle & \rightarrow^{\mathrm{N}*}_{\lambda}(x)\; \mu\alpha.\langle M^v|P^v{\cdot}\alpha\rangle & \downarrow^{\mathrm{N}}_{\lambda}(ih) \\ & \mu\alpha.\langle N^v|P^v{\cdot}\alpha\rangle & \downarrow^{\mathrm{N}}_{\lambda}\;(x)\; \mu\alpha.\langle P^v \,|\, \tilde{\mu}x.\langle N^v|x{\cdot}\alpha\rangle\rangle & \triangleq\; (VN)^v \end{array}$$

This is not allowed for CBV, since P^v need not be a value.

- We could argue that the encoding $\cdot^v$ actually represents of a CBV-reduction strategy variant on $\mathrm{s}\lambda\mu$ with the contextual rules:

$$\mathsf{C}_{\mathrm{v}} \;::=\; \lceil\rfloor \mid \mathsf{C}_{\mathrm{v}}V \mid M\mathsf{C}_{\mathrm{v}} \mid \mu\alpha.[\beta]\,\mathsf{C}_{\mathrm{v}}$$

which would force the evaluation of the parameter until it becomes a value, after which the term in function position gets reduced; this corresponds to a reduction like (where we assume that each P_i runs to a value V_i):

$$\begin{array}{l} (\lambda x.M)P_1P_2\cdots P_{n-1}P_n \;\rightarrow^{*}_{\mathrm{v}}\; (\lambda x.M)P_1P_2\cdots P_{n-1}V_n \;\rightarrow^{*}_{\mathrm{v}}\; (\lambda x.M)P_1P_2\cdots V_{n-1}V_n \;\rightarrow^{*}_{\mathrm{v}} \\ (\lambda x.M)P_1V_2\cdots V_{n-1}V_n \;\rightarrow_{\mathrm{v}}\; (\lambda x.M)V_1V_2\cdots V_{n-1}V_n \;\rightarrow_{\mathrm{v}}\; M\{V_1/x\}V_2\cdots V_{n-1}V_n \end{array}$$

which would perhaps be too great a deviation from a 'normal' CBV strategy.

We would then have:

$$\begin{array}{llll} (PM)^v \triangleq & \mu\alpha.\langle M^v \,|\, \tilde{\mu}x.\langle P^v|x{\cdot}\alpha\rangle\rangle & \downarrow^{\mathrm{v}}_{\lambda}\;(ih)\; \mu\alpha.\langle N^v \,|\, \tilde{\mu}x.\langle P^v|x{\cdot}\alpha\rangle\rangle & \triangleq\; (PN)^v \\ (MV)^v \triangleq & \mu\alpha.\langle V^v \,|\, \tilde{\mu}x.\langle M^n|x{\cdot}\alpha\rangle\rangle & \rightarrow^{\mathrm{N}*}_{\lambda}(x)\; \mu\alpha.\langle M^v|V^n{\cdot}\alpha\rangle & \downarrow^{\mathrm{N}}_{\lambda}(ih) \\ & \mu\alpha.\langle N^v|V^n{\cdot}\alpha\rangle & \downarrow^{\mathrm{N}}_{\lambda}\;(x)\; \mu\alpha.\langle V^v \,|\, \tilde{\mu}x.\langle N^n|x{\cdot}\alpha\rangle\rangle & \triangleq\; (NV)^v \end{array}$$

without the need to reduce inside the environment.[5]

Curien and Herbelin [12] state reduction preservation results for their encodings (formu-

[5] This might be well suited to model reduction in the Call by Push Value calculus [25], where reduction inside parameters is not permitted.

lated as if $\lambda\mu$-reduction is represented through $\overline{\lambda}\mu\tilde{\mu}$-reduction, modulo (μ)-expansion). These are stated with respect to the notion of CBN and CBV-reduction for $\overline{\lambda}\mu\tilde{\mu}$ that just remove the $(\mu),(\tilde{\mu})$ critical pair, so are sub-reduction systems, not the notions we have defined here, which are reduction strategies. We will see in the proofs of Lem. 5.9 and Thm. 5.10 that in order to model (μ_{R})-reduction, some reverse reduction steps are needed as well.

We will now show that we can strengthen the results of [12], and show that we can define *one* interpretation with which we can successfully represent all three notions of reduction and strategy. We will essentially show that our interpretation can be used to represent not only the (traditional) $\lambda\mu$ calculus, but also s$\lambda\mu$, and not only for the CBV reduction strategy, but also CBN, as well as unrestricted reduction.

We will first define our interpretation.

Definition 5.5 Interpretation $\llbracket\cdot\rrbracket^{\mathrm{s}}$ of s$\lambda\mu$ into $\overline{\lambda}\mu\tilde{\mu}$:

$$\begin{array}{rcl}
\llbracket x\rrbracket^{\mathrm{s}} & \stackrel{\Delta}{=} & x \\
\llbracket \lambda x.M\rrbracket^{\mathrm{s}} & \stackrel{\Delta}{=} & \lambda x.\llbracket M\rrbracket^{\mathrm{s}} \\
\llbracket MN\rrbracket^{\mathrm{s}} & \stackrel{\Delta}{=} & \mu\alpha.\langle \llbracket M\rrbracket^{\mathrm{s}} \,|\, \tilde{\mu}x.\langle \llbracket N\rrbracket^{\mathrm{s}} \,|\, \tilde{\mu}y.\langle x\,|\,y\cdot\alpha\rangle\rangle\rangle \\
\llbracket \mu\beta.[\gamma]M\rrbracket^{\mathrm{s}} & \stackrel{\Delta}{=} & \mu\beta.\langle \llbracket M\rrbracket^{\mathrm{s}} | \gamma\rangle
\end{array}$$

Notice that this interpretation also is a mapping from the λ-calculus to $\overline{\lambda}\mu\tilde{\mu}$.

It is straightforward to show that this interpretation respects assignable types:

Theorem 5.6 *If $\Gamma \vdash_{\lambda\mu} M : A \mid \Delta$, then $\Gamma \vdash_{\overline{\lambda}} \llbracket M\rrbracket^{\mathrm{s}} : A \mid \Delta$.*

Proof: (Ax): Then $M \equiv x$ and $x{:}A \in \Gamma$; since $\llbracket M\rrbracket^{\mathrm{s}} = x$, also $\Gamma \vdash_{\overline{\lambda}} x : A \mid \Delta$ by rule (Ax).

$(\to I)$: Then $M \equiv \lambda x.N$, $A \equiv B\to C$, and $\Gamma, x{:}B \vdash_{\lambda\mu} N : C \mid \Delta$. By induction, $\Gamma, x{:}B \vdash_{\overline{\lambda}} \llbracket N\rrbracket^{\mathrm{s}} : C \mid \Delta$, and by rule $(\to I)$, $\Gamma \vdash_{\overline{\lambda}} \lambda x.\llbracket N\rrbracket^{\mathrm{s}} : A \mid \Delta$.

$(\to E)$: Then $M \equiv PQ$, and there exists B such that $\Gamma \vdash_{\lambda\mu} P : B\to A \mid \Delta$ and $\Gamma \vdash_{\lambda\mu} Q : B \mid \Delta$. Then by induction, $\Gamma \vdash_{\overline{\lambda}} \llbracket P\rrbracket^{\mathrm{s}} : B\to A \mid \Delta$ and $\Gamma \vdash_{\overline{\lambda}} \llbracket Q\rrbracket^{\mathrm{s}} : B \mid \Delta$; by weakening, we also have $\Gamma \vdash_{\overline{\lambda}} \llbracket P\rrbracket^{\mathrm{s}} : B\to A \mid \alpha{:}A,\Delta$ and $\Gamma, x{:}B\to A \vdash_{\overline{\lambda}} \llbracket Q\rrbracket^{\mathrm{s}} : B \mid \alpha{:}A,\Delta$, and we can construct (where $\Gamma' = \Gamma, x{:}B\to A, y{:}B$):

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash_{\overline{\lambda}} \llbracket P\rrbracket^{\mathrm{s}} : B\to A \mid \alpha{:}A,\Delta\end{array} \quad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma, x{:}B\to A \vdash_{\overline{\lambda}} \llbracket Q\rrbracket^{\mathrm{s}} : B \mid \alpha{:}A,\Delta\end{array} \quad \dfrac{\dfrac{\dfrac{}{\Gamma' \vdash_{\overline{\lambda}} x : B\to A \mid \alpha{:}A,\Delta} \quad \dfrac{\dfrac{}{\Gamma' \vdash_{\overline{\lambda}} y : B \mid \alpha{:}A,\Delta} \quad \dfrac{}{\Gamma' \mid \alpha : A \vdash_{\overline{\lambda}} \alpha{:}A,\Delta}}{\Gamma' \mid y\cdot\alpha : B\to A \vdash_{\overline{\lambda}} \alpha{:}A,\Delta}}{\langle x|y\cdot\alpha\rangle : \Gamma' \vdash_{\overline{\lambda}} \alpha{:}A,\Delta}}{\Gamma, x{:}B\to A \mid \tilde{\mu}y.\langle x|y\cdot\alpha\rangle : B \vdash_{\overline{\lambda}} \alpha{:}A,\Delta}}{\langle \llbracket Q\rrbracket^{\mathrm{s}} \,|\, \tilde{\mu}y.\langle x|y\cdot\alpha\rangle\rangle : \Gamma, x{:}B\to A \vdash_{\overline{\lambda}} \alpha{:}A,\Delta}}{\Gamma \mid \tilde{\mu}x.\langle \llbracket Q\rrbracket^{\mathrm{s}} \,|\, \tilde{\mu}y.\langle x|y\cdot\alpha\rangle\rangle : B\to A \vdash_{\overline{\lambda}} \alpha{:}A,\Delta}}{\langle \llbracket P\rrbracket^{\mathrm{s}} \,|\, \tilde{\mu}x.\langle \llbracket Q\rrbracket^{\mathrm{s}} \,|\, \tilde{\mu}y.\langle x|y\cdot\alpha\rangle\rangle\rangle : \Gamma \vdash_{\overline{\lambda}} \alpha{:}A,\Delta}}{\Gamma \vdash_{\overline{\lambda}} \mu\alpha.\langle \llbracket P\rrbracket^{\mathrm{s}} \,|\, \tilde{\mu}x.\langle \llbracket Q\rrbracket^{\mathrm{s}} \,|\, \tilde{\mu}y.\langle x|y\cdot\alpha\rangle\rangle\rangle : A \mid \Delta}$$

(μ): We have two cases: $M \equiv \mu\alpha.[\beta]N$, $\Delta = \beta{:}B,\Delta'$, and $\Gamma \vdash_{\lambda\mu} N : B \mid \alpha{:}A,\beta{:}B,\Delta'$; then by induction we have $\Gamma \vdash_{\overline{\lambda}} \llbracket N\rrbracket^{\mathrm{s}} : B \mid \alpha{:}A,\beta{:}B,\Delta'$. We can construct:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash_{\overline{\lambda}} \llbracket N\rrbracket^{\mathrm{s}} : B \mid \alpha{:}A,\beta{:}B,\Delta'\end{array} \quad \dfrac{}{\Gamma \mid \beta : B \vdash_{\overline{\lambda}} \alpha{:}A,\beta{:}B,\Delta'}}{\langle \llbracket N\rrbracket^{\mathrm{s}} | \beta\rangle : \Gamma \vdash_{\overline{\lambda}} \alpha{:}A,\beta{:}B,\Delta'}}{\Gamma \vdash_{\overline{\lambda}} \mu\alpha.\langle \llbracket N\rrbracket^{\mathrm{s}} | \beta\rangle : A \mid \beta{:}B,\Delta'}$$

Or $M \equiv \mu\alpha.[\alpha]N$ and $\Gamma \vdash_{\lambda\mu} N : A \mid \alpha{:}A,\Delta$; then by induction we have $\Gamma \vdash_{\overline{\lambda}} \llbracket N\rrbracket^{\mathrm{s}} : A \mid \alpha{:}A,\Delta$. We can construct:

$$\dfrac{\dfrac{\overline{\quad\quad}}{\Gamma \vdash_{\overline{\lambda}} [\![N]\!]^{\mathrm{s}} : A \mid \alpha{:}A,\Delta} \quad \dfrac{}{\Gamma \mid \alpha : A \vdash_{\overline{\lambda}} \alpha{:}A,\Delta}}{\dfrac{\langle [\![N]\!]^{\mathrm{s}} | \alpha \rangle : \Gamma \vdash_{\overline{\lambda}} \alpha{:}A,\Delta}{\Gamma \vdash_{\overline{\lambda}} \mu\alpha.\langle [\![N]\!]^{\mathrm{s}} | \alpha \rangle : A \mid \Delta}}$$

□

Remark that we have:

$$\begin{array}{lcl} [\![PQ]\!]^{\mathrm{s}} & \triangleq & \mu\alpha.\langle [\![P]\!]^{\mathrm{s}} \,|\, \tilde{\mu}x.\langle [\![Q]\!]^{\mathrm{s}} \,|\, \tilde{\mu}y.\langle x | y{\cdot}\alpha \rangle\rangle\rangle \to \mu\alpha.\langle [\![Q]\!]^{\mathrm{s}} \,|\, \tilde{\mu}y.\langle [\![P]\!]^{\mathrm{s}} | y{\cdot}\alpha \rangle\rangle \to \mu\alpha.\langle [\![P]\!]^{\mathrm{s}} \,|\, [\![Q]\!]^{\mathrm{s}}{\cdot}\alpha \rangle \\ (PQ)^{v} & \triangleq & \mu\alpha.\langle Q^{v} \,|\, \tilde{\mu}y.\langle P^{v} | y{\cdot}\alpha \rangle\rangle \;\to\; \mu\alpha.\langle P^{v} \,|\, Q^{v}{\cdot}\alpha \rangle \\ (PQ)^{n} & \triangleq & \mu\alpha.\langle P^{n} \,|\, Q^{n}{\cdot}\alpha \rangle \end{array}$$

a relation that will be useful when doing the proofs.

We will show our encoding respects the three notions of reduction by showing, in Thm. 5.15, 5.16, and 5.17 that the encoding respects reduction through equality:

$$M \to_{\lambda\mu}^{\mathrm{s}*} N \;\Rightarrow\; [\![M]\!]^{\mathrm{s}} =_{\overline{\lambda}} [\![N]\!]^{\mathrm{s}} \tag{1}$$

$$M \to_{\lambda\mu}^{\mathrm{sv}*} N \;\Rightarrow\; [\![M]\!]^{\mathrm{s}} \downarrow_{\overline{\lambda}}^{\mathrm{v}} [\![N]\!]^{\mathrm{s}} \tag{2}$$

$$M \to_{\lambda\mu}^{\mathrm{sn}*} N \;\Rightarrow\; [\![M]\!]^{\mathrm{s}} \downarrow_{\overline{\lambda}}^{\mathrm{n}} [\![N]\!]^{\mathrm{s}} \tag{3}$$

For the basic steps in reduction this will be shown through Thm. 5.8 that shows that the encoding respects the (β)-reduction rule:

$$[\![(\lambda z.M)N]\!]^{\mathrm{s}} \;\to_{\overline{\lambda}}^{*}\; [\![M\{N/z\}]\!]^{\mathrm{s}}$$

Thm. 5.10 shows it respects (μ_{R})-reduction:

$$\begin{array}{lcl} [\![(\mu\delta.[\beta]M)N]\!]^{\mathrm{s}} & =_{\overline{\lambda}} & [\![\mu\gamma.[\beta]M\{N{\cdot}\gamma/\delta\}]\!]^{\mathrm{s}}, \quad (\beta \neq \delta) \\ [\![(\mu\delta.[\delta]M)N]\!]^{\mathrm{s}} & =_{\overline{\lambda}} & [\![\mu\gamma.[\gamma](M\{N{\cdot}\gamma/\delta\})N]\!]^{\mathrm{s}} \end{array}$$

and Thm. 5.14 shows it respects (μ_{L})-reduction:

$$\begin{array}{lcl} [\![N(\mu\alpha.[\delta]M)]\!]^{\mathrm{s}} & =_{\overline{\lambda}} & [\![\mu\gamma.[\delta]\{N{\cdot}\gamma/\alpha\}M]\!]^{\mathrm{s}}, \quad (\alpha \neq \delta) \\ [\![N(\mu\alpha.[\alpha]M)]\!]^{\mathrm{s}} & =_{\overline{\lambda}} & [\![\mu\gamma.[\gamma]N(\{N{\cdot}\gamma/\alpha\}M)]\!]^{\mathrm{s}} \end{array}$$

We will, for each of these last three results, argue that all the contractions that take place in the proofs would still be allowed when restricting to the CBV and CBN reduction strategies, so prove the three results (1), (2), and (3) simultaneously. The only exception to this is Thm. 5.10 the proof of which is not sound for CBN; Thm. 5.12 will show that result for CBN. Note that, for CBV, we need to check that: *1) only values are substituted through* $(\tilde{\mu})$*-reduction steps; 2) no reduction takes place in environments*, and for CBN that: *1) only stacks are substituted through* (μ_{R})*-reduction steps; 2) no reduction takes place in environments.*

We start by showing that the interpretation respects β-reduction, for which we first need to show it respects term substitution.

Lemma 5.7 $[\![M]\!]^{\mathrm{s}}\{[\![N]\!]^{\mathrm{s}}/z\} = [\![M\{N/z\}]\!]^{\mathrm{s}}$

Proof: By induction on the structure of terms.

$(M = z)$: $[\![z]\!]^{\mathrm{s}}\{[\![N]\!]^{\mathrm{s}}/z\} \triangleq z\{[\![N]\!]^{\mathrm{s}}/z\} \triangleq [\![N]\!]^{\mathrm{s}} \triangleq [\![z\{N/z\}]\!]^{\mathrm{s}}$

$(M = y,\, y \neq z)$: $[\![y]\!]^{\mathrm{s}}\{[\![N]\!]^{\mathrm{s}}/z\} \triangleq y\{[\![N]\!]^{\mathrm{s}}/z\} \triangleq y \triangleq [\![y]\!]^{\mathrm{s}} \triangleq [\![y\{N/z\}]\!]^{\mathrm{s}}$

$(M = \lambda y.P)$: $[\![\lambda y.P]\!]^{\mathrm{s}}\{[\![N]\!]^{\mathrm{s}}/z\} \triangleq \lambda y.[\![P]\!]^{\mathrm{s}}\{[\![N]\!]^{\mathrm{s}}/z\} = (ih)\, \lambda y.[\![P\{N/z\}]\!]^{\mathrm{s}} \triangleq [\![\lambda y.P\{N/z\}]\!]^{\mathrm{s}}$

$(M = PQ)$: $[\![PQ]\!]^{\mathrm{s}}\{[\![N]\!]^{\mathrm{s}}/z\} \triangleq \mu\alpha.\langle [\![P]\!]^{\mathrm{s}} \,|\, \tilde{\mu}x.\langle [\![Q]\!]^{\mathrm{s}} \,|\, \tilde{\mu}y.\langle x | y{\cdot}\alpha \rangle\rangle\rangle\{[\![N]\!]^{\mathrm{s}}/z\} \triangleq$

$$\begin{array}{ll} \mu\alpha.\langle [\![P]\!]^{\mathrm{s}}\{[\![N]\!]^{\mathrm{s}}/z\} \,|\, \tilde{\mu}x.\langle [\![Q]\!]^{\mathrm{s}}\{[\![N]\!]^{\mathrm{s}}/z\} \,|\, \tilde{\mu}y.\langle x | y{\cdot}\alpha \rangle\rangle\rangle & = (ih) \\ \mu\alpha.\langle [\![P\{N/z\}]\!]^{\mathrm{s}} \,|\, \tilde{\mu}x.\langle [\![Q\{N/z\}]\!]^{\mathrm{s}} \,|\, \tilde{\mu}y.\langle x | y{\cdot}\alpha \rangle\rangle\rangle & \triangleq \\ [\![(P\{N/z\})(Q\{N/z\})]\!]^{\mathrm{s}} \triangleq [\![(PQ)\{N/z\}]\!]^{\mathrm{s}} & \end{array}$$

$(M = \mu\beta.[\gamma]P)$: $[\![\mu\beta.[\gamma]P]\!]^{s}\{[\![N]\!]^{s}/z\} \stackrel{\Delta}{=} \mu\beta.\langle [\![P]\!]^{s}|\gamma\rangle\{[\![N]\!]^{s}/z\} \stackrel{\Delta}{=} \mu\beta.\langle [\![P]\!]^{s}\{[\![N]\!]^{s}/z\}\,|\,\gamma\rangle = (ih)$
$\mu\beta.\langle [\![P\{N/z\}]\!]^{s}\,|\,\gamma\rangle \stackrel{\Delta}{=} [\![\mu\beta.[\gamma]P\{N/z\}]\!]^{s} \stackrel{\Delta}{=} [\![(\mu\beta.[\gamma]P)\{N/z\}]\!]^{s}$ □

Reduction does not play a role in this result.

Theorem 5.8 $[\![(\lambda z.M)N]\!]^{s} \rightarrow^{+}_{\overline{\lambda}} [\![M\{N/z\}]\!]^{s}$.

Proof: $[\![(\lambda z.M)N]\!]^{s} \stackrel{\Delta}{=} \mu\alpha.\langle \lambda z.[\![M]\!]^{s}\,|\,\tilde{\mu}x.\langle [\![N]\!]^{s}\,|\,\tilde{\mu}y.\langle x|y{\cdot}\alpha\rangle\rangle\rangle \rightarrow_{\overline{\lambda}} (x)$
$\mu\alpha.\langle [\![N]\!]^{s}\,|\,\tilde{\mu}y.\langle \lambda z.[\![M]\!]^{s}|y{\cdot}\alpha\rangle\rangle \rightarrow_{\overline{\lambda}} (y)$ $\mu\alpha.\langle \lambda x.[\![M]\!]^{s}\,|\,[\![N]\!]^{s}{\cdot}\alpha\rangle \rightarrow_{\overline{\lambda}} (\lambda)$
$\mu\alpha.\langle [\![N]\!]^{s}\,|\,\tilde{\mu}z.\langle [\![M]\!]^{s}|\alpha\rangle\rangle \rightarrow_{\overline{\lambda}} (z)$ $\mu\alpha.\langle [\![M]\!]^{s}\{[\![N]\!]^{s}/z\}\,|\,\alpha\rangle = (5.7)$
$\mu\alpha.\langle [\![M\{N/z\}]\!]^{s}\,|\,\alpha\rangle \rightarrow_{\overline{\lambda}} (\eta\mu)$ $[\![M\{N/z\}]\!]^{s}$ □

When restricting to CBV, $[\![N]\!]^{s}$ and $\lambda z.[\![M]\!]^{s}$ are both values, so the $(\tilde{\mu})$ contractions over x, y and z would be permitted. There are no CBN considerations here. There are five reduction steps involved here, so the simulation of this single reduction step requires multiple steps in the image.

For right-structural reduction, the situation is slightly more complicated, in that we cannot show that, for example,

$$[\![(\mu\delta.[\delta]M)N]\!]^{s} \rightarrow^{*}_{\overline{\lambda}} [\![\mu\gamma.[\gamma](M\{N{\cdot}\gamma/\delta\})N]\!]^{s}.$$

For that, we would like to show (something like) $[\![M]\!]^{s}\{[\![N]\!]^{s}{\cdot}\gamma/\delta\} \rightarrow^{*}_{\overline{\lambda}} [\![M\{N{\cdot}\gamma/\delta\}]\!]^{s}$, as in the proof above (step 5.7) but cannot: in fact, the only relation we can show is with the components switched (so reduction takes place in the opposite direction) as is shown in the following lemma. This is directly related to the fact that the two μ abstractions are fundamentally different: in $\lambda\mu$, μ-reduction reconstructs the μ-abstraction, whereas in $\overline{\lambda}\mu\tilde{\mu}$ it disappears.

The proofs for the preservation of right-structural reduction come in two parts. First, in Thm. 5.10, we will show that the interpretation preserves full reduction and CBV, using Lem. 5.9; these proofs will make (μ)-reduction steps that are not allowed in CBN. This is followed by Thm. 5.12, where we show the result for CBN, using Lem. 5.11; these proofs will make $(\tilde{\mu})$-reduction steps that are not allowed in CBV.

In the following two results, we will write $[\![N{\cdot}\gamma]\!]^{s}$ for $\tilde{\mu}x.\langle [\![N]\!]^{s}\,|\,\tilde{\mu}y.\langle x|y{\cdot}\gamma\rangle\rangle$ for reasons of readability; notice that then $[\![PQ]\!]^{s} = \mu\alpha.\langle [\![P]\!]^{s}\,|\,[\![Q{\cdot}\alpha]\!]^{s}\rangle$.

Lemma 5.9 $[\![M\{N{\cdot}\gamma/\delta\}]\!]^{s} \rightarrow^{*}_{\overline{\lambda}} [\![M]\!]^{s}\{[\![N{\cdot}\gamma]\!]^{s}/\delta\}$.

Proof: $(M = z)$: $[\![z\{N{\cdot}\gamma/\delta\}]\!]^{s} = [\![z]\!]^{s} \stackrel{\Delta}{=} z = z\{[\![N{\cdot}\gamma]\!]^{s}/\delta\} \stackrel{\Delta}{=} [\![z]\!]^{s}\{[\![N{\cdot}\gamma]\!]^{s}/\delta\}$

$(M = \lambda z.P)$: $[\![(\lambda z.P)\{N{\cdot}\gamma/\delta\}]\!]^{s} = [\![\lambda z.P\{N{\cdot}\gamma/\delta\}]\!]^{s} \stackrel{\Delta}{=} \lambda z.[\![P\{N{\cdot}\gamma/\delta\}]\!]^{s} \rightarrow^{*}_{\overline{\lambda}} (ih)$
$\lambda z.[\![P]\!]^{s}\{[\![N{\cdot}\gamma]\!]^{s}/\delta\} = (\lambda z.[\![P]\!]^{s})\{[\![N{\cdot}\gamma]\!]^{s}/\delta\} \stackrel{\Delta}{=} [\![\lambda z.P]\!]^{s}\{[\![N{\cdot}\gamma]\!]^{s}/\delta\}$

$(M = PQ)$: $[\![(PQ)\{N{\cdot}\gamma/\delta\}]\!]^{s} = [\![(P\{N{\cdot}\gamma/\delta\})\,(Q\{N{\cdot}\gamma/\delta\})]\!]^{s} \stackrel{\Delta}{=}$
$\mu\alpha.\langle [\![P\{N{\cdot}\gamma/\delta\}]\!]^{s}\,|\,\tilde{\mu}x.\langle [\![Q\{N{\cdot}\gamma/\delta\}]\!]^{s}\,|\,\tilde{\mu}y.\langle x|y{\cdot}\alpha\rangle\rangle\rangle \rightarrow^{*}_{\overline{\lambda}} (ih)$
$\mu\alpha.\langle [\![P]\!]^{s}\{[\![N{\cdot}\gamma]\!]^{s}/\delta\}\,|\,\tilde{\mu}x.\langle [\![Q]\!]^{s}\{[\![N{\cdot}\gamma]\!]^{s}/\delta\}\,|\,\tilde{\mu}y.\langle x|y{\cdot}\alpha\rangle\rangle\rangle =$
$\mu\alpha.\langle [\![P]\!]^{s}\,|\,\tilde{\mu}x.\langle [\![Q]\!]^{s}\,|\,\tilde{\mu}y.\langle x|y{\cdot}\alpha\rangle\rangle\rangle\{[\![N{\cdot}\gamma]\!]^{s}/\delta\} \stackrel{\Delta}{=} [\![PQ]\!]^{s}\{[\![N{\cdot}\gamma]\!]^{s}/\delta\}$

$(M = \mu\beta.[\tau]P,\ \tau \neq \delta)$: $[\![(\mu\beta.[\tau]P)\{N{\cdot}\gamma/\delta\}]\!]^{s} = [\![\mu\beta.[\tau]P\{N{\cdot}\gamma/\delta\}]\!]^{s} \stackrel{\Delta}{=}$
$\mu\beta.\langle [\![P\{N{\cdot}\gamma/\delta\}]\!]^{s}\,|\,\tau\rangle \rightarrow^{*}_{\overline{\lambda}} (ih)$ $\mu\beta.\langle [\![P]\!]^{s}\{[\![N{\cdot}\gamma]\!]^{s}/\delta\}\,|\,\tau\rangle = \mu\beta.\langle [\![P]\!]^{s}|\tau\rangle\{[\![N{\cdot}\gamma]\!]^{s}/\delta\} \stackrel{\Delta}{=}$
$[\![\mu\beta.[\tau]P]\!]^{s}\{[\![N{\cdot}\gamma]\!]^{s}/\delta\}$

$(M = \mu\beta.[\delta]P)$: $[\![(\mu\beta.[\delta]P)\{N{\cdot}\gamma/\delta\}]\!]^{s} \stackrel{\Delta}{=} [\![\mu\beta.[\gamma](P\{N{\cdot}\gamma/\delta\})N]\!]^{s} \stackrel{\Delta}{=}$
$\mu\beta.\langle \mu\alpha.\langle [\![P\{N{\cdot}\gamma/\delta\}]\!]^{s}\,|\,[\![N{\cdot}\alpha]\!]^{s}\rangle\,|\,\gamma\rangle \rightarrow_{\overline{\lambda}} (\alpha)$ $\mu\beta.\langle [\![P\{N{\cdot}\gamma/\delta\}]\!]^{s}\,|\,[\![N{\cdot}\gamma]\!]^{s}\rangle \rightarrow^{*}_{\overline{\lambda}} (ih)$
$\mu\beta.\langle [\![P]\!]^{s}\{[\![N{\cdot}\gamma]\!]^{s}/\delta\}\,|\,[\![N{\cdot}\gamma]\!]^{s}\rangle = \mu\beta.\langle [\![P]\!]^{s}|\delta\rangle\{[\![N{\cdot}\gamma]\!]^{s}/\delta\} \stackrel{\Delta}{=}$
$[\![\mu\beta.[\delta]P]\!]^{s}\{[\![N{\cdot}\gamma]\!]^{s}/\delta\}$ □

So modelling right substitution could require reduction. The only contraction takes place for

$\llbracket \mu\beta.[\delta]P \rrbracket^{s}$, where a (μ)-step takes place. Since induction gets applied twice when M is an application, the reduction in general requires more than one step.

We write $M \downarrow_{\overline{\lambda}} N$ if there exists P such that $M \rightarrow^{*}_{\overline{\lambda}} P$ and $N \rightarrow^{*}_{\overline{\lambda}} P$.

Theorem 5.10 *i)* $\llbracket (\mu\delta.[\beta]M)N \rrbracket^{s} \downarrow_{\overline{\lambda}} \llbracket \mu\gamma.[\beta]M\{N\cdot\gamma/\delta\} \rrbracket^{s}$, $\beta \neq \delta$.
ii) $\llbracket (\mu\delta.[\delta]M)N \rrbracket^{s} \downarrow_{\overline{\lambda}} \llbracket \mu\gamma.[\gamma](M\{v\cdot\gamma/\delta\})N \rrbracket^{s}$.

Proof: *i*)
$$\llbracket (\mu\delta.[\beta]M)N \rrbracket^{s} \triangleq \mu\gamma.\langle \mu\delta.\langle \llbracket M \rrbracket^{s} | \beta\rangle \,|\, \llbracket N\cdot\gamma \rrbracket^{s}\rangle \rightarrow_{\overline{\lambda}} (\delta)$$
$$\mu\gamma.\langle \llbracket M \rrbracket^{s}\{\llbracket N\cdot\gamma \rrbracket^{s}/\delta\} \,|\, \beta\rangle \; {}^{*}_{\overline{\lambda}}\!\leftarrow (5.9)\; \mu\gamma.\langle \llbracket M\{N\cdot\gamma/\delta\} \rrbracket^{s} | \beta\rangle \triangleq \llbracket \mu\gamma.[\beta]M\{N\cdot\gamma/\delta\} \rrbracket^{s}$$

ii)
$$\llbracket (\mu\delta.[\delta]M)N \rrbracket^{s} \triangleq \mu\gamma.\langle \mu\delta.\langle \llbracket M \rrbracket^{s} | \delta\rangle \,|\, \llbracket N\cdot\gamma \rrbracket^{s}\rangle \rightarrow_{\overline{\lambda}} (\delta)$$
$$\mu\gamma.\langle \llbracket M \rrbracket^{s}\{\llbracket N\cdot\gamma \rrbracket^{s}/\delta\} \,|\, \llbracket N\cdot\gamma \rrbracket^{s}\rangle \; {}^{*}_{\overline{\lambda}}\!\leftarrow (5.9)\; \mu\gamma.\langle \llbracket M\{N\cdot\gamma/\delta\} \rrbracket \,|\, \llbracket N\cdot\gamma \rrbracket^{s}\rangle \; {}^{\mathrm{N}}_{\overline{\lambda}}\!\leftarrow (\alpha)$$
$$\mu\gamma.\langle \mu\alpha.\langle \llbracket M\{N\cdot\gamma/\delta\} \rrbracket^{s} \,|\, \llbracket N\cdot\alpha \rrbracket^{s}\rangle \,|\, \gamma\rangle \triangleq \llbracket \mu\gamma.[\gamma](M\{N\cdot\gamma/\delta\})N \rrbracket^{s} \qquad \square$$

The (μ)-contractions over δ pull in $\llbracket N\cdot\gamma \rrbracket^{s} = \tilde{\mu}x.\langle \llbracket N \rrbracket^{s} \,|\, \tilde{\mu}y.\langle x | y\cdot\gamma\rangle\rangle$, a $\tilde{\mu}$-term, so this step would not be allowed in CBN; reduction takes place in two directions. There are no CBV considerations here.

We will now show that the interpretation also respects (μ_{R})-reduction under CBN. First we show that result for right substitution.

Lemma 5.11 $\llbracket M\{N\cdot\gamma/\delta\} \rrbracket^{s} \rightarrow^{\mathrm{N}*}_{\overline{\lambda}} \llbracket M \rrbracket^{s}\{\llbracket N \rrbracket^{s}\cdot\gamma/\delta\}$.

Proof: $(M = z)$: $\llbracket z\{N\cdot\gamma/\delta\} \rrbracket^{s} \triangleq \llbracket z \rrbracket^{s} \triangleq \llbracket z \rrbracket^{s}\{\llbracket N \rrbracket^{s}\cdot\gamma/\delta\}$

$(M = \lambda z.P)$:
$$\llbracket (\lambda z.P)\{N\cdot\gamma/\delta\} \rrbracket^{s} \triangleq \lambda z.\llbracket P\{N\cdot\gamma/\delta\} \rrbracket^{s} \rightarrow^{\mathrm{N}*}_{\overline{\lambda}} (ih)\; \lambda z.\llbracket P \rrbracket^{s}\{\llbracket N \rrbracket^{s}\cdot\gamma/\delta\} =$$
$$(\lambda z.\llbracket P \rrbracket^{s})\{\llbracket N \rrbracket^{s}\cdot\gamma/\delta\} \triangleq \llbracket \lambda z.P \rrbracket^{s}\{\llbracket N \rrbracket^{s}\cdot\gamma/\delta\}$$

$(M = PQ)$:
$$\llbracket (PQ)\{N\cdot\gamma/\delta\} \rrbracket^{s} \triangleq \mu\alpha.\langle \llbracket P\{N\cdot\gamma/\delta\} \rrbracket^{s} \,|\, \tilde{\mu}x.\langle \llbracket Q\{N\cdot\gamma/\delta\} \rrbracket^{s} \,|\, \tilde{\mu}y.\langle x | y\cdot\alpha\rangle\rangle\rangle \rightarrow^{\mathrm{N}*}_{\overline{\lambda}} (ih)$$
$$\mu\alpha.\langle \llbracket P \rrbracket^{s}\{\llbracket N \rrbracket^{s}\cdot\gamma/\delta\} \,|\, \tilde{\mu}x.\langle \llbracket Q \rrbracket^{s}\{\llbracket N \rrbracket^{s}\cdot\gamma/\delta\} \,|\, \tilde{\mu}y.\langle x | y\cdot\alpha\rangle\rangle\rangle =$$
$$\mu\alpha.\langle \llbracket P \rrbracket^{s} \,|\, \tilde{\mu}x.\langle \llbracket Q \rrbracket^{s} \,|\, \tilde{\mu}y.\langle x | y\cdot\alpha\rangle\rangle\rangle\{\llbracket N \rrbracket^{s}\cdot\gamma/\delta\} \triangleq \llbracket PQ \rrbracket^{s}\{\llbracket N \rrbracket^{s}\cdot\gamma/\delta\}$$

$(M = \mu\beta.[\tau]P,\ \tau \neq \delta)$:
$$\llbracket (\mu\beta.[\tau]P)\{N\cdot\gamma/\delta\} \rrbracket^{s} = \llbracket \mu\beta.[\tau]P\{N\cdot\gamma/\delta\} \rrbracket^{s} \triangleq$$
$$\mu\beta.\langle \llbracket P\{N\cdot\gamma/\delta\} \rrbracket^{s} | \tau\rangle \rightarrow^{\mathrm{N}*}_{\overline{\lambda}} (ih)\; \mu\beta.\langle \llbracket P \rrbracket^{s}\{\llbracket N \rrbracket^{s}\cdot\gamma/\delta\} \,|\, \tau\rangle = \mu\beta.\langle \llbracket P \rrbracket^{s} | \tau\rangle\{\llbracket N \rrbracket^{s}\cdot\gamma/\delta\} \triangleq$$
$$\llbracket \mu\beta.[\tau]P \rrbracket^{s}\{\llbracket N \rrbracket^{s}\cdot\gamma/\delta\}$$

$(M = \mu\beta.[\delta]P)$:
$$\llbracket (\mu\beta.[\delta]P)\{N\cdot\gamma/\delta\} \rrbracket^{s} = \llbracket \mu\beta.[\gamma](P\{N\cdot\gamma/\delta\})N \rrbracket^{s} \triangleq$$
$$\mu\beta.\langle \mu\alpha.\langle \llbracket P\{N\cdot\gamma/\delta\} \rrbracket^{s} \,|\, \tilde{\mu}x.\langle \llbracket N \rrbracket^{s} \,|\, \tilde{\mu}y.\langle x | y\cdot\alpha\rangle\rangle\rangle \,|\, \gamma\rangle \rightarrow^{\mathrm{N}}_{\overline{\lambda}} (\alpha)$$
$$\mu\beta.\langle \llbracket P\{N\cdot\gamma/\delta\} \rrbracket^{s} \,|\, \tilde{\mu}x.\langle \llbracket N \rrbracket^{s} \,|\, \tilde{\mu}y.\langle x | y\cdot\gamma\rangle\rangle\rangle \rightarrow^{\mathrm{N}}_{\overline{\lambda}} (x)\; \mu\beta.\langle \llbracket N \rrbracket^{s} \,|\, \tilde{\mu}y.\langle \llbracket P\{N\cdot\gamma/\delta\} \rrbracket^{s} | y\cdot\gamma\rangle\rangle \rightarrow^{\mathrm{N}}_{\overline{\lambda}} (y)$$
$$\mu\beta.\langle \llbracket P\{N\cdot\gamma/\delta\} \rrbracket^{s} \,|\, \llbracket N \rrbracket^{s}\cdot\gamma\rangle \rightarrow^{\mathrm{N}*}_{\overline{\lambda}} (ih)\; \mu\beta.\langle \llbracket P \rrbracket^{s}\{\llbracket N \rrbracket^{s}\cdot\gamma/\delta\} \,|\, \llbracket N \rrbracket^{s}\cdot\gamma\rangle \triangleq$$
$$\mu\beta.\langle \llbracket P \rrbracket^{s} | \delta\rangle\{\llbracket N \rrbracket^{s}\cdot\gamma/\delta\} \triangleq \llbracket \mu\beta.[\delta]P \rrbracket^{s}\{\llbracket N \rrbracket^{s}\cdot\gamma/\delta\} \qquad \square$$

Notice that the only contractions are in the last part: one over α, a (μ)-step that pulls in γ, a stack which is a CBN reduction step. The $(\tilde{\mu})$-step over x pulls in $\llbracket P\{N\cdot\gamma/\delta\} \rrbracket^{s}$, which need not be a value when restricting to CBV, so this proof would not work for CBV. In CBN, $(\tilde{\mu})$-contractions are unrestricted.

With this result we can show that our encoding deals with CBN μ_{R}-reduction through equality.

Theorem 5.12 *i)* $\llbracket (\mu\delta.[\beta]M)N \rrbracket^{s} \downarrow^{\mathrm{N}}_{\overline{\lambda}} \llbracket \mu\gamma.[\beta]M\{N\cdot\gamma/\delta\} \rrbracket^{s}$, $\beta \neq \delta$.
ii) $\llbracket (\mu\delta.[\delta]M)N \rrbracket^{s} \downarrow^{\mathrm{N}}_{\overline{\lambda}} \llbracket \mu\gamma.[\gamma](M\{N\cdot\gamma/\delta\})N \rrbracket^{s}$.

Proof: *i*)

$$
\begin{array}{l}
[\![(\mu\delta.[\beta]M)N]\!]^{\mathrm{s}} \stackrel{\Delta}{=} \mu\gamma.\langle \mu\delta.\langle [\![M]\!]^{\mathrm{s}} | \beta\rangle \,|\, \tilde{\mu}x.\langle [\![N]\!]^{\mathrm{s}} \,|\, \tilde{\mu}y.\langle x | y\cdot\gamma\rangle\rangle\rangle \rightarrow_{\overline{\lambda}}^{\mathbf{N}} (x) \\
\mu\gamma.\langle [\![N]\!]^{\mathrm{s}} \,|\, \tilde{\mu}y.\langle \mu\delta.\langle [\![M]\!]^{\mathrm{s}} | \beta\rangle \,|\, y\cdot\gamma\rangle\rangle \rightarrow_{\overline{\lambda}}^{\mathbf{N}} (y)\ \mu\gamma.\langle \mu\delta.\langle [\![M]\!]^{\mathrm{s}} | \beta\rangle \,|\, [\![N]\!]^{\mathrm{s}}\cdot\gamma\rangle \rightarrow_{\overline{\lambda}}^{\mathbf{N}} (\delta) \\
\mu\gamma.\langle [\![M]\!]^{\mathrm{s}}\{[\![N]\!]^{\mathrm{s}}\cdot\gamma/\delta\} \,|\, \beta\rangle \ {}_{\overline{\lambda}}^{\mathbf{N}*}\!\!\leftarrow (5.11)\ \mu\gamma.\langle [\![M\{N\cdot\gamma/\delta\}]\!]^{\mathrm{s}} \,|\, \beta\rangle \stackrel{\Delta}{=} \\
[\![\mu\gamma.[\beta]M\{N\cdot\gamma/\delta\}]\!]^{\mathrm{s}}
\end{array}
$$

ii)

$$
\begin{array}{l}
[\![(\mu\delta.[\delta]M)N]\!]^{\mathrm{s}} \stackrel{\Delta}{=} \mu\alpha.\langle \mu\delta.\langle [\![M]\!]^{\mathrm{s}} | \delta\rangle \,|\, \tilde{\mu}x.\langle [\![N]\!]^{\mathrm{s}} \,|\, \tilde{\mu}y.\langle x | y\cdot\alpha\rangle\rangle\rangle \rightarrow_{\overline{\lambda}}^{\mathbf{N}} (x) \\
\mu\gamma.\langle [\![N]\!]^{\mathrm{s}} \,|\, \tilde{\mu}y.\langle \mu\delta.\langle [\![M]\!]^{\mathrm{s}} | \delta\rangle \,|\, y\cdot\gamma\rangle\rangle \rightarrow_{\overline{\lambda}}^{\mathbf{N}} (y)\ \mu\gamma.\langle \mu\delta.\langle [\![M]\!]^{\mathrm{s}} | \delta\rangle \,|\, [\![N]\!]^{\mathrm{s}}\cdot\gamma\rangle \rightarrow_{\overline{\lambda}}^{\mathbf{N}} (\delta) \\
\mu\gamma.\langle [\![M]\!]^{\mathrm{s}}\{[\![N]\!]^{\mathrm{s}}\cdot\gamma/\delta\} \,|\, [\![N]\!]^{\mathrm{s}}\cdot\gamma\rangle \ {}_{\overline{\lambda}}^{\mathbf{N}*}\!\!\leftarrow (5.11)\ \mu\gamma.\langle [\![M\{N\cdot\gamma/\delta\}]\!]^{\mathrm{s}} \,|\, [\![N]\!]^{\mathrm{s}}\cdot\gamma\rangle \ {}_{\overline{\lambda}}^{\mathbf{N}}\!\!\leftarrow (y) \\
\mu\gamma.\langle [\![N]\!]^{\mathrm{s}} \,|\, \tilde{\mu}y.\langle [\![M\{N\cdot\gamma/\delta\}]\!]^{\mathrm{s}} \,|\, y\cdot\gamma\rangle\rangle \ {}_{\overline{\lambda}}^{\mathbf{N}}\!\!\leftarrow (\alpha)\ \mu\gamma.\langle \mu\alpha.\langle [\![N]\!]^{\mathrm{s}} \,|\, \tilde{\mu}y.\langle [\![M\{N\cdot\gamma/\delta\}]\!]^{\mathrm{s}} \,|\, y\cdot\alpha\rangle\rangle \,|\, \gamma\rangle \ {}_{\overline{\lambda}}^{\mathbf{N}}\!\!\leftarrow (x) \\
\mu\gamma.\langle \mu\alpha.\langle [\![M\{N\cdot\gamma/\delta\}]\!]^{\mathrm{s}} \,|\, \tilde{\mu}x.\langle [\![N]\!]^{\mathrm{s}} \,|\, \tilde{\mu}y.\langle x | y\cdot\alpha\rangle\rangle\rangle \,|\, \gamma\rangle \stackrel{\Delta}{=} [\![\mu\gamma.[\gamma](M\{N\cdot\gamma/\delta\})N]\!]^{\mathrm{s}} \qquad \square
\end{array}
$$

Notice that the (μ)-contractions are over δ and α and pull in $[\![N]\!]^{\mathrm{s}}\cdot\gamma$ and γ, respectively, which are stacks, so these steps are allowed in CBN. Moreover, the $(\tilde{\mu})$-contraction over x pulls in $\mu\delta.\langle [\![M]\!]^{\mathrm{s}} | \delta\rangle$ which is not a value and $[\![M\{N\cdot\gamma/\delta\}]\!]^{\mathrm{s}}$, which need not be a value, so this proof does not hold for CBV.

The main difference between this proof and that for Thm. 5.10 is that in CBV, we can contract the (μ)-redex in

$$\mu\gamma.\langle \mu\delta.\langle [\![M]\!]^{\mathrm{s}} | \delta\rangle \,|\, \tilde{\mu}x.\langle [\![N]\!]^{\mathrm{s}} \,|\, \tilde{\mu}y.\langle x | y\cdot\gamma\rangle\rangle\rangle$$

over δ immediately, by pulling in the environment $\tilde{\mu}x.\langle [\![N]\!]^{\mathrm{s}} \,|\, \tilde{\mu}y.\langle x | y\cdot\gamma\rangle\rangle$. In CBN, this is not allowed since the latter term is not a stack; we therefore need to first contract the $(\tilde{\mu})$-redexes on x and y, creating the term $\mu\gamma.\langle \mu\delta.\langle [\![M]\!]^{\mathrm{s}} | \beta\rangle \,|\, [\![N]\!]^{\mathrm{s}}\cdot\gamma\rangle$ where now the (μ)-redex over δ can be contracted. But this is not allowed in CBV, since the $(\tilde{\mu})$-contraction over x pulls in $\mu\delta.\langle [\![M]\!]^{\mathrm{s}} | \delta\rangle$, which is not a value.

The following lemma shows that we can implement left-structural substitution in $\overline{\lambda}\mu\tilde{\mu}$ without extending the reduction relation. In the following two results, we will now write $[\![N\cdot\gamma]\!]^{\mathrm{s}}$ for $\tilde{\mu}x.\langle [\![N]\!]^{\mathrm{s}} | x\cdot\gamma\rangle$.

Lemma 5.13 $[\![\{N\cdot\gamma/\delta\}M]\!]^{\mathrm{s}} \rightarrow_{\overline{\lambda}}^{*} [\![M]\!]^{\mathrm{s}}\{\tilde{\mu}y.\langle [\![N]\!]^{\mathrm{s}} | y\cdot\gamma\rangle/\delta\} \rightarrow_{\overline{\lambda}}^{*} [\![M]\!]^{\mathrm{s}}\{[\![N\cdot\gamma]\!]^{\mathrm{s}}/\delta\}$.

Proof: By induction on the structure of terms.

$(M = x)$: $[\![\{N\cdot\gamma/\delta\}x]\!]^{\mathrm{s}} = [\![x]\!]^{\mathrm{s}} \stackrel{\Delta}{=} x = x\{[\![N\cdot\gamma]\!]^{\mathrm{s}}/\delta\} \stackrel{\Delta}{=} [\![x]\!]^{\mathrm{s}}\{[\![N\cdot\gamma]\!]^{\mathrm{s}}/\delta\}$

$(M = \lambda x.P)$:

$$
\begin{array}{l}
[\![\{N\cdot\gamma/\delta\}(\lambda x.P)]\!]^{\mathrm{s}} = [\![\lambda x.\{N\cdot\gamma/\delta\}P]\!]^{\mathrm{s}} \stackrel{\Delta}{=} \lambda x.[\![\{N\cdot\gamma/\delta\}P]\!]^{\mathrm{s}} \rightarrow_{\overline{\lambda}}^{*} (ih) \\
\lambda x.[\![P]\!]^{\mathrm{s}}\{[\![N\cdot\gamma]\!]^{\mathrm{s}}/\delta\} = (\lambda x.[\![P]\!]^{\mathrm{s}})\{[\![N\cdot\gamma]\!]^{\mathrm{s}}/\delta\} \stackrel{\Delta}{=} [\![\lambda x.P]\!]^{\mathrm{s}}\{[\![N\cdot\gamma]\!]^{\mathrm{s}}/\delta\}
\end{array}
$$

$(M = PQ)$:

$$
\begin{array}{l}
[\![\{N\cdot\gamma/\delta\}(PQ)]\!]^{\mathrm{s}} = [\![(\{N\cdot\gamma/\delta\}P)\,(\{N\cdot\gamma/\delta\}Q)]\!]^{\mathrm{s}} \stackrel{\Delta}{=} \\
\mu\alpha.\langle [\![\{N\cdot\gamma/\delta\}P]\!]^{\mathrm{s}} \,|\, \tilde{\mu}x.\langle [\![\{N\cdot\gamma/\delta\}Q]\!]^{\mathrm{s}} \,|\, \tilde{\mu}y.\langle x | y\cdot\alpha\rangle\rangle\rangle \rightarrow_{\overline{\lambda}}^{*} (ih) \\
\mu\alpha.\langle [\![P]\!]^{\mathrm{s}}\{[\![N\cdot\gamma]\!]^{\mathrm{s}}/\delta\} \,|\, \tilde{\mu}x.\langle [\![Q]\!]^{\mathrm{s}}\{[\![N\cdot\gamma]\!]^{\mathrm{s}}/\delta\} \,|\, \tilde{\mu}y.\langle x | y\cdot\alpha\rangle\rangle\rangle = \\
\mu\alpha.\langle [\![P]\!]^{\mathrm{s}} \,|\, \tilde{\mu}x.\langle [\![Q]\!]^{\mathrm{s}} \,|\, \tilde{\mu}y.\langle x | y\cdot\alpha\rangle\rangle\rangle\{[\![N\cdot\gamma]\!]^{\mathrm{s}}/\delta\} \stackrel{\Delta}{=} [\![PQ]\!]^{\mathrm{s}}\{[\![N\cdot\gamma]\!]^{\mathrm{s}}/\delta\}
\end{array}
$$

$(M = \mu\alpha.[\beta]P,\ \delta \neq \beta)$:

$$
\begin{array}{l}
[\![\{N\cdot\gamma/\delta\}(\mu\alpha.[\beta]P)]\!]^{\mathrm{s}} = [\![\mu\alpha.[\beta]\{N\cdot\gamma/\delta\}P]\!]^{\mathrm{s}} \stackrel{\Delta}{=} \\
\mu\alpha.\langle [\![\{N\cdot\gamma/\delta\}P]\!]^{\mathrm{s}} | \beta\rangle \rightarrow_{\overline{\lambda}}^{*} (ih)\ \mu\alpha.\langle [\![P]\!]^{\mathrm{s}}\{[\![N\cdot\gamma]\!]^{\mathrm{s}}/\delta\} \,|\, \beta\rangle = \mu\alpha.\langle [\![P]\!]^{\mathrm{s}} | \beta\rangle\{[\![N\cdot\gamma]\!]^{\mathrm{s}}/\delta\} \stackrel{\Delta}{=} \\
[\![\mu\alpha.[\beta]P]\!]^{\mathrm{s}}\{[\![N\cdot\gamma]\!]^{\mathrm{s}}/\delta\}
\end{array}
$$

$(M = \mu\alpha.[\delta]P)$:

$$
\begin{array}{l}
[\![\{N\cdot\gamma/\delta\}(\mu\alpha.[\delta]P)]\!]^{\mathrm{s}} \stackrel{\Delta}{=} [\![\mu\alpha.[\gamma]N\,(\{N\cdot\gamma/\delta\}P)]\!]^{\mathrm{s}} \stackrel{\Delta}{=} \\
\mu\alpha.\langle \mu\beta.\langle [\![N]\!]^{\mathrm{s}} \,|\, \tilde{\mu}x.\langle [\![\{N\cdot\gamma/\delta\}P]\!]^{\mathrm{s}} \,|\, \tilde{\mu}y.\langle x | y\cdot\beta\rangle\rangle\rangle \,|\, \gamma\rangle \rightarrow_{\overline{\lambda}} (\beta) \\
\mu\alpha.\langle [\![N]\!]^{\mathrm{s}} \,|\, \tilde{\mu}x.\langle [\![\{N\cdot\gamma/\delta\}P]\!]^{\mathrm{s}} \,|\, \tilde{\mu}y.\langle x | y\cdot\gamma\rangle\rangle\rangle \rightarrow_{\overline{\lambda}} (x)\ \mu\alpha.\langle [\![\{N\cdot\gamma/\delta\}P]\!]^{\mathrm{s}} \,|\, [\![N\cdot\gamma]\!]^{\mathrm{s}}\rangle \rightarrow_{\overline{\lambda}}^{*} (ih) \\
\mu\alpha.\langle [\![P]\!]^{\mathrm{s}}\{[\![N\cdot\gamma]\!]^{\mathrm{s}}/\delta\} \,|\, [\![N\cdot\gamma]\!]^{\mathrm{s}}\rangle = \mu\alpha.\langle [\![P]\!]^{\mathrm{s}} | \delta\rangle\{[\![N\cdot\gamma]\!]^{\mathrm{s}}/\delta\} \stackrel{\Delta}{=} \\
[\![\mu\alpha.[\delta]P]\!]^{\mathrm{s}}\{[\![N\cdot\gamma]\!]^{\mathrm{s}}/\delta\} \qquad \square
\end{array}
$$

The $(\tilde{\mu})$-step over x pulls in $[\![N]\!]^{\mathrm{s}}$, which would be a value in CBV; the (μ)-step over β pulls in γ, so would be allowed in CBN.

With this result, we can now show that $\lambda\mu$'s reduction rule (μ_{L}) can be simulated in $\overline{\lambda}\mu\tilde{\mu}$.

Theorem 5.14 *i)* $[\![N\,(\mu\alpha.[\delta]M)]\!]^{\mathrm{s}} \downarrow_{\overline{\lambda}} [\![\mu\gamma.[\delta]\{N{\cdot}\gamma/\alpha\}M]\!]^{\mathrm{s}}$*, with* $\alpha \neq \delta$.
ii) $[\![N\,(\mu\alpha.[\alpha]M)]\!]^{\mathrm{s}} \downarrow_{\overline{\lambda}} [\![\mu\gamma.[\gamma]N\,(\{N{\cdot}\gamma/\alpha\}M)]\!]^{\mathrm{s}}$.

Proof: *i*) $[\![N\,(\mu\alpha.[\delta]M)]\!]^{\mathrm{s}} \stackrel{\Delta}{=} \mu\gamma.\langle [\![N]\!]^{\mathrm{s}} \,|\, \tilde{\mu}x.\langle \mu\alpha.\langle [\![M]\!]^{\mathrm{s}} | \delta\rangle \,|\, \tilde{\mu}y.\langle x | y{\cdot}\gamma\rangle\rangle\rangle \rightarrow_{\overline{\lambda}} (x)$
$\mu\gamma.\langle \mu\alpha.\langle [\![M]\!]^{\mathrm{s}} | \delta\rangle \,|\, \tilde{\mu}y.\langle [\![N]\!]^{\mathrm{s}} | y{\cdot}\gamma\rangle\rangle \rightarrow_{\overline{\lambda}} (\alpha)\ \mu\gamma.\langle [\![M]\!]^{\mathrm{s}}\{[\![N{\cdot}\gamma]\!]^{\mathrm{s}}/\alpha\} \,|\, \delta\rangle \quad {}_{\overline{\lambda}}^{*}\!\!\leftarrow (5.13)$
$\mu\gamma.\langle [\![\{N{\cdot}\gamma/\alpha\}M]\!]^{\mathrm{s}} \,|\, \delta\rangle \stackrel{\Delta}{=} [\![\mu\gamma.[\delta]\{N{\cdot}\gamma/\alpha\}M]\!]^{\mathrm{s}}$

ii) $[\![N\,(\mu\alpha.[\alpha]M)]\!]^{\mathrm{s}} \stackrel{\Delta}{=} \mu\gamma.\langle [\![N]\!]^{\mathrm{s}} \,|\, \tilde{\mu}x.\langle \mu\alpha.\langle [\![M]\!]^{\mathrm{s}} | \alpha\rangle \,|\, \tilde{\mu}y.\langle x | y{\cdot}\gamma\rangle\rangle\rangle \rightarrow_{\overline{\lambda}} (x)$
$\mu\gamma.\langle \mu\alpha.\langle [\![M]\!]^{\mathrm{s}} | \alpha\rangle \,|\, \tilde{\mu}y.\langle [\![N]\!]^{\mathrm{s}} | y{\cdot}\gamma\rangle\rangle \rightarrow_{\overline{\lambda}} (\alpha)\ \mu\gamma.\langle [\![M]\!]^{\mathrm{s}}\{[\![N{\cdot}\gamma]\!]^{\mathrm{s}}/\alpha\} \,|\, [\![N{\cdot}\gamma]\!]^{\mathrm{s}}\rangle \quad {}_{\overline{\lambda}}^{*}\!\!\leftarrow (5.13)$
$\mu\gamma.\langle [\![\{N{\cdot}\gamma/\alpha\}M]\!]^{\mathrm{s}} \,|\, [\![N{\cdot}\gamma]\!]^{\mathrm{s}}\rangle \ {}_{\overline{\lambda}}\!\!\leftarrow (x)\ \mu\gamma.\langle [\![N]\!]^{\mathrm{s}} \,|\, \tilde{\mu}x.\langle [\![\{N{\cdot}\gamma/\alpha\}M]\!]^{\mathrm{s}} \,|\, \tilde{\mu}y.\langle x | y{\cdot}\gamma\rangle\rangle\rangle \ {}_{\overline{\lambda}}\!\!\leftarrow (\delta)$
$\mu\gamma.\langle \mu\delta.\langle [\![N]\!]^{\mathrm{s}} \,|\, \tilde{\mu}x.\langle [\![\{N{\cdot}\gamma/\alpha\}M]\!]^{\mathrm{s}} \,|\, \tilde{\mu}y.\langle x | y{\cdot}\delta\rangle\rangle\rangle \,|\, \gamma\rangle \stackrel{\Delta}{=} [\![\mu\gamma.[\gamma]N(\{N{\cdot}\gamma/\alpha\}M)]\!]^{\mathrm{s}}$ □

The $(\tilde{\mu})$-reductions on x are permitted in CBV, since then N and $[\![N]\!]^{\mathrm{s}}$ will be values. The (μ) reduction over δ pulls in γ so would be permitted in CBN, but the reductions on α pull in a $\tilde{\mu}$-term $[\![N{\cdot}\gamma]\!]^{\mathrm{s}} = \tilde{\mu}x.\langle [\![N]\!]^{\mathrm{s}} | x{\cdot}\gamma\rangle$, which is not allowed in CBN, which shows that this interpretation is not suited for CBN. Fortunately, μ_{L} is not part of CBN reduction in s$\lambda\mu$, so this is not an issue.

We can now state our main result for our encoding.

Theorem 5.15 (Preservation of reduction) *If* $M \rightarrow_{\lambda\mu}^{\mathrm{s}*} N$*, then* $[\![M]\!]^{\mathrm{s}} \downarrow_{\overline{\lambda}} [\![N]\!]^{\mathrm{s}}$.

Proof: $((\lambda z.M)N \rightarrow_{\lambda\mu} M\{N/z\})$: By Theorem 5.8.
$((\mu\alpha.\mathrm{C})N \rightarrow_{\lambda\mu} \mu\gamma.\mathrm{C}\{N{\cdot}\gamma/\alpha\})$: By Theorem 5.10.
$(N\,(\mu\alpha.\mathrm{C}) \rightarrow_{\lambda\mu} \mu\gamma.\{N{\cdot}\gamma/\alpha\}\mathrm{C})$: By Theorem 5.14.
$(\mu\alpha.[\beta]\mu\gamma.[\delta]M \rightarrow \mu\alpha.[\delta]M\{\beta/\gamma\},\ \gamma \neq \delta)$: $[\![\mu\alpha.[\beta]\mu\gamma.[\delta]M]\!]^{\mathrm{s}} \stackrel{\Delta}{=} \mu\alpha.\langle \mu\gamma.\langle [\![M]\!]^{\mathrm{s}} | \delta\rangle \,|\, \beta\rangle \rightarrow_{\overline{\lambda}}$
$\mu\alpha.\langle [\![M]\!]^{\mathrm{s}}\{\beta/\gamma\} \,|\, \delta\rangle \stackrel{\Delta}{=} \mu\alpha.\langle [\![M\{\beta/\gamma\}]\!]^{\mathrm{s}} \,|\, \delta\rangle \stackrel{\Delta}{=} [\![\mu\alpha.[\delta]M\{\beta/\gamma\}]\!]^{\mathrm{s}}$
$(\mu\alpha.[\beta]\mu\gamma.[\gamma]M \rightarrow \mu\alpha.[\beta]M\{\beta/\gamma\})$: $[\![\mu\alpha.[\beta]\mu\gamma.[\gamma]M]\!]^{\mathrm{s}} \stackrel{\Delta}{=} \mu\alpha.\langle \mu\gamma.\langle [\![M]\!]^{\mathrm{s}} | \gamma\rangle \,|\, \beta\rangle \rightarrow_{\overline{\lambda}}$
$\mu\alpha.\langle [\![M]\!]^{\mathrm{s}}\{\beta/\gamma\} \,|\, \beta\rangle \stackrel{\Delta}{=} \mu\alpha.\langle [\![M\{\beta/\gamma\}]\!]^{\mathrm{s}} \,|\, \beta\rangle \stackrel{\Delta}{=} [\![\mu\alpha.[\beta]M\{\beta/\gamma\}]\!]^{\mathrm{s}}$
$(\mu\alpha.[\alpha]M \rightarrow_{\lambda\mu} M\ (\alpha \notin M))$: $[\![\mu\alpha.[\alpha]M]\!]^{\mathrm{s}} \stackrel{\Delta}{=} \mu\alpha.\langle [\![M]\!]^{\mathrm{s}} | \alpha\rangle \rightarrow_{\overline{\lambda}} (\eta\mu)\ [\![M]\!]^{\mathrm{s}}$
$(P \rightarrow_{\lambda\mu} Q \Rightarrow \lambda x.P \rightarrow_{\lambda\mu} \lambda x.Q)$: $[\![\lambda x.P]\!]^{\mathrm{s}} \stackrel{\Delta}{=} \lambda x.[\![P]\!]^{\mathrm{s}} \downarrow_{\overline{\lambda}} (ih)\ \lambda x.[\![Q]\!]^{\mathrm{s}} \stackrel{\Delta}{=} [\![\lambda x.Q]\!]^{\mathrm{s}}$
$(P \rightarrow_{\lambda\mu} Q \Rightarrow PR \rightarrow_{\lambda\mu} QR)$: $[\![PR]\!]^{\mathrm{s}} \stackrel{\Delta}{=} \mu\alpha.\langle [\![P]\!]^{\mathrm{s}} \,|\, \tilde{\mu}x.\langle [\![R]\!]^{\mathrm{s}} \,|\, \tilde{\mu}y.\langle x | y{\cdot}\alpha\rangle\rangle\rangle \downarrow_{\overline{\lambda}} (ih)$
$\mu\alpha.\langle [\![Q]\!]^{\mathrm{s}} \,|\, \tilde{\mu}x.\langle [\![R]\!]^{\mathrm{s}} \,|\, \tilde{\mu}y.\langle x | y{\cdot}\alpha\rangle\rangle\rangle \stackrel{\Delta}{=} [\![QR]\!]^{\mathrm{s}}$
$(P \rightarrow_{\lambda\mu} Q \Rightarrow RP \rightarrow_{\lambda\mu} RQ)$: $[\![RP]\!]^{\mathrm{s}} \stackrel{\Delta}{=} \mu\alpha.\langle [\![R]\!]^{\mathrm{s}} \,|\, \tilde{\mu}x.\langle [\![P]\!]^{\mathrm{s}} \,|\, \tilde{\mu}y.\langle x | y{\cdot}\alpha\rangle\rangle\rangle \downarrow_{\overline{\lambda}} (ih)$
$\mu\alpha.\langle [\![R]\!]^{\mathrm{s}} \,|\, \tilde{\mu}x.\langle [\![Q]\!]^{\mathrm{s}} \,|\, \tilde{\mu}y.\langle x | y{\cdot}\alpha\rangle\rangle\rangle \stackrel{\Delta}{=} [\![RQ]\!]^{\mathrm{s}}$
$(P \rightarrow_{\lambda\mu} Q \Rightarrow \mu\alpha.[\beta]P \rightarrow_{\lambda\mu} \mu\alpha.[\beta]Q)$: $[\![\mu\alpha.[\beta]P]\!]^{\mathrm{s}} \stackrel{\Delta}{=} \mu\alpha.\langle [\![P]\!]^{\mathrm{s}} | \beta\rangle \downarrow_{\overline{\lambda}} (ih)\ \mu\alpha.\langle [\![Q]\!]^{\mathrm{s}} | \beta\rangle \stackrel{\Delta}{=}$
$[\![\mu\alpha.[\beta]Q]\!]^{\mathrm{s}}$ □

As argued above, the proofs and the observations made on them also justify the following two results.

Theorem 5.16 (Preservation of CBV-reduction) *If* $M \rightarrow_{\lambda\mu}^{\mathrm{sv}*} N$*, then* $[\![M]\!]^{\mathrm{s}} \downarrow_{\overline{\lambda}}^{\mathrm{v}} [\![N]\!]^{\mathrm{s}}$.

Proof: $((\lambda z.M)V \rightarrow_{\lambda\mu}^{\mathrm{v}} M\{V/z\})$: By Theorem 5.8.
$((\mu\alpha.\mathrm{C})V \rightarrow_{\lambda\mu}^{\mathrm{v}} \mu\gamma.\mathrm{C}\{V{\cdot}\gamma/\alpha\})$: By Theorem 5.10.
$(N\,(\mu\alpha.\mathrm{C}) \rightarrow_{\lambda\mu}^{\mathrm{v}} \mu\gamma.\{N{\cdot}\gamma/\alpha\}\mathrm{C})$: By Theorem 5.14.
$(\mu\alpha.[\beta]\mu\gamma.[\delta]M \rightarrow_{\overline{\lambda}}^{\mathrm{v}} \mu\alpha.[\delta]M\{\beta/\gamma\},\ \gamma \neq \delta)$: $[\![\mu\alpha.[\beta]\mu\gamma.[\delta]M]\!]^{\mathrm{s}} \stackrel{\Delta}{=} \mu\alpha.\langle \mu\gamma.\langle [\![M]\!]^{\mathrm{s}} | \delta\rangle \,|\, \beta\rangle \rightarrow_{\overline{\lambda}}^{\mathrm{v}}$
$\mu\alpha.\langle [\![M]\!]^{\mathrm{s}}\{\beta/\gamma\} \,|\, \delta\rangle \stackrel{\Delta}{=} \mu\alpha.\langle [\![M\{\beta/\gamma\}]\!]^{\mathrm{s}} \,|\, \delta\rangle \stackrel{\Delta}{=} [\![\mu\alpha.[\delta]M\{\beta/\gamma\}]\!]^{\mathrm{s}}$

$(\mu\alpha.[\beta]\mu\gamma.[\gamma]M \rightarrow^{\text{v}}_{\lambda\mu} \mu\alpha.[\beta]M\{\beta/\gamma\})$: $[\![\mu\alpha.[\beta]\mu\gamma.[\gamma]M]\!]^{\text{s}} \triangleq \mu\alpha.\langle\mu\gamma.\langle[\![M]\!]^{\text{s}}|\gamma\rangle\,|\,\beta\rangle \rightarrow^{\text{v}}_{\lambda}$ $\mu\alpha.\langle[\![M]\!]^{\text{s}}\{\beta/\gamma\}\,|\,\beta\rangle \triangleq \mu\alpha.\langle[\![M\{\beta/\gamma\}]\!]^{\text{s}}\,|\,\beta\rangle \triangleq [\![\mu\alpha.[\beta]M\{\beta/\gamma\}]\!]^{\text{s}}$

$(\mu\alpha.[\alpha]M \rightarrow^{\text{v}}_{\lambda\mu} M\ (\alpha \notin M))$: $[\![\mu\alpha.[\alpha]M]\!]^{\text{s}} \triangleq \mu\alpha.\langle[\![M]\!]^{\text{s}}|\alpha\rangle \rightarrow^{\text{v}}_{\lambda}(\eta\mu)\ [\![M]\!]^{\text{s}}$

$(P \rightarrow^{\text{v}}_{\lambda\mu} Q \Rightarrow PR \rightarrow^{\text{v}}_{\lambda\mu} QR)$: $[\![PR]\!]^{\text{s}} \triangleq \mu\alpha.\langle[\![P]\!]^{\text{s}}\,|\,\tilde{\mu}x.\langle[\![R]\!]^{\text{s}}\,|\,\tilde{\mu}y.\langle x|y{\cdot}\alpha\rangle\rangle\rangle \downarrow^{\text{v}}_{\lambda}(ih)$ $\mu\alpha.\langle[\![Q]\!]^{\text{s}}\,|\,\tilde{\mu}x.\langle[\![R]\!]^{\text{s}}\,|\,\tilde{\mu}y.\langle x|y{\cdot}\alpha\rangle\rangle\rangle \triangleq [\![QR]\!]^{\text{s}}$.

$(P \rightarrow_{\lambda\mu} Q \Rightarrow VP \rightarrow_{\lambda\mu} VQ)$: $[\![VP]\!]^{\text{s}} \triangleq \mu\alpha.\langle[\![V]\!]^{\text{s}}\,|\,\tilde{\mu}x.\langle[\![P]\!]^{\text{s}}\,|\,\tilde{\mu}y.\langle x|y{\cdot}\alpha\rangle\rangle\rangle \rightarrow^{\text{v}}_{\lambda}(x)$ $\mu\alpha.\langle[\![P]\!]^{\text{s}}\,|\,\tilde{\mu}y.\langle[\![V]\!]^{\text{s}}|y{\cdot}\alpha\rangle\rangle \downarrow^{\text{v}}_{\lambda}(ih)\ \mu\alpha.\langle[\![Q]\!]^{\text{s}}\,|\,\tilde{\mu}y.\langle[\![V]\!]^{\text{s}}|y{\cdot}\alpha\rangle\rangle \ {}^{\text{v}}_{\lambda}\!\leftarrow (x)$ $\mu\alpha.\langle[\![V]\!]^{\text{s}}\,|\,\tilde{\mu}x.\langle[\![Q]\!]^{\text{s}}\,|\,\tilde{\mu}y.\langle x|y{\cdot}\alpha\rangle\rangle\rangle \triangleq [\![VQ]\!]^{\text{s}}$.

$(P \rightarrow_{\lambda\mu} Q \Rightarrow \mu\alpha.[\beta]P \rightarrow_{\lambda\mu} \mu\alpha.[\beta]Q)$: $[\![\mu\alpha.[\beta]P]\!]^{\text{s}} \triangleq \mu\alpha.\langle[\![P]\!]^{\text{s}}|\beta\rangle \downarrow^{\text{v}}_{\lambda}(ih)\ \mu\alpha.\langle[\![Q]\!]^{\text{s}}|\beta\rangle \triangleq$ $[\![\mu\alpha.[\beta]Q]\!]^{\text{s}}$. □

Notice that the $(\tilde{\mu})$-contractions over x are allowed in the penultimate part, since $[\![V]\!]^{\text{s}}$ is a value.

Theorem 5.17 (Preservation of cbn-reduction) *If $M \rightarrow^{\text{SN}*}_{\lambda\mu} N$, then $[\![M]\!]^{\text{s}} \downarrow^{\text{N}}_{\lambda} [\![N]\!]^{\text{s}}$.*

Proof: $((\lambda z.M)N \rightarrow^{\text{N}}_{\lambda\mu} M\{N/z\})$: By Theorem 5.8.

$((\mu\alpha.\text{C})N \rightarrow^{\text{N}}_{\lambda\mu} \mu\gamma.\text{C}\{N{\cdot}\gamma/\alpha\})$: By Theorem 5.12.

$(\mu\alpha.[\beta]\mu\gamma.[\delta]M \rightarrow^{\text{N}}_{\lambda} \mu\alpha.[\delta]M\{\beta/\gamma\},\ \gamma \neq \delta)$: $[\![\mu\alpha.[\beta]\mu\gamma.[\delta]M]\!]^{\text{s}} \triangleq \mu\alpha.\langle\mu\gamma.\langle[\![M]\!]^{\text{s}}|\delta\rangle\,|\,\beta\rangle \rightarrow^{\text{N}}_{\lambda}$ $\mu\alpha.\langle[\![M]\!]^{\text{s}}\{\beta/\gamma\}\,|\,\delta\rangle \triangleq \mu\alpha.\langle[\![M\{\beta/\gamma\}]\!]^{\text{s}}\,|\,\delta\rangle \triangleq [\![\mu\alpha.[\delta]M\{\beta/\gamma\}]\!]^{\text{s}}$

$(\mu\alpha.[\beta]\mu\gamma.[\gamma]M \rightarrow^{\text{N}}_{\lambda} \mu\alpha.[\beta]M\{\beta/\gamma\})$: $[\![\mu\alpha.[\beta]\mu\gamma.[\gamma]M]\!]^{\text{s}} \triangleq \mu\alpha.\langle\mu\gamma.\langle[\![M]\!]^{\text{s}}|\gamma\rangle\,|\,\beta\rangle \rightarrow^{\text{N}}_{\lambda}$ $\mu\alpha.\langle[\![M]\!]^{\text{s}}\{\beta/\gamma\}\,|\,\beta\rangle \triangleq \mu\alpha.\langle[\![M\{\beta/\gamma\}]\!]^{\text{s}}\,|\,\beta\rangle \triangleq [\![\mu\alpha.[\beta]M\{\beta/\gamma\}]\!]^{\text{s}}$

$(\mu\alpha.[\alpha]M \rightarrow^{\text{N}}_{\lambda\mu} M\ (\alpha \notin M))$: $[\![\mu\alpha.[\alpha]M]\!]^{\text{s}} \triangleq \mu\alpha.\langle[\![M]\!]^{\text{s}}|\alpha\rangle \rightarrow^{\text{N}}_{\lambda}(\eta\mu)\ [\![M]\!]^{\text{s}}$

$(P \rightarrow^{\text{N}}_{\lambda\mu} Q \Rightarrow PR \rightarrow^{\text{N}}_{\lambda\mu} QR)$: $[\![PR]\!]^{\text{s}} \triangleq \mu\alpha.\langle[\![P]\!]^{\text{s}}\,|\,\tilde{\mu}x.\langle[\![R]\!]^{\text{s}}\,|\,\tilde{\mu}y.\langle x|y{\cdot}\alpha\rangle\rangle\rangle \rightarrow^{\text{N}}_{\lambda}(ih)$ $\mu\alpha.\langle[\![Q]\!]^{\text{s}}\,|\,\tilde{\mu}x.\langle[\![R]\!]^{\text{s}}\,|\,\tilde{\mu}y.\langle x|y{\cdot}\alpha\rangle\rangle\rangle \triangleq [\![QR]\!]^{\text{s}}$.

$(P \rightarrow^{\text{N}}_{\lambda\mu} Q \Rightarrow \mu\alpha.[\beta]P \rightarrow^{\text{N}}_{\lambda\mu} \mu\alpha.[\beta]Q)$: $[\![\mu\alpha.[\beta]P]\!]^{\text{s}} \triangleq \mu\alpha.\langle[\![P]\!]^{\text{s}}|\beta\rangle \downarrow^{\text{N}}_{\lambda}(ih)\ \mu\alpha.\langle[\![Q]\!]^{\text{s}}|\beta\rangle \triangleq$ $[\![\mu\alpha.[\beta]Q]\!]^{\text{s}}$. □

Notice that reduction only takes places in terms, never in contexts.

We have shown that there exists a single interpretation from s$\lambda\mu$ to $\overline{\lambda}\mu\tilde{\mu}$ that respects reduction, and the cbn and cbv reduction strategies, thus establishing a strong relation between s$\lambda\mu$, $\lambda\mu$, and $\lambda\mu$v on one side, and $\overline{\lambda}\mu\tilde{\mu}$ on the other, as well as between the respective cbn and cbv strategies in s$\lambda\mu$ and $\overline{\lambda}\mu\tilde{\mu}$.

6 How about left μ contraction in $\overline{\lambda}\mu\tilde{\mu}$?

We have remarked above that the term $\mu\gamma.\langle\lambda x.t\,|\,\mu\alpha.\langle t'|\beta\rangle{\cdot}\gamma\rangle$ is not a $\overline{\lambda}\mu\tilde{\mu}$ critical pair, whereas its s$\lambda\mu$-counterpart $(\lambda x.M)\,(\mu\alpha.[\beta]N)$ is a s$\lambda\mu$ critical pair. Notice that

$$(\lambda x.M)\,(\mu\alpha.[\beta]N) \ \rightarrow^{\text{SV}}_{\lambda\mu}\ \mu\gamma.[\beta]\{\lambda x.M{\cdot}\gamma/\alpha\}N,$$

but $((\lambda x.M)\,(\mu\alpha.[\beta]N))^{v} = \mu\gamma.\langle\lambda x.M^{v}\,|\,\mu\alpha.\langle N^{v}|\beta\rangle{\cdot}\gamma\rangle$ is not reducible under cbn or cbv. We have shown that we can represent (implement) this reduction step under $[\![\cdot]\!]^{\text{s}}$, but there is an alternative path to this.

Definition 6.1 We extend $\overline{\lambda}\mu\tilde{\mu}$'s notion of reduction by adding the reduction rule:

$$(\mu_{l}):\ \langle t\,|\,\mu\alpha.c{\cdot}\gamma\rangle \ \rightarrow\ c\{\tilde{\mu}y.\langle t|y{\cdot}\gamma\rangle/\alpha\}$$

This rule is added for full reduction, but excluded from cbn and restricted in the normal way for cbv.

We can show that this reduction makes logical sense: first we show that the substitution $\cdot\{\tilde{\mu}y.\langle t|y\cdot\gamma\rangle/\alpha\}$ is sound.

Lemma 6.2 *i) If* $\Gamma \vdash_{\overline{\lambda}} t : A{\to}B \mid \gamma{:}B,\Delta$ *and* $c : \Gamma \vdash_{\overline{\lambda}} \alpha{:}A,\Delta$, *then* $c\{\tilde{\mu}y.\langle t|y\cdot\gamma\rangle/\alpha\} : \Gamma \vdash_{\overline{\lambda}} \gamma{:}B,\Delta$.
ii) If $\Gamma \vdash_{\overline{\lambda}} t : A{\to}B \mid \gamma{:}B,\Delta$ *and* $\Gamma \vdash_{\overline{\lambda}} t' : C \mid \alpha{:}A,\Delta$, *then* $\Gamma \vdash_{\overline{\lambda}} t'\{\tilde{\mu}y.\langle t|y\cdot\gamma\rangle/\alpha\} : C \mid \gamma{:}B,\Delta$.
iii) If $\Gamma \vdash_{\overline{\lambda}} t : A{\to}B \mid \gamma{:}B,\Delta$ *and* $\Gamma \mid e : C \vdash_{\overline{\lambda}} \alpha{:}A,\Delta$, *then* $\Gamma \mid e\{\tilde{\mu}y.\langle t|y\cdot\gamma\rangle/\alpha\} : C \vdash_{\overline{\lambda}} \gamma{:}B,\Delta$.

Proof: By induction on the definition of type assignment.

(cut): Then $c = \langle t|v\rangle$, and both Then by induction, both $\Gamma \vdash_{\overline{\lambda}} t\{\tilde{\mu}y.\langle t|y\cdot\gamma\rangle/\alpha\} : A \mid \gamma{:}B,\Delta$ and $\Gamma \mid e : A \vdash_{\overline{\lambda}} \gamma{:}B,\Delta$. By rule (cut) we have

$$\frac{\Gamma \vdash_{\overline{\lambda}} t\{\tilde{\mu}y.\langle t|y\cdot\gamma\rangle/\alpha\} : A \mid \gamma{:}B,\Delta \quad \Gamma \mid e\{\tilde{\mu}y.\langle t|y\cdot\gamma\rangle/\alpha\} : A \vdash_{\overline{\lambda}} \gamma{:}B,\Delta}{\langle t\{\tilde{\mu}y.\langle t|y\cdot\gamma\rangle/\alpha\} \,|\, e\{\tilde{\mu}y.\langle t|y\cdot\gamma\rangle/\alpha\}\rangle : \Gamma \vdash_{\overline{\lambda}} \Delta}$$

and $\langle t\{\tilde{\mu}y.\langle t|y\cdot\gamma\rangle/\alpha\} \,|\, e\{\tilde{\mu}y.\langle t|y\cdot\gamma\rangle/\alpha\}\rangle = \langle t|e\rangle\{\tilde{\mu}y.\langle t|y\cdot\gamma\rangle/\alpha\}$.

(Ax_R): Then $t = x$; since $x{:}C \in \Gamma$, by rule (Ax_R) also $\Gamma \vdash_{\overline{\lambda}} x : C \mid \gamma{:}B,\Delta$

(Ax_L): Then $e = \beta$, and $\beta{:}C \in \Delta$. We have two cases:

$(\alpha = \beta)$: Then $C = A$; we can construct:

$$\cfrac{\cfrac{\cfrac{\cfrac{\Box}{\Gamma \vdash t : A{\to}B \mid \gamma{:}B,\Delta}}{\Gamma,y{:}A \vdash t : A{\to}B \mid \gamma{:}B,\Delta}\,(Wk) \quad \cfrac{\cfrac{}{\Gamma,y{:}A \vdash y : A \mid \gamma{:}B,\Delta}\,(Ax_R) \quad \cfrac{}{\Gamma,y{:}A \mid \gamma : B \vdash \gamma{:}B,\Delta}\,(Ax_L)}{\Gamma,y{:}A \mid y\cdot\gamma : A{\to}B \vdash \gamma{:}B,\Delta}\,({\to}L)}{\langle t|y\cdot\gamma\rangle : \Gamma,y{:}A \vdash \gamma{:}B,\Delta}\,(cut)}{\Gamma \mid \tilde{\mu}y.\langle t|y\cdot\gamma\rangle : A \vdash \gamma{:}B,\Delta}\,(\tilde{\mu})$$

and $\alpha\{\tilde{\mu}y.\langle t|y\cdot\gamma\rangle/\alpha\} = \tilde{\mu}y.\langle t|y\cdot\gamma\rangle$.

$(\alpha \neq \beta)$: Since $\beta{:}C \in \Delta$, by rule (Ax_L) also $\Gamma \mid \beta : C \vdash_{\overline{\lambda}} \gamma{:}B,\Delta$.

$({\to}L)$, $({\to}R)$, (μ), $(\tilde{\mu})$: By induction. □

With this result, we can now show:

Theorem 6.3 (Soundness for rule (μ_l)) *If* $\langle t \,|\, \mu\alpha.c\cdot\gamma\rangle : \Gamma \vdash_{\overline{\lambda}} \gamma{:}B,\Delta$, *then* $c\{\tilde{\mu}y.\langle t|y\cdot\gamma\rangle/\alpha\} : \Gamma \vdash_{\overline{\lambda}} \gamma{:}B,\Delta$.

Proof: If $\langle t \,|\, \mu\alpha.c\cdot\gamma\rangle : \Gamma \vdash_{\overline{\lambda}} \Delta$, then the derivation is shaped like:

$$\cfrac{\cfrac{\Box}{\Gamma \vdash t : A{\to}B \mid \gamma{:}B,\Delta} \quad \cfrac{\cfrac{\cfrac{\Box}{c : \Gamma \vdash \alpha{:}A,\gamma{:}B,\Delta}}{\Gamma \vdash \mu\alpha.c : A \mid \gamma{:}B,\Delta} \quad \cfrac{}{\Gamma \mid \gamma : B \vdash \gamma{:}B,\Delta}}{\Gamma \mid \mu\alpha.c\cdot\gamma : A{\to}B \vdash \gamma{:}B,\Delta}}{\langle t \,|\, \mu\alpha.c\cdot\gamma\rangle : \Gamma \vdash \gamma{:}B,\Delta}$$

Then, in particular, we have $\Gamma \vdash_{\overline{\lambda}} t : A{\to}B \mid \gamma{:}B,\Delta$ and $c : \Gamma \vdash_{\overline{\lambda}} \alpha{:}A,\gamma{:}B,\Delta$, and by Lem. 6.2, we have $c\{\tilde{\mu}y.\langle t|y\cdot\gamma\rangle/\alpha\} : \Gamma \vdash_{\overline{\lambda}} \gamma{:}B,\Delta$. □

So this new rule is sound. We have shown above that we have full representation of the three notions of reductions we focus on for standard $\overline{\lambda}\mu\tilde{\mu}$, so do not need to extend reduction on $\overline{\lambda}\mu\tilde{\mu}$ to achieve this. It will nonetheless be interesting to see if adding this reduction step to $\overline{\lambda}\mu\tilde{\mu}$ would yield different interpretations of s$\lambda\mu$; we leave this for future work.

7 Interpreting $\overline{\lambda}\mu\tilde{\mu}$ in $\mathcal{X}_{\text{is}}$

[23] presents a translation of $\overline{\lambda}\mu\tilde{\mu}$ into $\mathcal{X}$ (called $\lambda\xi$ there) which preserves the typing and shows that it respects reduction; since here we use a slight variation, that uses substitution

rather than a renaming cut when dealing with $\mu\alpha.c$ or $\tilde{\mu}x.c$, we need to give new proofs for this result.

We could say that, in $\overline{\lambda}\mu\tilde{\mu}$, not all inputs (normally present as term variables) and outputs (context variables) are explicitly named; for example, a term like $\lambda x.t$ has no named output, which we can make explicit through η-expanding it into $\mu\beta.\langle\lambda x.t|\beta\rangle$; similarly, $t{\cdot}e$ has no named input, which we can make explicit through $\tilde{\mu}x.\langle x|t{\cdot}e\rangle$. This is the essence of the translation we define now: it interprets terms under an output, and environments under an input, making the implicit names explicit. The first version was defined in [23] through:

$$(\langle t|e\rangle)^{\xi} = (t)^{\xi}_{\alpha}\,\widehat{\alpha}\dagger\widehat{x}\,(e)^{\xi}_{x}\ (\alpha, x\ \mathit{fresh})$$

$$\begin{array}{rclrcl}
(x)^{\xi}_{\alpha} &=& \langle x{\cdot}\alpha\rangle & (\alpha)^{\xi}_{x} &=& \langle x{\cdot}\alpha\rangle \\
(\lambda x.t)^{\xi}_{\alpha} &=& \widehat{x}\,(t)^{\xi}_{\beta}\widehat{\beta}\cdot\alpha\ (\beta\ \mathit{fresh}) & (t{\cdot}e)^{\xi}_{x} &=& (t)^{\xi}_{\gamma}\,\widehat{\gamma}\,[x]\,\widehat{z}\,(e)^{\xi}_{z}\ (\gamma, z\ \mathit{fresh}) \\
(\mu\beta.c)^{\xi}_{\alpha} &=& (c)^{\xi}\{\alpha/\beta\} & (\tilde{\mu}y.c)^{\xi}_{x} &=& (c))^{\xi}\{x/y\}
\end{array}$$

In [5], the results were obtained for an interpretation where the last two alternatives are defined through:

$$\begin{array}{rcl}
(\mu\beta.c)^{\xi}_{\alpha} &=& (c)^{\xi}\,\widehat{\beta}\dagger\widehat{x}\,\langle x{\cdot}\alpha\rangle \\
(\tilde{\mu}y.c)^{\xi}_{x} &=& \langle x{\cdot}\beta\rangle\,\widehat{\beta}\dagger\widehat{y}\,(c)^{\xi}
\end{array}$$

showing that even here implicit substitutions are not needed. Since we can show

$$\begin{array}{rclcl}
(c)^{\xi}\,\widehat{\beta}\dagger\widehat{x}\,\langle x{\cdot}\alpha\rangle &\to^{*}_{\mathcal{X}_{\mathrm{IS}}}& (c)^{\xi}\{\beta\not\dagger\widehat{x}\,\langle x{\cdot}\alpha\rangle\} &\to^{*}_{\mathcal{X}_{\mathrm{IS}}}(4.6)& (c)^{\xi}\{\alpha/\beta\} \\
\langle x{\cdot}\beta\rangle\,\widehat{\beta}\dagger\widehat{y}\,(c)^{\xi} &\to^{*}_{\mathcal{X}_{\mathrm{IS}}}& \{\langle x{\cdot}\beta\rangle\,\widehat{\beta}\!\!\not\dagger y\}(c)^{\xi} &\to^{*}_{\mathcal{X}_{\mathrm{IS}}}(4.6)& (c)^{\xi}\{x/y\}
\end{array}$$

and in Lem. 7.3 and 7.4 we want the be able to simulate $\overline{\lambda}\mu\tilde{\mu}$-substitution through $\mathcal{X}_{\mathrm{IS}}$-substitution, we avoid adding the extra cuts, but use implicit substitution in the interpretation.

A disadvantage of this encoding is that it treats $\langle x|\alpha\rangle$ as a cut, rather than as $\langle x{\cdot}\alpha\rangle$. However, we can show:

$$\begin{array}{rclclcl}
(\langle y|\beta\rangle)^{\xi} &\triangleq& (y)^{\xi}_{\alpha}\,\widehat{\alpha}\dagger\widehat{x}\,(\beta)^{\xi}_{x} &\triangleq& \langle y{\cdot}\alpha\rangle\,\widehat{\alpha}\dagger\widehat{x}\,\langle x{\cdot}\beta\rangle &\to_{\mathcal{X}}(\mathit{cap})& \langle y{\cdot}\beta\rangle \\
(\langle y|t{\cdot}e\rangle)^{\xi} &\triangleq& (y)^{\xi}_{\alpha}\,\widehat{\alpha}\dagger\widehat{x}\,(t{\cdot}e)^{\xi}_{x} &\triangleq& \langle y{\cdot}\alpha\rangle\,\widehat{\alpha}\dagger\widehat{x}\,((t)^{\xi}_{\gamma}\,\widehat{\gamma}\,[x]\,\widehat{z}\,(e)^{\xi}_{z}) && \\
&&&\to_{\mathcal{X}}(\mathit{imp})& (t)^{\xi}_{\gamma}\,\widehat{\gamma}\,[y]\,\widehat{z}\,(e)^{\xi}_{z} &\triangleq& (t{\cdot}e)^{\xi}_{y} \\
(\langle y|\tilde{\mu}z.c\rangle)^{\xi} &\triangleq& (y)^{\xi}_{\alpha}\,\widehat{\alpha}\dagger\widehat{x}\,(\tilde{\mu}z.c)^{\xi}_{x} &\triangleq& \langle y{\cdot}\alpha\rangle\,\widehat{\alpha}\dagger\widehat{x}\,(c)^{\xi}\{x/z\} && \\
&&&\to_{\mathcal{X}}(4.6)& (c)^{\xi}\{y/z\} &\triangleq& (\tilde{\mu}z.c)^{\xi}_{y} \\
(\langle\lambda y.t|\beta\rangle)^{\xi} &\triangleq& (\lambda y.t)^{\xi}_{\alpha}\,\widehat{\alpha}\dagger\widehat{x}\,(\beta)^{\xi}_{x} &\triangleq& (\widehat{y}\,(t)^{\xi}_{\beta}\widehat{\beta}\cdot\alpha)\,\widehat{\alpha}\dagger\widehat{x}\,(\beta)^{\xi}_{x} && \\
&&&\to_{\mathcal{X}}(\mathit{exp})& \widehat{y}\,(t)^{\xi}_{\beta}\widehat{\beta}\cdot\beta &\triangleq& (\lambda y.t)^{\xi}_{y} \\
(\langle\mu\gamma.c|\beta\rangle)^{\xi} &\triangleq& (\mu\gamma.c)^{\xi}_{\alpha}\,\widehat{\alpha}\dagger\widehat{x}\,(\beta)^{\xi}_{x} &\triangleq& (c)^{\xi}\{\alpha/\gamma\}\,\widehat{\alpha}\dagger\widehat{x}\,(\beta)^{\xi}_{x} && \\
&&&\to_{\mathcal{X}}(4.6)& (c)^{\xi}\{\beta/\gamma\} &\triangleq& (\mu\gamma.c)^{\xi}_{\beta}
\end{array}$$

The interpretation we will consider here is an optimised version, that uses these observations and will induce less reduction in the image: this will yield stronger results.

Definition 7.1 (Translation of $\overline{\lambda}\mu\tilde{\mu}$ into $\mathcal{X}_{\mathrm{IS}}$)

$$\begin{array}{rclrcl}
[\![\langle y|\beta\rangle]\!]^{\overline{\lambda}} &=& \langle y{\cdot}\beta\rangle & [\![\langle y|e\rangle]\!]^{\overline{\lambda}} &=& [\![e]\!]^{\overline{\lambda}}_{y} \\
[\![\langle t|e\rangle]\!]^{\overline{\lambda}} &=& [\![t]\!]^{\overline{\lambda}}_{\alpha}\,\widehat{\alpha}\dagger\widehat{x}\,[\![e]\!]^{\overline{\lambda}}_{x}\ (\alpha, x\ \mathit{fresh, otherwise}) & [\![\langle t|\beta\rangle]\!]^{\overline{\lambda}} &=& [\![t]\!]^{\overline{\lambda}}_{\beta} \\
[\![x]\!]^{\overline{\lambda}}_{\alpha} &=& \langle x{\cdot}\alpha\rangle & [\![\alpha]\!]^{\overline{\lambda}}_{x} &=& \langle x{\cdot}\alpha\rangle \\
[\![\lambda x.t]\!]^{\overline{\lambda}}_{\alpha} &=& \widehat{x}\,[\![t]\!]^{\overline{\lambda}}_{\beta}\widehat{\beta}\cdot\alpha\ (\beta\ \mathit{fresh}) & [\![t{\cdot}e]\!]^{\overline{\lambda}}_{x} &=& [\![t]\!]^{\overline{\lambda}}_{\gamma}\,\widehat{\gamma}\,[x]\,\widehat{z}\,[\![e]\!]^{\overline{\lambda}}_{z}\ (\gamma, z\ \mathit{fresh}) \\
[\![\mu\beta.c]\!]^{\overline{\lambda}}_{\alpha} &=& [\![c]\!]^{\overline{\lambda}}\{\alpha/\beta\} & [\![\tilde{\mu}y.c]\!]^{\overline{\lambda}}_{x} &=& [\![c]\!]^{\overline{\lambda}}\{x/y\}
\end{array}$$

There are overlapping cases in this definition. However, we have

$$[\![\langle y|\beta\rangle]\!]^{\overline{\lambda}} = [\![\beta]\!]^{\overline{\lambda}}_{y} = [\![y]\!]^{\overline{\lambda}}_{\beta} = \langle y{\cdot}\beta\rangle$$

so this does not inconvenience the proofs. This translation will only create a cut when interpreting a command that does not involve a term or context variable.

If t is a $\overline{\lambda}\mu\tilde{\mu}$-value that does not contain α, then $[\![t]\!]^{\overline{\lambda}}_{\alpha}$ introduces α, and if e is a $\overline{\lambda}\mu\tilde{\mu}$-stack that does not contain x, then $[\![e]\!]^{\overline{\lambda}}_{x}$ introduces x:

$$\begin{array}{rclcrcl} [\![x]\!]^{\overline{\lambda}}_{\alpha} & = & \langle x{\cdot}\alpha\rangle & \qquad & [\![\alpha]\!]^{\overline{\lambda}}_{x} & = & \langle x{\cdot}\alpha\rangle \\ [\![\lambda x.t]\!]^{\overline{\lambda}}_{\alpha} & = & \widehat{x}\,[\![t]\!]^{\overline{\lambda}}_{\beta}\widehat{\beta}\cdot\alpha & & [\![t{\cdot}e']\!]^{\overline{\lambda}}_{x} & = & [\![t]\!]^{\overline{\lambda}}_{\gamma}\,\widehat{\gamma}\,[x]\,\widehat{z}\,[\![e']\!]^{\overline{\lambda}}_{z} \end{array}$$

We will need this observation later.

This interpretation preserves typeability:

Lemma 7.2 i) If $\Gamma \mid t : A \vdash_{\overline{\lambda}} \Delta$, then $[\![t]\!]^{\overline{\lambda}}_{\alpha} :\cdot\, \Gamma \vdash_{\mathcal{X}} \alpha{:}A,\Delta$.
ii) If $\Gamma \vdash_{\overline{\lambda}} e : A \mid \Delta$, then $[\![e]\!]^{\overline{\lambda}}_{x} :\cdot\, \Gamma, x{:}A \vdash_{\mathcal{X}} \Delta$.
iii) If $c : \Gamma \vdash_{\overline{\lambda}} \Delta$, then $[\![c]\!]^{\overline{\lambda}} :\cdot\, \Gamma \vdash_{\mathcal{X}} \Delta$.

Proof: This follows from a similar result for the original interpretation as shown in [5], the type preservation result shown there, and the observation we made above. □

We will now show the relation between implicit substitution in $\mathcal{X}_{\text{IS}}$ and in $\overline{\lambda}\mu\tilde{\mu}$. A similar result was show in [23], but with respect to $\mathcal{X}$. We show the proof in full since we need to know if reduction will stay within CBN or CBV.

First we show that $\mathcal{X}_{\text{IS}}$ successfully encodes $\overline{\lambda}\mu\tilde{\mu}$'s context substitution $\cdot\{e/\alpha\}$ through right substitution $\cdot\{\alpha \not\dagger \widehat{x}\,[\![e]\!]^{\overline{\lambda}}_{x}\}$.

Lemma 7.3 i) $[\![c]\!]^{\overline{\lambda}}\{\alpha \not\dagger \widehat{x}\,[\![e']\!]^{\overline{\lambda}}_{x}\} = [\![c\,\{e'/\alpha\}]\!]^{\overline{\lambda}}$.
ii) If $\alpha \neq \beta$, then $[\![t]\!]^{\overline{\lambda}}_{\beta}\{\alpha \not\dagger \widehat{x}\,[\![e']\!]^{\overline{\lambda}}_{x}\} = [\![t\,\{e'/\alpha\}]\!]^{\overline{\lambda}}_{\beta}$.
iii) $[\![e]\!]^{\overline{\lambda}}_{y}\{\alpha \not\dagger \widehat{x}\,[\![e']\!]^{\overline{\lambda}}_{x}\} = [\![e\,\{e'/\alpha\}]\!]^{\overline{\lambda}}_{y}$.

Proof: By simultaneous induction on the structure of nets.

i) $(c = \langle y|\alpha\rangle)$: $[\![\langle y|\alpha\rangle]\!]^{\overline{\lambda}}\{\alpha \not\dagger \widehat{x}\,[\![e']\!]^{\overline{\lambda}}_{x}\} \triangleq \langle y{\cdot}\alpha\rangle\{\alpha \not\dagger \widehat{x}\,[\![e']\!]^{\overline{\lambda}}_{x}\} = (d_{\text{R}})\ [\![e']\!]^{\overline{\lambda}}_{x}\{y/x\} = [\![e']\!]^{\overline{\lambda}}_{y} = [\![\langle y|e'\rangle]\!]^{\overline{\lambda}} = [\![\langle y|\alpha\rangle\,\{e'/\alpha\}]\!]^{\overline{\lambda}}$

$(c = \langle y|\beta\rangle,\ \alpha \neq \beta)$: $[\![\langle y|\beta\rangle]\!]^{\overline{\lambda}}\{\alpha \not\dagger \widehat{x}\,[\![e']\!]^{\overline{\lambda}}_{x}\} \triangleq \langle y{\cdot}\beta\rangle\{\alpha \not\dagger \widehat{x}\,[\![e']\!]^{\overline{\lambda}}_{x}\} = \langle y{\cdot}\beta\rangle = (gc_{\text{R}})\ [\![\langle y|\beta\rangle]\!]^{\overline{\lambda}} = [\![\langle y|\beta\rangle\,\{e'/\alpha\}]\!]^{\overline{\lambda}}$

$(c = \langle \lambda z.t|\alpha\rangle)$: $[\![\langle \lambda z.t|\alpha\rangle]\!]^{\overline{\lambda}}\{\alpha \not\dagger \widehat{x}\,[\![e']\!]^{\overline{\lambda}}_{x}\} \triangleq (\widehat{z}\,[\![t]\!]^{\overline{\lambda}}_{\beta}\widehat{\beta}\cdot\alpha)\{\alpha \not\dagger \widehat{x}\,[\![e']\!]^{\overline{\lambda}}_{x}\} = (exp\text{-}out_{\text{R}})$ $(\widehat{z}\,([\![t]\!]^{\overline{\lambda}}_{\beta}\{\alpha \not\dagger \widehat{x}\,[\![e']\!]^{\overline{\lambda}}_{x}\})\,\widehat{\beta}\cdot\gamma)\,\widehat{\gamma}\dagger\widehat{x}\,[\![e']\!]^{\overline{\lambda}}_{x} = (ih, \alpha \neq \beta)\ (\widehat{z}\,[\![t\,\{e'/\alpha\}]\!]^{\overline{\lambda}}\widehat{\beta}\cdot\gamma)\,\widehat{\gamma}\dagger\widehat{x}\,[\![e']\!]^{\overline{\lambda}}_{x} \triangleq [\![\langle \lambda z.t\,\{e'/\alpha\}\,|e'\rangle]\!]^{\overline{\lambda}} \triangleq [\![\langle \lambda z.t|\alpha\rangle\,\{e'/\alpha\}]\!]^{\overline{\lambda}}$

$(c = \langle \mu\beta.c|\alpha\rangle)$: $[\![\langle \mu\beta.c|\alpha\rangle]\!]^{\overline{\lambda}}\{\alpha \not\dagger \widehat{x}\,[\![e']\!]^{\overline{\lambda}}_{x}\} \triangleq [\![\mu\beta.c]\!]^{\overline{\lambda}}_{\alpha}\{\alpha \not\dagger \widehat{x}\,[\![e']\!]^{\overline{\lambda}}_{x}\} \triangleq [\![c]\!]^{\overline{\lambda}}\{\alpha/\beta\}\{\alpha \not\dagger \widehat{x}\,[\![e']\!]^{\overline{\lambda}}_{x}\} \triangleq ([\![c]\!]^{\overline{\lambda}}\{\alpha \not\dagger \widehat{x}\,[\![e']\!]^{\overline{\lambda}}_{x}\})\{\alpha/\beta\}\{\alpha \not\dagger \widehat{x}\,[\![e']\!]^{\overline{\lambda}}_{x}\} = (ih)$ $[\![c\,\{e'/\alpha\}]\!]^{\overline{\lambda}}\{\alpha/\beta\}\{\alpha \not\dagger \widehat{x}\,[\![e']\!]^{\overline{\lambda}}_{x}\} \triangleq [\![\mu\beta.c\,\{e'/\alpha\}]\!]^{\overline{\lambda}}_{\alpha}\{\alpha \not\dagger \widehat{x}\,[\![e']\!]^{\overline{\lambda}}_{x}\} \triangleq [\![\langle \mu\beta.c\,\{e'/\alpha\}\,|\,\alpha\rangle]\!]^{\overline{\lambda}}\{\alpha \not\dagger \widehat{x}\,[\![e']\!]^{\overline{\lambda}}_{x}\} \triangleq (ih)\ [\![\langle \mu\beta.c\,\{e'/\alpha\}\,|\,e'\rangle]\!]^{\overline{\lambda}} \triangleq [\![\langle \mu\beta.c|\alpha\rangle\,\{e'/\alpha\}]\!]^{\overline{\lambda}}$

$(c = \langle y|e\rangle)$: $[\![\langle y|e\rangle]\!]^{\overline{\lambda}}\{\alpha \not\dagger \widehat{x}\,[\![e']\!]^{\overline{\lambda}}_{x}\} \triangleq \langle y{\cdot}e\rangle\{\alpha \not\dagger \widehat{x}\,[\![e']\!]^{\overline{\lambda}}_{x}\} \triangleq [\![e]\!]^{\overline{\lambda}}_{y}\{\alpha \not\dagger \widehat{x}\,[\![e']\!]^{\overline{\lambda}}_{x}\} = (ih)$ $[\![e\,\{e'/\alpha\}]\!]^{\overline{\lambda}}_{y} \triangleq [\![\langle y\,|\,e\{e'/\alpha\}\rangle]\!]^{\overline{\lambda}} \triangleq [\![\langle y|e\rangle\,\{e'/\alpha\}]\!]^{\overline{\lambda}}$

$(c = \langle t|e\rangle)$: $[\![\langle t|e\rangle]\!]^{\overline{\lambda}}\{\alpha \not\dagger \widehat{x}\,[\![e']\!]^{\overline{\lambda}}_{x}\} \triangleq ([\![t]\!]^{\overline{\lambda}}_{\beta}\,\widehat{\beta}\dagger\widehat{y}\,[\![e]\!]^{\overline{\lambda}}_{y})\{\alpha \not\dagger \widehat{x}\,[\![e']\!]^{\overline{\lambda}}_{x}\} = (cut_{\text{R}})$ $([\![t]\!]^{\overline{\lambda}}_{\beta}\{\alpha \not\dagger \widehat{x}\,[\![e']\!]^{\overline{\lambda}}_{x}\})\,\widehat{\beta}\dagger\widehat{y}\,([\![e]\!]^{\overline{\lambda}}_{y}\{\alpha \not\dagger \widehat{x}\,[\![e']\!]^{\overline{\lambda}}_{x}\}) = (ih, \alpha \neq \beta)$ $[\![t\,\{e'/\alpha\}]\!]^{\overline{\lambda}}_{\beta}\,\widehat{\beta}\dagger\widehat{y}\,[\![e\,\{e'/\alpha\}]\!]^{\overline{\lambda}}_{y} \triangleq [\![\langle t\,\{e'/\alpha\}\,|\,e\,\{e'/\alpha\}\rangle]\!]^{\overline{\lambda}} \triangleq [\![\langle t|e\rangle\,\{e'/\alpha\}]\!]^{\overline{\lambda}}$

ii) $(t = y)$: $[\![y]\!]^{\overline{\lambda}}_{\beta}\{\alpha \not\dagger \widehat{x}\,[\![e']\!]^{\overline{\lambda}}_{x}\} \triangleq \langle y{\cdot}\beta\rangle\{\alpha \not\dagger \widehat{x}\,[\![e']\!]^{\overline{\lambda}}_{x}\} = (gc_{\text{R}})\ \langle y{\cdot}\beta\rangle \triangleq [\![y]\!]^{\overline{\lambda}}_{\beta} \triangleq [\![y\,\{e'/\alpha\}]\!]^{\overline{\lambda}}_{\beta}$

$(t = \lambda y.t)$: $\llbracket \lambda y.t\rrbracket^{\overline{\lambda}}_\beta\{\alpha\widehat{/}\widehat{x}\,\llbracket e'\rrbracket^{\overline{\lambda}}_x\} \;\triangleq\; (\widehat{y}\,\llbracket t\rrbracket^{\overline{\lambda}}_\delta\widehat{\delta}\cdot\beta)\{\alpha\widehat{/}\widehat{x}\,\llbracket e'\rrbracket^{\overline{\lambda}}_x\} \;=(\mathit{exp\text{-}out}_{\mathrm{R}})$
$\widehat{y}\,(\llbracket t\rrbracket^{\overline{\lambda}}_\delta\{\alpha\widehat{/}\widehat{x}\,\llbracket e'\rrbracket^{\overline{\lambda}}_x\})\widehat{\delta}\cdot\beta \;=(ih,\alpha\neq\delta)\; \widehat{y}\,\llbracket t\{e'/\alpha\}\rrbracket^{\overline{\lambda}}_\delta\widehat{\delta}\cdot\beta \;\triangleq$
$\llbracket \lambda y.t\{e'/\alpha\}\rrbracket^{\overline{\lambda}}_\beta \;\triangleq\; \llbracket(\lambda y.t)\{e'/\alpha\}\rrbracket^{\overline{\lambda}}_\beta$

$(t = \mu\gamma.c)$: $\llbracket \mu\gamma.c\rrbracket^{\overline{\lambda}}_\beta\{\alpha\widehat{/}\widehat{x}\,\llbracket e'\rrbracket^{\overline{\lambda}}_x\} \;\triangleq\; \llbracket c\rrbracket^{\overline{\lambda}}\{\beta/\gamma\}\{\alpha\widehat{/}\widehat{x}\,\llbracket e'\rrbracket^{\overline{\lambda}}_x\} \;=(ih,\alpha\neq\beta,\gamma)$
$\llbracket c\{e'/\alpha\}\rrbracket^{\overline{\lambda}}\{\beta/\gamma\} \;\triangleq\; \llbracket \mu\gamma.c\{e'/\alpha\}\rrbracket^{\overline{\lambda}}_\beta \;\triangleq\; \llbracket(\mu\gamma.c)\{e'/\alpha\}\rrbracket^{\overline{\lambda}}_\beta$

iii) $(e=\alpha)$: $\llbracket\alpha\rrbracket^{\overline{\lambda}}_y\{\alpha\widehat{/}\widehat{x}\,\llbracket e'\rrbracket^{\overline{\lambda}}_x\} \;\triangleq\; \langle y{\cdot}\alpha\rangle\{\alpha\widehat{/}\widehat{x}\,\llbracket e'\rrbracket^{\overline{\lambda}}_x\} \;=(d_{\mathrm{R}})\; \llbracket e'\rrbracket^{\overline{\lambda}}_x\{y/x\} \;\triangleq$
$\llbracket e'\rrbracket^{\overline{\lambda}}_y \;\triangleq\; \llbracket\alpha\{e'/\alpha\}\rrbracket^{\overline{\lambda}}_y$

$(e=\beta\neq\alpha)$: $\llbracket\beta\rrbracket^{\overline{\lambda}}_y\{\alpha\widehat{/}\widehat{x}\,\llbracket e'\rrbracket^{\overline{\lambda}}_x\} \;=\; \langle y{\cdot}\beta\rangle\{\alpha\widehat{/}\widehat{x}\,\llbracket e'\rrbracket^{\overline{\lambda}}_x\} \;=(gc_{\mathrm{R}})\; \langle y{\cdot}\beta\rangle \;\triangleq\; \llbracket\beta\rrbracket^{\overline{\lambda}}_y \;\triangleq\; \llbracket\beta\{e'/\alpha\}\rrbracket^{\overline{\lambda}}_y$

$(e=t{\cdot}e'')$: $\llbracket t{\cdot}e''\rrbracket^{\overline{\lambda}}_y\{\alpha\widehat{/}\widehat{x}\,\llbracket e'\rrbracket^{\overline{\lambda}}_x\} \;\triangleq\; (\llbracket t\rrbracket^{\overline{\lambda}}_\gamma\,\widehat{\gamma}\,[y]\,\widehat{z}\,\llbracket e''\rrbracket^{\overline{\lambda}}_z)\{\alpha\widehat{/}\widehat{x}\,\llbracket e'\rrbracket^{\overline{\lambda}}_x\} \;=(\mathit{imp}_{\mathrm{R}})$
$(\llbracket t\rrbracket^{\overline{\lambda}}_\gamma\{\alpha\widehat{/}\widehat{x}\,\llbracket e'\rrbracket^{\overline{\lambda}}_x\})\,\widehat{\gamma}\,[y]\,\widehat{z}\,(\llbracket e''\rrbracket^{\overline{\lambda}}_z\{\alpha\widehat{/}\widehat{x}\,\llbracket e'\rrbracket^{\overline{\lambda}}_x\}) \;=(ih,\alpha\neq\gamma)$
$\llbracket t\{e'/\alpha\}\rrbracket^{\overline{\lambda}}_\gamma\,\widehat{\gamma}\,[y]\,\widehat{z}\,\llbracket e''\{e'/\alpha\}\rrbracket^{\overline{\lambda}}_z \;\triangleq\; \llbracket(t\{e'/\alpha\}\cdot e''\{e'/\alpha\})\rrbracket^{\overline{\lambda}}_y \;\triangleq\; \llbracket(t{\cdot}e'')\{e'/\alpha\}\rrbracket^{\overline{\lambda}}_y$

$(e=\tilde{\mu}y.c)$: $\llbracket\tilde{\mu}z.c\rrbracket^{\overline{\lambda}}_y\{\alpha\widehat{/}\widehat{x}\,\llbracket e'\rrbracket^{\overline{\lambda}}_x\} \;\triangleq\; \llbracket c\rrbracket^{\overline{\lambda}}\{y/z\}\{\alpha\widehat{/}\widehat{x}\,\llbracket e'\rrbracket^{\overline{\lambda}}_x\} \;\triangleq\; (\llbracket c\rrbracket^{\overline{\lambda}}\{\alpha\widehat{/}\widehat{x}\,\llbracket e'\rrbracket^{\overline{\lambda}}_x\})\{y/z\}$
$=(ih)\; \llbracket c\{e'/\alpha\}\rrbracket^{\overline{\lambda}}\{y/z\} \;\triangleq\; \llbracket\tilde{\mu}z.c\{e'/\alpha\}\rrbracket^{\overline{\lambda}}_y \;\triangleq\; \llbracket(\tilde{\mu}z.c)\{e'/\alpha\}\rrbracket^{\overline{\lambda}}_y$ ☐

Notice that no reduction steps are used in this proof.

Likewise, we can show that $\mathcal{X}_{\mathrm{IS}}$ successfully encodes $\overline{\lambda}\mu\tilde{\mu}$'s term substitution $\cdot\{t/x\}$ through left substitution $\{\llbracket t\rrbracket^{\overline{\lambda}}_\alpha\,\widehat{\alpha}\widehat{\setminus}x\}\cdot$.

Lemma 7.4 *i)* $\{\llbracket t'\rrbracket^{\overline{\lambda}}_\alpha\,\widehat{\alpha}\widehat{\setminus}x\}\llbracket c\rrbracket^{\overline{\lambda}} = \llbracket c\{t'/x\}\rrbracket^{\overline{\lambda}}$.
ii) $\{\llbracket t'\rrbracket^{\overline{\lambda}}_\alpha\,\widehat{\alpha}\widehat{\setminus}x\}\llbracket t\rrbracket^{\overline{\lambda}}_\beta = \llbracket t\{t'/x\}\rrbracket^{\overline{\lambda}}_\beta$.
iii) *If* $z\neq x$, $\{\llbracket t'\rrbracket^{\overline{\lambda}}_\alpha\,\widehat{\alpha}\widehat{\setminus}x\}\llbracket e\rrbracket^{\overline{\lambda}}_z = \llbracket e\{t'/x\}\rrbracket^{\overline{\lambda}}_z$.

Proof: By simultaneous induction on the structure of nets.

i) $(c=\langle y|\alpha\rangle)$: $\{\llbracket t'\rrbracket^{\overline{\lambda}}_\alpha\,\widehat{\alpha}\widehat{\setminus}x\}\llbracket\langle x|\beta\rangle\rrbracket^{\overline{\lambda}} \;\triangleq\; \{\llbracket t'\rrbracket^{\overline{\lambda}}_\alpha\,\widehat{\alpha}\widehat{\setminus}x\}\langle x{\cdot}\beta\rangle \;\triangleq(d_{\mathrm{L}})\; \llbracket t'\rrbracket^{\overline{\lambda}}_\alpha\{\beta/\alpha\} \;=\; \llbracket t'\rrbracket^{\overline{\lambda}}_\beta \;=$
$\llbracket\langle t'|\beta\rangle\rrbracket^{\overline{\lambda}} \;=\; \llbracket\langle x|\beta\rangle\{t'/x\}\rrbracket^{\overline{\lambda}}$

$(c=\langle y|\alpha\rangle,\, y\neq x)$: $\{\llbracket t'\rrbracket^{\overline{\lambda}}_\alpha\,\widehat{\alpha}\widehat{\setminus}x\}\llbracket\langle y|\beta\rangle\rrbracket^{\overline{\lambda}} \;\triangleq\; \{\llbracket t'\rrbracket^{\overline{\lambda}}_\alpha\,\widehat{\alpha}\widehat{\setminus}x\}\langle y{\cdot}\beta\rangle \;\triangleq(gc_{\mathrm{L}})\; \langle y{\cdot}\beta\rangle \;=$
$\llbracket\langle y|\beta\rangle\rrbracket^{\overline{\lambda}} \;=\; \llbracket\langle y|\beta\rangle\{t'/x\}\rrbracket^{\overline{\lambda}}$

$(c=\langle t|\alpha\rangle)$: $\{\llbracket t'\rrbracket^{\overline{\lambda}}_\alpha\,\widehat{\alpha}\widehat{\setminus}x\}\llbracket\langle t|\beta\rangle\rrbracket^{\overline{\lambda}} \;\triangleq\; \{\llbracket t'\rrbracket^{\overline{\lambda}}_\alpha\,\widehat{\alpha}\widehat{\setminus}x\}\llbracket t\rrbracket^{\overline{\lambda}}_\beta \;=(ih)\; \llbracket t\{t'/x\}\rrbracket^{\overline{\lambda}}_\beta \;\triangleq$
$\llbracket\langle t\{t'/x\}\,|\,\beta\rangle\rrbracket^{\overline{\lambda}} \;\triangleq\; \llbracket\langle t|\beta\rangle\{t'/x\}\rrbracket^{\overline{\lambda}}$

$(c=\langle y|e\rangle)$: $\{\llbracket t'\rrbracket^{\overline{\lambda}}_\alpha\,\widehat{\alpha}\widehat{\setminus}x\}\llbracket\langle y|e\rangle\rrbracket^{\overline{\lambda}} \;\triangleq\; \{\llbracket t'\rrbracket^{\overline{\lambda}}_\alpha\,\widehat{\alpha}\widehat{\setminus}x\}\langle y{\cdot}e\rangle \;\triangleq\; \{\llbracket t'\rrbracket^{\overline{\lambda}}_\alpha\,\widehat{\alpha}\widehat{\setminus}x\}\llbracket e\rrbracket^{\overline{\lambda}}_y \;=(ih)$
$\llbracket e\{t'/x\}\rrbracket^{\overline{\lambda}}_y \;\triangleq\; \llbracket\langle y\,|\,e\{t'/x\}\rangle\rrbracket^{\overline{\lambda}} \;\triangleq\; \llbracket\langle y|e\rangle\{t'/x\}\rrbracket^{\overline{\lambda}}$

$(c=\langle y|t{\cdot}e\rangle)$: $\{\llbracket t'\rrbracket^{\overline{\lambda}}_\alpha\,\widehat{\alpha}\widehat{\setminus}x\}\llbracket\langle y|t{\cdot}e\rangle\rrbracket^{\overline{\lambda}} \;\triangleq\; \{\llbracket t'\rrbracket^{\overline{\lambda}}_\alpha\,\widehat{\alpha}\widehat{\setminus}x\}\llbracket t{\cdot}e\rrbracket^{\overline{\lambda}}_y \;\triangleq$
$\{\llbracket t'\rrbracket^{\overline{\lambda}}_\alpha\,\widehat{\alpha}\widehat{\setminus}x\}(\llbracket t\rrbracket^{\overline{\lambda}}_\gamma\,\widehat{\gamma}\,[y]\,\widehat{z}\,\llbracket e\rrbracket^{\overline{\lambda}}_z) \;\triangleq\; (\{\llbracket t'\rrbracket^{\overline{\lambda}}_\alpha\,\widehat{\alpha}\widehat{\setminus}x\}\llbracket t\rrbracket^{\overline{\lambda}}_\gamma)\,\widehat{\gamma}\,[y]\,\widehat{z}\,(\{\llbracket t'\rrbracket^{\overline{\lambda}}_\alpha\,\widehat{\alpha}\widehat{\setminus}x\}\llbracket e\rrbracket^{\overline{\lambda}}_z) \;=(ih)$
$\llbracket t\{t'/x\}\rrbracket^{\overline{\lambda}}_\gamma\,\widehat{\gamma}\,[y]\,\widehat{z}\,\llbracket e\{t'/x\}\rrbracket^{\overline{\lambda}}_z \;\triangleq\; \llbracket\langle y\,|\,t\{t'/x\}{\cdot}e\{t'/x\}\rangle\rrbracket^{\overline{\lambda}} \;=$
$\llbracket\langle y|t{\cdot}e\rangle\{t'/x\}\rrbracket^{\overline{\lambda}}$

$(c=\langle x|\tilde{\mu}z.c\rangle)$: $\{\llbracket t'\rrbracket^{\overline{\lambda}}_\alpha\,\widehat{\alpha}\widehat{\setminus}x\}\llbracket\langle x|\tilde{\mu}z.c\rangle\rrbracket^{\overline{\lambda}} \;\triangleq\; \{\llbracket t'\rrbracket^{\overline{\lambda}}_\alpha\,\widehat{\alpha}\widehat{\setminus}x\}\llbracket\tilde{\mu}z.c\rrbracket^{\overline{\lambda}}_x \;\triangleq$
$\{\llbracket t'\rrbracket^{\overline{\lambda}}_\alpha\,\widehat{\alpha}\widehat{\setminus}x\}(\llbracket c\rrbracket^{\overline{\lambda}}\{x/z\}) \;=\; \{\llbracket t'\rrbracket^{\overline{\lambda}}_\alpha\,\widehat{\alpha}\widehat{\setminus}x\}((\{\llbracket t'\rrbracket^{\overline{\lambda}}_\alpha\,\widehat{\alpha}\widehat{\setminus}x\}\llbracket c\rrbracket^{\overline{\lambda}})\{x/z\}) \;=(ih)$
$\{\llbracket t'\rrbracket^{\overline{\lambda}}_\alpha\,\widehat{\alpha}\widehat{\setminus}x\}(\llbracket c\{t'/x\}\rrbracket^{\overline{\lambda}}\{x/z\}) \;\triangleq\; \{\llbracket t'\rrbracket^{\overline{\lambda}}_\alpha\,\widehat{\alpha}\widehat{\setminus}x\}(\llbracket\tilde{\mu}z.c\{t'/x\}\rrbracket^{\overline{\lambda}}_x \;\triangleq$
$\{\llbracket t'\rrbracket^{\overline{\lambda}}_\alpha\,\widehat{\alpha}\widehat{\setminus}x\}(\llbracket\langle x\,|\,\tilde{\mu}z.c\{t'/x\}\rangle\rrbracket^{\overline{\lambda}} \;=(ih)\; \llbracket\langle t'\,|\,\tilde{\mu}z.c\{t'/x\}\rangle\rrbracket^{\overline{\lambda}} \;=$
$\llbracket\langle x|\tilde{\mu}z.c\rangle\{t'/x\}\rrbracket^{\overline{\lambda}}$

$(c=\langle y|\tilde{\mu}z.c\rangle,\, y\neq x)$: $\{\llbracket t'\rrbracket^{\overline{\lambda}}_\alpha\,\widehat{\alpha}\widehat{\setminus}x\}\llbracket\langle y|\tilde{\mu}z.c\rangle\rrbracket^{\overline{\lambda}} \;\triangleq\; \{\llbracket t'\rrbracket^{\overline{\lambda}}_\alpha\,\widehat{\alpha}\widehat{\setminus}x\}\llbracket\tilde{\mu}z.c\rrbracket^{\overline{\lambda}}_y \;\triangleq$
$\{\llbracket t'\rrbracket^{\overline{\lambda}}_\alpha\,\widehat{\alpha}\widehat{\setminus}x\}\llbracket c\rrbracket^{\overline{\lambda}}\{y/z\} \;=\; \{\llbracket t'\rrbracket^{\overline{\lambda}}_\alpha\,\widehat{\alpha}\widehat{\setminus}x\}\llbracket c\rrbracket^{\overline{\lambda}}\{y/z\} \;=(ih)\; \llbracket c\{t'/x\}\rrbracket^{\overline{\lambda}}\{z/y\} \;\triangleq$
$\llbracket\langle y\,|\,\tilde{\mu}z.c\{t'/x\}\rangle\rrbracket^{\overline{\lambda}} \;=\; \llbracket\langle y|\tilde{\mu}z.c\rangle\{t'/x\}\rrbracket^{\overline{\lambda}}$

$(c = \langle t|e\rangle)$: $\{[\![t']\!]^{\overline{\lambda}}_{\alpha}\,\widehat{\alpha}\,\dagger\,\widehat{x}\}[\![\langle t|e\rangle]\!]^{\overline{\lambda}} \triangleq \{[\![t']\!]^{\overline{\lambda}}_{\alpha}\,\widehat{\alpha}\,\dagger\,\widehat{x}\}([\![t]\!]^{\overline{\lambda}}_{\beta}\,\widehat{\beta}\dagger\widehat{y}\,[\![e]\!]^{\overline{\lambda}}_{y}) = (cut_{\text{L}})$
$(\{[\![t']\!]^{\overline{\lambda}}_{\alpha}\,\widehat{\alpha}\,\dagger\,\widehat{x}\}[\![t]\!]^{\overline{\lambda}}_{\beta})\,\widehat{\beta}\dagger\widehat{y}\,(\{[\![t']\!]^{\overline{\lambda}}_{\alpha}\,\widehat{\alpha}\,\dagger\,\widehat{x}\}[\![e]\!]^{\overline{\lambda}}_{y}) = (ih, x \neq y)$
$[\![t\{t'/x\}]\!]^{\overline{\lambda}}_{\beta}\,\widehat{\beta}\dagger\widehat{y}\,[\![e\{t'/x\}]\!]^{\overline{\lambda}}_{y} = [\![\langle t\{t'/x\}\,|\,e\{t'/x\}\rangle]\!]^{\overline{\lambda}} \triangleq [\![\langle t|e\rangle\{t'/x\}]\!]^{\overline{\lambda}}$

ii) $(t = x)$: $\{[\![t']\!]^{\overline{\lambda}}_{\alpha}\,\widehat{\alpha}\,\dagger\,\widehat{x}\}[\![x]\!]^{\overline{\lambda}}_{\beta} \triangleq \{[\![t']\!]^{\overline{\lambda}}_{\alpha}\,\widehat{\alpha}\,\dagger\,\widehat{x}\}\langle x\cdot\beta\rangle = (d_{\text{L}})\ [\![t']\!]^{\overline{\lambda}}_{\alpha}\{\beta/\alpha\} = [\![t']\!]^{\overline{\lambda}}_{\beta} \triangleq [\![x\{t'/x\}]\!]^{\overline{\lambda}}_{\beta}$

$(t = y \neq x)$: $\{[\![t']\!]^{\overline{\lambda}}_{\alpha}\,\widehat{\alpha}\,\dagger\,\widehat{x}\}[\![y]\!]^{\overline{\lambda}}_{\beta} \triangleq \{[\![t']\!]^{\overline{\lambda}}_{\alpha}\,\widehat{\alpha}\,\dagger\,\widehat{x}\}\langle y\cdot\beta\rangle = (gc_{\text{L}})\ \langle y\cdot\beta\rangle \triangleq [\![y]\!]^{\overline{\lambda}}_{\beta} \triangleq [\![y\{t'/x\}]\!]^{\overline{\lambda}}_{\beta}$

$(t = \lambda y.t'')$: $\{[\![t']\!]^{\overline{\lambda}}_{\alpha}\,\widehat{\alpha}\,\dagger\,\widehat{x}\}[\![\lambda y.t'']\!]^{\overline{\lambda}}_{\beta} \triangleq \{[\![t']\!]^{\overline{\lambda}}_{\alpha}\,\widehat{\alpha}\,\dagger\,\widehat{x}\}(\widehat{y}\,[\![t'']\!]^{\overline{\lambda}}_{\delta}\widehat{\delta}\cdot\beta) = (exp_{\text{L}})$
$\widehat{y}\,(\{[\![t']\!]^{\overline{\lambda}}_{\alpha}\,\widehat{\alpha}\,\dagger\,\widehat{x}\}[\![t'']\!]^{\overline{\lambda}}_{\delta})\,\widehat{\delta}\cdot\beta = (ih)\ \widehat{y}\,[\![t''\{t'/x\}]\!]^{\overline{\lambda}}_{\delta}\widehat{\delta}\cdot\beta \triangleq [\![\lambda y.t''\{t'/x\}]\!]^{\overline{\lambda}}_{\beta} \triangleq$
$[\![(\lambda y.t'')\{t'/x\}]\!]^{\overline{\lambda}}_{\beta}$

$(t = \mu\gamma.c)$: $\{[\![t']\!]^{\overline{\lambda}}_{\alpha}\,\widehat{\alpha}\,\dagger\,\widehat{x}\}[\![\mu\gamma.c]\!]^{\overline{\lambda}}_{\beta} \triangleq \{[\![t']\!]^{\overline{\lambda}}_{\alpha}\,\widehat{\alpha}\,\dagger\,\widehat{x}\}[\![c]\!]^{\overline{\lambda}}\{\beta/\gamma\} = (ih)$
$[\![c\{t'/x\}]\!]^{\overline{\lambda}}\{\beta/\gamma\} \triangleq [\![\mu\gamma.c\{t'/x\}]\!]^{\overline{\lambda}}_{\beta} \triangleq [\![(\mu\gamma.c)\{t'/x\}]\!]^{\overline{\lambda}}_{\beta}$

iii) $(e = \beta)$: $\{[\![t']\!]^{\overline{\lambda}}_{\alpha}\,\widehat{\alpha}\,\dagger\,\widehat{x}\}[\![\beta]\!]^{\overline{\lambda}}_{z} \triangleq \{[\![t']\!]^{\overline{\lambda}}_{\alpha}\,\widehat{\alpha}\,\dagger\,\widehat{x}\}\langle z\cdot\beta\rangle = (gc_{\text{L}})\ \langle z\cdot\beta\rangle \triangleq [\![z]\!]^{\overline{\lambda}}_{\beta} \triangleq [\![z\{t'/x\}]\!]^{\overline{\lambda}}_{\beta}$

$(e = t\cdot e)$: $\{[\![t']\!]^{\overline{\lambda}}_{\alpha}\,\widehat{\alpha}\,\dagger\,\widehat{x}\}[\![t\cdot e]\!]^{\overline{\lambda}}_{z} \triangleq \{[\![t']\!]^{\overline{\lambda}}_{\alpha}\,\widehat{\alpha}\,\dagger\,\widehat{x}\}([\![t]\!]^{\overline{\lambda}}_{\beta}\,\widehat{\beta}\,[z]\,\widehat{y}\,[\![e]\!]^{\overline{\lambda}}_{y}) = (imp\text{-}in_{\text{L}})$
$(\{[\![t']\!]^{\overline{\lambda}}_{\alpha}\,\widehat{\alpha}\,\dagger\,\widehat{x}\}[\![t]\!]^{\overline{\lambda}}_{\beta})\,\widehat{\beta}\,[z]\,\widehat{y}\,(\{[\![t']\!]^{\overline{\lambda}}_{\alpha}\,\widehat{\alpha}\,\dagger\,\widehat{x}\}[\![e]\!]^{\overline{\lambda}}_{y}) = (ih, x \neq y)$
$[\![t\{t'/x\}]\!]^{\overline{\lambda}}_{\beta}\,\widehat{\beta}\,[z]\,\widehat{y}\,[\![e\{t'/x\}]\!]^{\overline{\lambda}}_{y} \triangleq [\![t\{t'/x\}\cdot e\{t'/x\}]\!]^{\overline{\lambda}}_{z} \triangleq [\![(t\cdot e)\{t'/x\}]\!]^{\overline{\lambda}}_{z}$

$(e = \tilde{\mu}y.c)$: $\{[\![t']\!]^{\overline{\lambda}}_{x}\,\widehat{\alpha}\,\dagger\,\widehat{x}\}[\![\tilde{\mu}y.c]\!]^{\overline{\lambda}}_{z} \triangleq \{[\![t']\!]^{\overline{\lambda}}_{x}\,\widehat{\alpha}\,\dagger\,\widehat{x}\}[\![c]\!]^{\overline{\lambda}}\{z/y\} \triangleq (\{[\![t']\!]^{\overline{\lambda}}_{x}\,\widehat{\alpha}\,\dagger\,\widehat{x}\}[\![c]\!]^{\overline{\lambda}})\{z/y\}$
$= (ih)\ [\![c\{t'/x\}]\!]^{\overline{\lambda}}\{z/y\} \triangleq [\![\tilde{\mu}y.c\{t'/x\}]\!]^{\overline{\lambda}}_{z} \triangleq [\![(\tilde{\mu}y.c)\{t'/x\}]\!]^{\overline{\lambda}}_{z}$ □

As above, no reduction steps are used in this proof.

We now strengthen these results by stating that this encoding preserves evaluations:

Theorem 7.5 (SIMULATION OF $\rightarrow_{\overline{\lambda}}$) • *If* $c \rightarrow_{\overline{\lambda}} c'$ *then* $[\![c]\!]^{\overline{\lambda}} \rightarrow^{+}_{\mathcal{X}_{\text{IS}}} [\![c']\!]^{\overline{\lambda}}$.

- *If* $t \rightarrow_{\overline{\lambda}} t'$ *then* $[\![t]\!]^{\overline{\lambda}}_{\alpha} \rightarrow^{+}_{\mathcal{X}_{\text{IS}}} [\![c']\!]^{\overline{\lambda}}_{\alpha}$.
- *If* $e \rightarrow_{\overline{\lambda}} e'$ *then* $[\![e]\!]^{\overline{\lambda}}_{x} \rightarrow^{+}_{\mathcal{X}_{\text{IS}}} [\![e']\!]^{\overline{\lambda}}_{x}$.

Proof: Simultaneously by induction of the definition of $\rightarrow_{\overline{\lambda}}$.

(λ): $[\![\langle\lambda y.t_1|t_2\cdot e\rangle]\!]^{\overline{\lambda}} \triangleq (\widehat{y}\,[\![t_1]\!]^{\overline{\lambda}}_{\beta}\widehat{\beta}\cdot\alpha)\,\widehat{\alpha}\dagger\widehat{x}\,([\![t_2]\!]^{\overline{\lambda}}_{\gamma}\,\widehat{\gamma}\,[x]\,\widehat{z}\,[\![e]\!]^{\overline{\lambda}}_{z}) \rightarrow_{\mathcal{X}} (exp\text{-}imp)$
$[\![t_2]\!]^{\overline{\lambda}}_{\gamma}\,\widehat{\gamma}\dagger\widehat{y}\,([\![t_1]\!]^{\overline{\lambda}}_{\beta}\,\widehat{\beta}\dagger\widehat{z}\,[\![e]\!]^{\overline{\lambda}}_{z}) \triangleq [\![t_2]\!]^{\overline{\lambda}}_{\gamma}\,\widehat{\gamma}\dagger\widehat{y}\,[\![\langle t_1|e\rangle]\!]^{\overline{\lambda}} =_{\alpha} [\![t_2]\!]^{\overline{\lambda}}_{\gamma}\,\widehat{\gamma}\dagger\widehat{z}\,[\![\langle t_1|e\rangle]\!]^{\overline{\lambda}}\{z/y\} \triangleq$
$[\![t_2]\!]^{\overline{\lambda}}_{\gamma}\,\widehat{\gamma}\dagger\widehat{z}\,[\![\tilde{\mu}y.\langle t_1|e\rangle]\!]^{\overline{\lambda}}_{z} \triangleq [\![\langle t_2\,|\,\tilde{\mu}y.\langle t_1|e\rangle\rangle]\!]^{\overline{\lambda}}$

(μ): $[\![\langle\mu\beta.c|e\rangle]\!]^{\overline{\lambda}} \triangleq [\![\mu\beta.c]\!]^{\overline{\lambda}}_{\alpha}\,\widehat{\alpha}\dagger\widehat{x}\,[\![e]\!]^{\overline{\lambda}}_{x} \triangleq [\![c]\!]^{\overline{\lambda}}\{\alpha/\beta\}\,\widehat{\alpha}\dagger\widehat{x}\,[\![e]\!]^{\overline{\lambda}}_{x} =_{\alpha} (\alpha\ fresh)$
$[\![c]\!]^{\overline{\lambda}}_{\beta}\,\widehat{\beta}\dagger\widehat{x}\,[\![e]\!]^{\overline{\lambda}}_{x} \rightarrow_{\mathcal{X}} (sub_{\text{R}})\ [\![c]\!]^{\overline{\lambda}}\{\beta\not\widehat{x}\,[\![e]\!]^{\overline{\lambda}}_{x}\} = (7.3)\ [\![c\{e/\beta\}]\!]^{\overline{\lambda}}$

$(\tilde{\mu})$: $[\![\langle t|\tilde{\mu}y.c\rangle]\!]^{\overline{\lambda}} \triangleq [\![t]\!]^{\overline{\lambda}}_{\alpha}\,\widehat{\alpha}\dagger\widehat{x}\,[\![\tilde{\mu}y.c]\!]^{\overline{\lambda}}_{x} \triangleq [\![t]\!]^{\overline{\lambda}}_{\alpha}\,\widehat{\alpha}\dagger\widehat{x}\,[\![c]\!]^{\overline{\lambda}}\{x/y\} =_{\alpha} (x\ fresh)$
$[\![t]\!]^{\overline{\lambda}}_{\alpha}\,\widehat{\alpha}\dagger\widehat{y}\,[\![c]\!]^{\overline{\lambda}} \rightarrow_{\mathcal{X}} (sub_{\text{L}})\ \{[\![t]\!]^{\overline{\lambda}}_{\alpha}\,\widehat{\alpha}\,\dagger\,\widehat{y}\}[\![c]\!]^{\overline{\lambda}} = (7.4)\ [\![c\{t/y\}]\!]^{\overline{\lambda}}$

$(\eta\mu)$: $[\![\mu\alpha.\langle t|\alpha\rangle]\!]^{\overline{\lambda}}_{\beta} \triangleq [\![\langle t|\alpha\rangle]\!]^{\overline{\lambda}}\{\beta/\alpha\} \triangleq [\![t]\!]^{\overline{\lambda}}_{\gamma}\,\widehat{\gamma}\dagger\widehat{z}\,\langle z\cdot\alpha\rangle\{\beta/\alpha\} \triangleq [\![t]\!]^{\overline{\lambda}}_{\gamma}\,\widehat{\gamma}\dagger\widehat{z}\,\langle z\cdot\beta\rangle$

Now either γ is introduced in $[\![t]\!]^{\overline{\lambda}}_{\gamma}$, and we have

$$[\![t]\!]^{\overline{\lambda}}_{\gamma}\,\widehat{\gamma}\dagger\widehat{z}\,\langle z\cdot\beta\rangle \rightarrow_{\mathcal{X}} [\![t]\!]^{\overline{\lambda}}_{\beta}$$

by either (*cap*) or (*exp*), or it is not, and we have

$$[\![t]\!]^{\overline{\lambda}}_{\gamma}\,\widehat{\gamma}\dagger\widehat{z}\,\langle z\cdot\beta\rangle \rightarrow_{\mathcal{X}} (sub_{\text{R}})\ [\![t]\!]^{\overline{\lambda}}_{\gamma}\{\gamma\not\widehat{z}\,\langle z\cdot\beta\rangle\} = (4.6)\ [\![t]\!]^{\overline{\lambda}}_{\beta}$$

$(\eta\tilde{\mu})$: $[\![\tilde{\mu}x.\langle x|e\rangle]\!]^{\overline{\lambda}}_{y} \triangleq [\![\langle x|e\rangle]\!]^{\overline{\lambda}}\{y/x\} \triangleq (\langle x\cdot\gamma\rangle\,\widehat{\gamma}\dagger\widehat{z}\,[\![e]\!]^{\overline{\lambda}}_{z})\{y/x\} = \langle y\cdot\gamma\rangle\,\widehat{\gamma}\dagger\widehat{z}\,[\![e]\!]^{\overline{\lambda}}_{z}$

Now either z is introduced in $[\![e]\!]^{\overline{\lambda}}_{z}$, and we have

$$\langle y\cdot\alpha\rangle\,\widehat{\alpha}\dagger\widehat{z}\,[\![e]\!]^{\overline{\lambda}}_{z} \rightarrow_{\mathcal{X}} [\![e]\!]^{\overline{\lambda}}_{z}$$

by either (*cap*) or (*imp*), or it is not, and we have

$$\langle y\cdot\alpha\rangle\,\widehat{\alpha}\dagger\widehat{z}\,[\![e]\!]^{\overline{\lambda}}_{z} \rightarrow_{\mathcal{X}} (sub_{\text{L}})\ \{\langle y\cdot\alpha\rangle\,\widehat{\alpha}\,\dagger\,z\}[\![e]\!]^{\overline{\lambda}}_{z} = (4.6)\ [\![e]\!]^{\overline{\lambda}}_{y}$$

The contextual rules all follow by straightforward induction. □

Notice that we cannot model the η reduction rule $\lambda x.\mu\beta.\langle t|x\cdot\beta\rangle \rightarrow_{\mathcal{X}} t\ (x,\beta \notin fv(t))$, since the 'surrounding' λ-abstraction produces an export term that can never be removed; $\mathcal{X}_{\text{IS}}$ itself is not extensional.

We can also show that the CBN strategy is respected. We need to check that $(sub^{\text{N}}_{\text{R}})$ gets correctly applied, and no reduction takes place in the environment.

Theorem 7.6 (Simulation of $\rightarrow^{\text{N}}_{\overline{\lambda}}$) • *If $c \rightarrow^{\text{N}}_{\overline{\lambda}} c'$ then $[\![c]\!]^{\overline{\lambda}} \rightarrow^{\text{N}+}_{\mathcal{X}_{\text{IS}}} [\![c']\!]^{\overline{\lambda}}$.*

- *If $t \rightarrow^{\text{N}}_{\overline{\lambda}} t'$ then $[\![t]\!]^{\overline{\lambda}}_{\alpha} \rightarrow^{\text{N}+}_{\mathcal{X}_{\text{IS}}} [\![t']\!]^{\overline{\lambda}}_{\alpha}$.*

Proof: Simultaneously by induction of the definition of reduction. The proof is mostly as that of Thm. 7.5, which showed this result for full reduction; we will highlight the differences.

(λ): No CBN issues.

(μ): $[\![\langle\mu\beta.c|S\rangle]\!]^{\overline{\lambda}} \triangleq [\![\mu\beta.c]\!]^{\overline{\lambda}}_{\alpha}\,\widehat{\alpha} \dagger \widehat{x}\,[\![S]\!]^{\overline{\lambda}}_{x} \triangleq [\![c]\!]^{\overline{\lambda}}\{\alpha/\beta\}\,\widehat{\alpha} \dagger \widehat{x}\,[\![S]\!]^{\overline{\lambda}}_{x} =_{\alpha} [\![c]\!]^{\overline{\lambda}}\,\widehat{\beta} \dagger \widehat{x}\,[\![S]\!]^{\overline{\lambda}}_{x}$

Notice that $[\![S]\!]^{\overline{\lambda}}_{x}$ introduces x; also, $c = \langle t|e\rangle$, so $[\![c]\!]^{\overline{\lambda}}_{\beta}$ is a cut and does not introduce β, and we have

$$[\![c]\!]^{\overline{\lambda}}_{\beta}\,\widehat{\beta} \dagger \widehat{x}\,[\![S]\!]^{\overline{\lambda}}_{x} \rightarrow_{\mathcal{X}} (sub^{\text{N}}_{\text{R}})\ [\![c]\!]^{\overline{\lambda}}\{\beta \not\!\widehat{x}\,[\![S]\!]^{\overline{\lambda}}_{x}\} = (7.3)\ [\![c\{S/\beta\}]\!]^{\overline{\lambda}}$$

Since $[\![S]\!]^{\overline{\lambda}}_{x}$ introduces x, $(sub^{\text{N}}_{\text{R}})$ is permitted.

$(\tilde{\mu})$: No CBN issues.

$(\eta\mu)$: The (sub_{R}) step in the proof of Thm. 7.5 becomes $(sub^{\text{N}}_{\text{R}})$ since $\langle z\cdot\beta\rangle$ introduces z.

$(\eta\tilde{\mu})$: No CBN issues.

$(t \rightarrow^{\text{N}}_{\overline{\lambda}} t' \Rightarrow \langle t|e\rangle \rightarrow^{\text{N}}_{\overline{\lambda}} \langle t'|e\rangle)$: Reduction takes place inside a term in a cut, which is allowed in CBN.

$(c \rightarrow^{\text{N}}_{\overline{\lambda}} c' \Rightarrow \mu\alpha.c \rightarrow^{\text{N}}_{\overline{\lambda}} \mu\alpha.c')$: Reduction inside a cut is allowed in CBN.

The other rules are not part of $\rightarrow^{\text{N}}_{\overline{\lambda}}$. □

We can also show that the CBV strategy is respected. Now we need to check that now $(sub^{\text{V}}_{\text{L}})$ gets correctly applied, and again no reduction takes place in the environment.

Theorem 7.7 (Simulation of $\rightarrow^{\text{V}}_{\overline{\lambda}}$) • *If $c \rightarrow^{\text{V}}_{\overline{\lambda}} c'$ then $[\![c]\!]^{\overline{\lambda}} \rightarrow^{\text{V}+}_{\mathcal{X}_{\text{IS}}} [\![c']\!]^{\overline{\lambda}}$.*

- *If $t \rightarrow^{\text{V}}_{\overline{\lambda}} t'$ then $[\![t]\!]^{\overline{\lambda}}_{\alpha} \rightarrow^{\text{V}+}_{\mathcal{X}_{\text{IS}}} [\![c']\!]^{\overline{\lambda}}_{\alpha}$.*

Proof: Simultaneously by induction of the definition of reduction. The proof is mostly as that for Thm. 7.5; we will highlight the differences.

(λ): No CBV issues.

(μ): No CBV issues.

$(\tilde{\mu})$: $[\![\langle V|\tilde{\mu}y.c\rangle]\!]^{\overline{\lambda}} \triangleq [\![V]\!]^{\overline{\lambda}}_{\alpha}\,\widehat{\alpha} \dagger \widehat{x}\,[\![\tilde{\mu}y.c]\!]^{\overline{\lambda}}_{x} \triangleq [\![V]\!]^{\overline{\lambda}}_{\alpha}\,\widehat{\alpha} \dagger \widehat{x}\,[\![c]\!]^{\overline{\lambda}}\{x/y\} =_{\alpha} [\![V]\!]^{\overline{\lambda}}_{\alpha}\,\widehat{\alpha} \dagger \widehat{y}\,[\![c]\!]^{\overline{\lambda}}$

Notice that $[\![V]\!]^{\overline{\lambda}}_{\alpha}$ introduces α; also, $c = \langle t|e\rangle$, so $[\![c]\!]^{\overline{\lambda}}$ is a cut that does not introduce y, and we have

$$[\![V]\!]^{\overline{\lambda}}_{\alpha}\,\widehat{\alpha} \dagger \widehat{y}\,[\![c]\!]^{\overline{\lambda}} \rightarrow_{\mathcal{X}} (sub^{\text{V}}_{\text{L}})\ \{[\![V]\!]^{\overline{\lambda}}_{\alpha}\,\widehat{\alpha}\!\nwarrow\! y\}[\![c]\!]^{\overline{\lambda}} = (7.4)\ [\![c\{V/y\}]\!]^{\overline{\lambda}}$$

$(\eta\mu)$: No CBV issues.

$(\eta\tilde{\mu})$: The (sub_{L}) step in the proof of Thm. 7.5 becomes $(sub^{\text{V}}_{\text{L}})$ since $\langle y\cdot\alpha\rangle$ introduces α.

$(t \rightarrow_{\overline{\lambda}} t' \Rightarrow \langle t|e\rangle \rightarrow_{\overline{\lambda}} \langle t'|e\rangle)$: Allowed in CBV, as reduction takes place in a term in a cut.

$(c \rightarrow_{\overline{\lambda}} c' \Rightarrow \mu\alpha.c \rightarrow_{\overline{\lambda}} \mu\alpha.c')$: Allowed in CBV.

The other rules are not part of $\rightarrow^{\text{V}}_{\overline{\lambda}}$. □

So $[\![\cdot]\!]^{\overline{\lambda}}$, the natural encoding of $\overline{\lambda}\mu\tilde{\mu}$ in $\mathcal{X}$ (and $\mathcal{X}_{\text{IS}}$), that names the implicit term and context variables, strongly connects the two calculi: not only the standard $\overline{\lambda}\mu\tilde{\mu}$ reduction is embedded into the reduction of $\mathcal{X}_{\text{IS}}$, but also the CBN and CBV reduction strategies of $\overline{\lambda}\mu\tilde{\mu}$ are respected by their $\mathcal{X}_{\text{IS}}$ counterparts.

8 Embedding $\mathcal{X}_{\text{is}}$ in $\overline{\lambda}\mu\tilde{\mu}$

In this section we will study the reverse of the previous section, and investigate if the natural interpretation of $\mathcal{X}$ into $\overline{\lambda}\mu\tilde{\mu}$, as first suggested in [12], respects the three notions of reduction we focus on in this paper.

Definition 8.1 (Translation of $\mathcal{X}$ into $\overline{\lambda}\mu\tilde{\mu}$ [12, 23]) The interpretation of terms of $\mathcal{X}$ into commands of $\overline{\lambda}\mu\tilde{\mu}$ is defined by:

$$
\begin{array}{rcl}
\lceil \langle x{\cdot}\alpha\rangle \rfloor^{\mathcal{X}} & \triangleq & \langle x \,|\, \alpha\rangle \\
\lceil \widehat{x} P \widehat{\alpha} \cdot \beta \rfloor^{\mathcal{X}} & \triangleq & \langle \lambda x.\mu\alpha.\lceil P \rfloor^{\mathcal{X}} \,|\, \beta\rangle \\
\lceil P \widehat{\alpha}\,[y]\,\widehat{x} Q \rfloor^{\mathcal{X}} & \triangleq & \langle y \,|\, \mu\alpha.\lceil P \rfloor^{\mathcal{X}} \cdot \tilde{\mu}x.\lceil Q \rfloor^{\mathcal{X}} \rangle \\
\lceil P \widehat{\alpha} \dagger \widehat{x} Q \rfloor^{\mathcal{X}} & \triangleq & \langle \mu\alpha.\lceil P \rfloor^{\mathcal{X}} \,|\, \tilde{\mu}x.\lceil Q \rfloor^{\mathcal{X}} \rangle
\end{array}
$$

In fact, this is the origin of $\mathcal{X}$: in Remark 4.1 of [12], Curien and Herbelin give a hint on a way to connect $\mathsf{LK}_{\mu\tilde{\mu}}$ (as presented there) and LK. The proofs of LK embed in $\mathsf{LK}_{\mu\tilde{\mu}}$ by considering the following sub-syntax of $\overline{\lambda}\mu\tilde{\mu}$:

$$
c \;::=\; \langle x \mid \alpha\rangle \mid \langle \lambda x.\mu\alpha.c \mid \beta\rangle \mid \langle y \mid \mu\alpha.c{\cdot}\tilde{\mu}x.c\rangle \mid \langle \mu\alpha.c \mid \tilde{\mu}x.c\rangle
$$

Later it was discovered that this corresponded closely to Urban's approach in [31]; however, the approaches differ.

As can be expected, the interpretations from $\overline{\lambda}\mu\tilde{\mu}$ to $\mathcal{X}_{\text{IS}}$ and back are strongly related, as that they act as each other's inverse, albeit with some reductions involved, as can be expected. These mainly deal with converting implicit names to explicit names, as discussed above.

First we look at $\overline{\lambda}\mu\tilde{\mu} \mapsto \mathcal{X}_{\text{IS}} \mapsto \overline{\lambda}\mu\tilde{\mu}$, and show that $\lceil \cdot \rfloor^{\mathcal{X}}$ is $[\![\cdot]\!]^{\overline{\lambda}}$'s left-inverse up to extensionality. We will write $=_\eta$ for the equivalence relation on $\overline{\lambda}\mu\tilde{\mu}$ terms generated by rules $(\eta\mu)$ and $(\eta\tilde{\mu})$.

Theorem 8.2 *$\lceil [\![c]\!]^{\overline{\lambda}} \rfloor^{\mathcal{X}} =_\eta c$, $\mu\alpha.\lceil [\![t]\!]^{\overline{\lambda}}_{\alpha} \rfloor^{\mathcal{X}} =_\eta t$, and $\tilde{\mu}x.\lceil [\![e]\!]^{\overline{\lambda}}_{x} \rfloor^{\mathcal{X}} =_\eta e$.*

Proof: Simultaneous by induction.

- – $\lceil [\![\langle y|\beta\rangle]\!]^{\overline{\lambda}} \rfloor^{\mathcal{X}} = \lceil \langle y{\cdot}\beta\rangle \rfloor^{\mathcal{X}} = \langle y|\beta\rangle$
  – $\lceil [\![\langle \lambda y.t|\beta\rangle]\!]^{\overline{\lambda}} \rfloor^{\mathcal{X}} = \lceil [\![\lambda y.t]\!]^{\overline{\lambda}}_{\beta} \rfloor^{\mathcal{X}} = \langle \lambda y.\mu\gamma.\lceil [\![t]\!]^{\overline{\lambda}}_{\gamma} \rfloor^{\mathcal{X}} \,|\, \beta\rangle =_\eta (ih)\ \langle \lambda y.t|\beta\rangle$
  – $\lceil [\![\langle \mu\alpha.c|\beta\rangle]\!]^{\overline{\lambda}} \rfloor^{\mathcal{X}} = \lceil [\![\mu\alpha.c]\!]^{\overline{\lambda}}_{\beta} \rfloor^{\mathcal{X}} = \lceil [\![c]\!]^{\overline{\lambda}}\{\beta/\alpha\} \rfloor^{\mathcal{X}} = \lceil [\![c]\!]^{\overline{\lambda}} \rfloor^{\mathcal{X}}\{\beta/\alpha\} =_\eta (ih)$
  $c\{\beta/\alpha\}\ (\eta\mu) \leftarrow \langle \mu\alpha.c|\beta\rangle$
  – $\lceil [\![\langle y|t{\cdot}e\rangle]\!]^{\overline{\lambda}} \rfloor^{\mathcal{X}} = \lceil [\![t{\cdot}e]\!]^{\overline{\lambda}}_{y} \rfloor^{\mathcal{X}} = \lceil [\![t]\!]^{\overline{\lambda}}_{\gamma}\, \widehat{\gamma}\,[y]\,\widehat{z}\, [\![e]\!]^{\overline{\lambda}}_{z} \rfloor^{\mathcal{X}} = \langle y \,|\, \mu\gamma.\lceil [\![t]\!]^{\overline{\lambda}}_{\gamma} \rfloor^{\mathcal{X}}.\tilde{\mu}z.\lceil [\![e]\!]^{\overline{\lambda}}_{z} \rfloor^{\mathcal{X}} \rangle =_\eta (ih)\ \langle y|t{\cdot}e\rangle$
  – $\lceil [\![\langle y|\tilde{\mu}z.c\rangle]\!]^{\overline{\lambda}} \rfloor^{\mathcal{X}} = \lceil [\![\tilde{\mu}z.c]\!]^{\overline{\lambda}}_{y} \rfloor^{\mathcal{X}} = \lceil [\![c]\!]^{\overline{\lambda}}\{y/z\} \rfloor^{\mathcal{X}} = \lceil [\![c]\!]^{\overline{\lambda}} \rfloor^{\mathcal{X}}\{y/z\} =_\eta (ih)$
  $c\{y/z\}\ (\eta\tilde{\mu}) \leftarrow \langle y|\tilde{\mu}z.c\rangle$
  – $\lceil [\![\langle t|e\rangle]\!]^{\overline{\lambda}} \rfloor^{\mathcal{X}} = \lceil [\![t]\!]^{\overline{\lambda}}_{\alpha}\, \widehat{\alpha} \dagger \widehat{x}\, [\![e]\!]^{\overline{\lambda}}_{x} \rfloor^{\mathcal{X}} = \langle \mu\alpha.\lceil [\![t]\!]^{\overline{\lambda}}_{\alpha} \rfloor^{\mathcal{X}} \,|\, \tilde{\mu}x.\lceil [\![e]\!]^{\overline{\lambda}}_{x} \rfloor^{\mathcal{X}} \rangle =_\eta (ih)\ \langle t|e\rangle$
- – $\mu\alpha.\lceil [\![x]\!]^{\overline{\lambda}}_{\alpha} \rfloor^{\mathcal{X}} = \mu\alpha.\lceil \langle x{\cdot}\alpha\rangle \rfloor^{\mathcal{X}} = \mu\alpha.\langle x|\alpha\rangle \rightarrow_{\overline{\lambda}} (\eta\mu)\ x$
  – $\mu\alpha.\lceil [\![\lambda x.t]\!]^{\overline{\lambda}}_{\alpha} \rfloor^{\mathcal{X}} = \mu\alpha.\lceil \widehat{x}\, [\![t]\!]^{\overline{\lambda}}_{\beta}\, \widehat{\beta} \cdot \alpha \rfloor^{\mathcal{X}} = \mu\alpha.\langle \lambda x.\mu\beta.\lceil [\![t]\!]^{\overline{\lambda}}_{\beta} \rfloor^{\mathcal{X}} \,|\, \alpha\rangle =_\eta (ih)$
  $\mu\alpha.\langle \lambda x.t|\alpha\rangle \rightarrow_{\overline{\lambda}} (\eta\mu)\ \lambda x.t.$
  – $\mu\alpha.\lceil [\![\mu\beta.c]\!]^{\overline{\lambda}}_{\alpha} \rfloor^{\mathcal{X}} = \mu\alpha.\lceil [\![c]\!]^{\overline{\lambda}}\{\alpha/\beta\} \rfloor^{\mathcal{X}} = \mu\alpha.\lceil [\![c]\!]^{\overline{\lambda}} \rfloor^{\mathcal{X}}\{\beta/\alpha\} =_\eta (ih)\ \mu\alpha.c\{\beta/\alpha\} =_\alpha \mu\beta.c$
- – $\tilde{\mu}x.\lceil [\![\alpha]\!]^{\overline{\lambda}}_{x} \rfloor^{\mathcal{X}} = \tilde{\mu}x.\lceil \langle x{\cdot}\alpha\rangle \rfloor^{\mathcal{X}} = \tilde{\mu}x.\langle x|\alpha\rangle \rightarrow_{\overline{\lambda}} (\eta\tilde{\mu})\ \alpha$
  – $\tilde{\mu}x.\lceil [\![t{\cdot}e]\!]^{\overline{\lambda}}_{x} \rfloor^{\mathcal{X}} = \tilde{\mu}x.\lceil [\![t]\!]^{\overline{\lambda}}_{\gamma}\, \widehat{\gamma}\,[x]\,\widehat{z}\, [\![e]\!]^{\overline{\lambda}}_{z} \rfloor^{\mathcal{X}} = \tilde{\mu}x.\langle x \,|\, \mu\gamma.\lceil [\![t]\!]^{\overline{\lambda}}_{\gamma} \rfloor^{\mathcal{X}} \cdot \tilde{\mu}z.\lceil [\![e]\!]^{\overline{\lambda}}_{z} \rfloor^{\mathcal{X}} \rangle =_\eta (ih)$
  $\tilde{\mu}x.\langle x|t{\cdot}e\rangle \rightarrow_{\overline{\lambda}} (\eta\tilde{\mu})\ t{\cdot}e$
  – $\tilde{\mu}x.\lceil [\![\tilde{\mu}y.c]\!]^{\overline{\lambda}}_{x} \rfloor^{\mathcal{X}} = \tilde{\mu}x.\lceil [\![c]\!]^{\overline{\lambda}}\{x/y\} \rfloor^{\mathcal{X}} =_\alpha \tilde{\mu}x.\lceil [\![c]\!]^{\overline{\lambda}} \rfloor^{\mathcal{X}}\{x/y\} =_\eta (ih)\ \tilde{\mu}x.c\{x/y\} =_\alpha \tilde{\mu}y.c$ □

Notice that the only reduction steps needed here are $(\eta\mu)$ and $(\eta\tilde{\mu})$, in both directions, so the compositions of encodings gives identity modulo η-reduction, *i.e.* extensional equality.

Had we stuck to the interpretation as defined in [23], then reduction would have been involved.

Example 8.3
- $(\lceil \langle y{\cdot}\beta\rangle \rfloor^{x})^{\xi} = (\langle y|\beta\rangle)^{\xi} = (y)^{\xi}_{\alpha}\,\widehat{\alpha} \dagger \widehat{x}\,(\beta)^{\xi}_{x} = \langle y{\cdot}\alpha\rangle\,\widehat{\alpha} \dagger \widehat{x}\,\langle x{\cdot}\beta\rangle \rightarrow_{\mathcal{X}_{\text{IS}}} \langle y{\cdot}\beta\rangle$
- $(\lceil \widehat{y}P\widehat{\gamma}\cdot\beta \rfloor^{x})^{\xi} = (\langle \lambda y.\mu\gamma.\lceil P \rfloor^{x} | \beta\rangle)^{\xi} = (\lambda y.\mu\gamma.\lceil P \rfloor^{x})^{\xi}_{\alpha}\,\widehat{\alpha} \dagger \widehat{x}\,(\beta)^{\xi}_{x} =$
  $(\widehat{y}\,(\mu\gamma.\lceil P \rfloor^{x})^{\xi}_{\delta}\widehat{\delta}\cdot\alpha)\,\widehat{\alpha} \dagger \widehat{x}\,\langle x{\cdot}\beta\rangle \rightarrow_{\mathcal{X}_{\text{IS}}} \widehat{y}\,(\mu\gamma.\lceil P \rfloor^{x})^{\xi}_{\delta}\widehat{\delta}\cdot\beta = \widehat{y}\,(\lceil P \rfloor^{x}\{\delta/\gamma\})^{\xi}\widehat{\delta}\cdot\beta =_{\alpha}$
  $\widehat{y}\,(\lceil P \rfloor^{x})^{\xi}\widehat{\gamma}\cdot\beta \rightarrow^{*}_{\mathcal{X}_{\text{IS}}} (ih)\ \widehat{y}P\widehat{\gamma}\cdot\beta$
- $(\lceil P\widehat{\alpha}\,[y]\,\widehat{x}Q \rfloor^{x})^{\xi} = (\langle y\,|\,\mu\alpha.\lceil P \rfloor^{x}\cdot\tilde{\mu}x.\lceil Q \rfloor^{x}\rangle)^{\xi} = \langle y{\cdot}b\rangle\,\widehat{\beta} \dagger \widehat{z}\,((\lceil P \rfloor^{x})^{\xi}\,\widehat{\alpha}\,[z]\,\widehat{x}\,(\lceil Q \rfloor^{x})^{\xi}) \rightarrow_{\mathcal{X}_{\text{IS}}}$
  $(\lceil P \rfloor^{x})^{\xi}\,\widehat{\alpha}\,[y]\,\widehat{x}\,(\lceil Q \rfloor^{x})^{\xi} \rightarrow^{*}_{\mathcal{X}_{\text{IS}}} (ih)\ P\widehat{\alpha}\,[y]\,\widehat{x}Q$

Notice that, in these cases, the rules (*cap*), (*exp*), and (*imp*) are used.

Now we look at $\mathcal{X}_{\text{IS}} \mapsto \overline{\lambda}\mu\tilde{\mu} \mapsto \mathcal{X}_{\text{IS}}$, and show that $\lceil\cdot\rfloor^{x}$ is $[\![\cdot]\!]^{\overline{\lambda}}$'s right-inverse.

Theorem 8.4 $[\![\lceil P \rfloor^{x}]\!]^{\overline{\lambda}} = P$.

Proof: Simultaneous by induction.

- $[\![\lceil \langle y{\cdot}\beta\rangle \rfloor^{x}]\!]^{\overline{\lambda}} \triangleq [\![\langle y|\beta\rangle]\!]^{\overline{\lambda}} \triangleq \langle y{\cdot}\beta\rangle$
- $[\![\lceil \widehat{y}P\widehat{\gamma}\cdot\beta \rfloor^{x}]\!]^{\overline{\lambda}} \triangleq [\![\langle \lambda y.\mu\gamma.\lceil P \rfloor^{x} | \beta\rangle]\!]^{\overline{\lambda}} \triangleq [\![\lambda y.\mu\gamma.\lceil P \rfloor^{x}]\!]^{\overline{\lambda}}_{\beta} \triangleq \widehat{y}\,[\![\mu\gamma.\lceil P \rfloor^{x}]\!]^{\overline{\lambda}}_{\delta}\widehat{\delta}\cdot\beta \triangleq$
  $\widehat{y}\,[\![\lceil P \rfloor^{x}\{\delta/\gamma\}]\!]^{\overline{\lambda}}\widehat{\delta}\cdot\beta =_{\alpha} \widehat{y}\,[\![\lceil P \rfloor^{x}]\!]^{\overline{\lambda}}\widehat{\gamma}\cdot\beta = (ih)\ \widehat{y}P\widehat{\gamma}\cdot\beta$
- $[\![\lceil P\widehat{\alpha}\,[y]\,\widehat{x}Q \rfloor^{x}]\!]^{\overline{\lambda}} \triangleq [\![\langle y\,|\,\mu\alpha.\lceil P \rfloor^{x}\cdot\tilde{\mu}x.\lceil Q \rfloor^{x}\rangle]\!]^{\overline{\lambda}} \triangleq [\![\lceil P \rfloor^{x}]\!]^{\overline{\lambda}}\,\widehat{\alpha}\,[y]\,\widehat{x}\,[\![\lceil Q \rfloor^{x}]\!]^{\overline{\lambda}} = (ih)\ P\widehat{\alpha}\,[y]\,\widehat{x}Q$
- $[\![\lceil P\widehat{\alpha} \dagger \widehat{x}Q \rfloor^{x}]\!]^{\overline{\lambda}} \triangleq [\![\langle \mu\alpha.\lceil P \rfloor^{x}\,|\,\tilde{\mu}x.\lceil Q \rfloor^{x}\rangle]\!]^{\overline{\lambda}} \triangleq [\![\mu\alpha.\lceil P \rfloor^{x}]\!]^{\overline{\lambda}}_{\beta}\,\widehat{\beta} \dagger \widehat{z}\,[\![\tilde{\mu}x.\lceil Q \rfloor^{x}]\!]^{\overline{\lambda}}_{z} \triangleq$
  $[\![\lceil P \rfloor^{x}\{\beta/\alpha\}]\!]^{\overline{\lambda}}\,\widehat{\beta} \dagger \widehat{z}\,[\![\lceil Q \rfloor^{x}\{z/x\}]\!]^{\overline{\lambda}} =_{\alpha} [\![\lceil P \rfloor^{x}]\!]^{\overline{\lambda}}\,\widehat{\alpha} \dagger \widehat{x}\,[\![\lceil Q \rfloor^{x}]\!]^{\overline{\lambda}} = (ih)\ P\widehat{\alpha} \dagger \widehat{x}Q$ □

We will now show that reduction in $\mathcal{X}_{\text{IS}}$ is respected by the interpretation $\lceil\cdot\rfloor^{x}$, for which, as suggested above, we need to extend $\overline{\lambda}\mu\tilde{\mu}$.

Definition 8.5 (Extended $\overline{\lambda}\mu\tilde{\mu}$) We define $\overline{\lambda}\mu\tilde{\mu}^{\mathrm{E}}$ by adding the rule:

$$(\lambda') : \ \langle \lambda y.t | t'{\cdot}e\rangle \ \rightarrow\ \langle \mu\gamma.\langle t'\,|\,\tilde{\mu}y.\langle t|\gamma\rangle\rangle\,|\,e\rangle \quad (\gamma\ \textit{fresh})$$

We will first show that the two implicit substitutions of $\mathcal{X}_{\text{IS}}$ are respected by the interpretation; we need to involve reduction for these results as well, but in *both* directions.

We will write $=_{\mu\tilde{\mu}}$ for the equivalence relation on $\overline{\lambda}\mu\tilde{\mu}$ generated by rule (μ) and $(\tilde{\mu}$.

Lemma 8.6 $\lceil Q\{\alpha\not\widehat{x}P\} \rfloor^{x} =_{\mu\tilde{\mu}} \lceil Q \rfloor^{x}\{\tilde{\mu}x.\lceil P \rfloor^{x}/\alpha\}$.

Proof: By induction on the structure of terms in $\mathcal{X}_{\text{IS}}$.

$(Q = \langle y{\cdot}\beta\rangle)$: We have two cases:

$(\alpha = \beta)$: $\lceil \langle y{\cdot}\alpha\rangle\{\alpha\not\widehat{x}P\} \rfloor^{x} \triangleq \lceil P\{y/x\} \rfloor^{x} = \lceil P \rfloor^{x}\{y/x\}\ \ (\tilde{\mu})_{\overline{\lambda}}\!\leftarrow\ \langle y\,|\,\tilde{\mu}x.\lceil P \rfloor^{x}\rangle =$
$\langle y|\alpha\rangle\,\{\tilde{\mu}x.\lceil P \rfloor^{x}/\alpha\} \triangleq \lceil \langle y{\cdot}\alpha\rangle \rfloor^{x}\{\tilde{\mu}x.\lceil P \rfloor^{x}/\alpha\}$

$(\alpha \neq \beta)$: $\lceil \langle y{\cdot}\beta\rangle\{\alpha\not\widehat{x}P\} \rfloor^{x} \triangleq \lceil \langle y{\cdot}\beta\rangle \rfloor^{x} \triangleq \lceil \langle y{\cdot}\beta\rangle \rfloor^{x}\{\tilde{\mu}x.\lceil P \rfloor^{x}/\alpha\}$

$(Q = \widehat{y}R\widehat{\beta}\cdot\alpha)$: $\lceil (\widehat{y}R\widehat{\beta}\cdot\alpha)\{\alpha\not\widehat{x}P\} \rfloor^{x} \triangleq \lceil (\widehat{y}\,(R\{\alpha\not\widehat{x}P\})\widehat{\beta}\cdot\gamma)\,\widehat{\gamma} \dagger \widehat{x}P \rfloor^{x} \triangleq$
$\langle \mu\gamma.\langle \lambda y.\mu\beta.\lceil R\{\alpha\not\widehat{x}P\} \rfloor^{x} | \gamma\rangle\,|\,\tilde{\mu}x.\lceil P \rfloor^{x}\rangle \rightarrow_{\overline{\lambda}} (\mu)\ \langle \lambda y.\mu\beta.\lceil R\{\alpha\not\widehat{x}P\} \rfloor^{x}\,|\,\tilde{\mu}x.\lceil P \rfloor^{x}\rangle =_{\mu\tilde{\mu}} (ih)$
$\langle \lambda y.\mu\beta.\lceil R \rfloor^{x}\{\tilde{\mu}x.\lceil P \rfloor^{x}/\alpha\}\,|\,\tilde{\mu}x.\lceil P \rfloor^{x}\rangle = \langle \lambda y.\mu\beta.\lceil R \rfloor^{x} | \alpha\rangle\,\{\tilde{\mu}x.\lceil P \rfloor^{x}/\alpha\} \triangleq$
$\lceil \widehat{y}R\widehat{\beta}\cdot\alpha \rfloor^{x}\{\tilde{\mu}x.\lceil P \rfloor^{x}/\alpha\}$

$(Q = \widehat{y}R\widehat{\beta}\cdot\gamma,\ \textit{with}\ \gamma \neq \alpha)$: $\lceil (\widehat{y}R\widehat{\beta}\cdot\gamma)\{\alpha\not\widehat{x}P\} \rfloor^{x} \triangleq \lceil \widehat{y}\,(R\{\alpha\not\widehat{x}P\})\widehat{\beta}\cdot\gamma \rfloor^{x} \triangleq$
$\langle \lambda y.\mu\beta.\lceil R\{\alpha\not\widehat{x}P\} \rfloor^{x} | \gamma\rangle =_{\mu\tilde{\mu}} (ih)\ \langle \lambda y.\mu\beta.\lceil R \rfloor^{x}\{\tilde{\mu}x.\lceil P \rfloor^{x}/\alpha\}\,|\,\gamma\rangle \triangleq \langle \lambda y.\mu\beta.\lceil R \rfloor^{x} | \gamma\rangle\,\{\tilde{\mu}x.\lceil P \rfloor^{x}/\alpha\}$

The other cases follow, as the last one, by induction. □

Notice that reduction is limited to two steps, using rules $(\tilde{\mu})$ and (μ), in both directions.

Similarly, we have:

Lemma 8.7 $\lceil \{P\widehat{\alpha}\not\searrow x\}Q \rfloor^{x} =_{\mu\tilde{\mu}} \lceil Q \rfloor^{x}\{\mu\alpha.\lceil P \rfloor^{x}/x\}$.

Proof: By induction on the structure of terms.

$(Q = \langle y{\cdot}\beta\rangle)$: We have two cases:

$(y = x)$: $\lceil\{P\widehat{\alpha}\nwarrow x\}\langle x{\cdot}\beta\rangle\rfloor^{x} \triangleq \lceil P\{\beta/\alpha\}\rfloor^{x} = \lceil P\rfloor^{x}\{\beta/\alpha\}\ (\mu)_{\overline{\lambda}}\!\leftarrow \langle \mu\alpha.\lceil P\rfloor^{x} \,|\, \beta\rangle =$
$\langle x\,|\,\beta\rangle\{\mu\alpha.\lceil P\rfloor^{x}/x\} \triangleq \lceil\langle x{\cdot}\beta\rangle\rfloor^{x}\{\mu\alpha.\lceil P\rfloor^{x}/x\}$

$(y \neq x)$: $\lceil\{P\widehat{\alpha}\nwarrow x\}\langle y{\cdot}\beta\rangle\rfloor^{x} \triangleq \lceil\langle y{\cdot}\beta\rangle\rfloor^{x} \triangleq \lceil\langle y{\cdot}\beta\rangle\rfloor^{x}\{\mu\alpha.\lceil P\rfloor^{x}/x\}$

$(Q = R\widehat{\beta}\,[x]\,\widehat{y}S)$: $\lceil\{P\widehat{\alpha}\nwarrow x\}(R\widehat{\beta}\,[x]\,\widehat{y}S)\rfloor^{x} \triangleq$

$$
\begin{array}{ll}
\lceil P\widehat{\alpha}\dagger\widehat{z}\,((\{P\widehat{\alpha}\nwarrow x\}R)\,\widehat{\beta}\,[z]\,\widehat{y}\,(\{P\widehat{\alpha}\nwarrow x\}S))\rfloor^{x} & \triangleq \\
\langle \mu\alpha.\lceil P\rfloor^{x} \,|\, \tilde{\mu}z.\langle z\,|\,\mu\beta.\lceil\{P\widehat{\alpha}\nwarrow x\}R\rfloor^{x}\cdot\tilde{\mu}y.\lceil\{P\widehat{\alpha}\nwarrow x\}S\rfloor^{x}\rangle\rangle & \to_{\overline{\lambda}}(\tilde{\mu}) \\
\langle \mu\alpha.\lceil P\rfloor^{x} \,|\, \mu\beta.\lceil\{P\widehat{\alpha}\nwarrow x\}R\rfloor^{x}\cdot\tilde{\mu}y.\lceil\{P\widehat{\alpha}\nwarrow x\}S\rfloor^{x}\rangle & =_{\mu\tilde{\mu}}(ih) \\
\langle \mu\alpha.\lceil P\rfloor^{x} \,|\, \mu\beta.\lceil R\rfloor^{x}\{\mu\alpha.\lceil P\rfloor^{x}/x\}\cdot\tilde{\mu}y.[\![S]\!]^{s}_{}\{\mu\alpha.\lceil P\rfloor^{x}/x\}\rangle & = \\
\langle x\,|\,\mu\beta.\lceil R\rfloor^{x}\cdot\tilde{\mu}y.\lceil S\rfloor^{x}\rangle\{\mu\alpha.\lceil P\rfloor^{x}/x\} & = \ \lceil R\widehat{\beta}\,[x]\,\widehat{y}S\rfloor^{x}\{\mu\alpha.\lceil P\rfloor^{x}/x\}
\end{array}
$$

The other cases follow by induction. □

Also here reduction is limited to two steps, using rules $(\tilde{\mu})$ and (μ), but in opposite direction with respect to the previous proof.

We can now show that the interpretation respects reduction.

Theorem 8.8 *If* $P \to_{\mathcal{X}} Q$, *then* $\lceil P\rfloor^{x} =_{\overline{\lambda}} \lceil Q\rfloor^{x}$.

Proof: (cap): $\lceil\langle y{\cdot}\alpha\rangle\,\widehat{\alpha}\dagger\widehat{x}\,\langle x{\cdot}\beta\rangle\rfloor^{x} \triangleq \langle \mu\alpha.\langle y|\alpha\rangle \,|\, \tilde{\mu}x.\langle x|\beta\rangle\rangle \to(\mu)\ \langle y\,|\,\tilde{\mu}x.\langle x|\beta\rangle\rangle \to(\tilde{\mu})$
$\langle y|\beta\rangle \triangleq \lceil\langle y{\cdot}\beta\rangle\rfloor^{x}$

(exp): $\lceil(\widehat{y}P\widehat{\beta}\cdot\alpha)\,\widehat{\alpha}\dagger\widehat{x}\,\langle x{\cdot}\gamma\rangle\rfloor^{x} \triangleq \langle \mu\alpha.\langle \lambda y.\mu\beta.\lceil P\rfloor^{x}|\alpha\rangle \,|\, \tilde{\mu}x.\langle x|\gamma\rangle\rangle \to(\mu)$
$\langle \lambda y.\mu\beta.\lceil P\rfloor^{x} \,|\, \tilde{\mu}x.\langle x|\gamma\rangle\rangle \to(\tilde{\mu})\ \langle \lambda y.\mu\beta.\lceil P\rfloor^{x}\,|\,\gamma\rangle \triangleq \lceil\widehat{y}P\widehat{\beta}\cdot\gamma\rfloor^{x}$

(imp): $\lceil\langle y{\cdot}\alpha\rangle\,\widehat{\alpha}\dagger\widehat{x}\,(Q\widehat{\beta}\,[x]\,\widehat{z}R)\rfloor^{x} \triangleq \langle \mu\alpha.\langle y|\alpha\rangle \,|\, \tilde{\mu}x.\langle x\,|\,\mu\beta.\lceil Q\rfloor^{x}\cdot\tilde{\mu}z.\lceil R\rfloor^{x}\rangle\rangle \to(\mu)$
$\langle y\,|\,\tilde{\mu}x.\langle x\,|\,\mu\beta.\lceil Q\rfloor^{x}\cdot\tilde{\mu}z.\lceil R\rfloor^{x}\rangle\rangle \to(\tilde{\mu})\ \langle y\,|\,\mu\beta.\lceil Q\rfloor^{x}\cdot\tilde{\mu}z.\lceil R\rfloor^{x}\rangle \triangleq \lceil Q\widehat{\beta}\,[y]\,\widehat{z}R\rfloor^{x}$

$(exp\text{-}imp)$: $\lceil(\widehat{y}P\widehat{\beta}\cdot\alpha)\,\widehat{\alpha}\dagger\widehat{x}\,(Q\widehat{\gamma}\,[x]\,\widehat{z}R)\rfloor^{x} \triangleq$
$\langle \mu\alpha.\langle \lambda y.\mu\beta.\lceil P\rfloor^{x}|\alpha\rangle \,|\, \tilde{\mu}x.\langle x\,|\,\mu\gamma.\lceil Q\rfloor^{x}\cdot\tilde{\mu}z.\lceil R\rfloor^{x}\rangle\rangle \to(\mu)$
$\langle \lambda y.\mu\beta.\lceil P\rfloor^{x} \,|\, \tilde{\mu}x.\langle x\,|\,\mu\gamma.\lceil Q\rfloor^{x}\cdot\tilde{\mu}z.\lceil R\rfloor^{x}\rangle\rangle \to(\tilde{\mu})\ \langle \lambda y.\mu\beta.\lceil P\rfloor^{x}\,|\,\mu\gamma.\lceil Q\rfloor^{x}\cdot\tilde{\mu}z.\lceil R\rfloor^{x}\rangle \to(\lambda)$
$\langle \mu\gamma.\lceil Q\rfloor^{x} \,|\, \tilde{\mu}y.\langle \mu\beta.\lceil P\rfloor^{x} \,|\, \tilde{\mu}z.\lceil R\rfloor^{x}\rangle\rangle \triangleq \lceil Q\widehat{\gamma}\dagger\widehat{y}\,(P\widehat{\beta}\dagger\widehat{z}R)\rfloor^{x}$

$(exp\text{-}imp)$: $\lceil(\widehat{y}P\widehat{\beta}\cdot\alpha)\,\widehat{\alpha}\dagger\widehat{x}\,(Q\widehat{\gamma}\,[x]\,\widehat{z}R)\rfloor^{x} \triangleq$
$\langle \mu\alpha.\langle \lambda y.\mu\beta.\lceil P\rfloor^{x}|\alpha\rangle \,|\, \tilde{\mu}x.\langle x\,|\,\mu\gamma.\lceil Q\rfloor^{x}\cdot\tilde{\mu}z.\lceil R\rfloor^{x}\rangle\rangle \to(\mu)$
$\langle \lambda y.\mu\beta.\lceil P\rfloor^{x} \,|\, \tilde{\mu}x.\langle x\,|\,\mu\gamma.\lceil Q\rfloor^{x}\cdot\tilde{\mu}z.\lceil R\rfloor^{x}\rangle\rangle \to(\tilde{\mu})\ \langle \lambda y.\mu\beta.\lceil P\rfloor^{x}\,|\,\mu\gamma.\lceil Q\rfloor^{x}\cdot\tilde{\mu}z.\lceil R\rfloor^{x}\rangle \to(\lambda')$
$\langle \mu\delta.\langle \mu\gamma.\lceil Q\rfloor^{x} \,|\, \tilde{\mu}y.\langle \mu\beta.\lceil P\rfloor^{x}|\delta\rangle\rangle \,|\, \tilde{\mu}z.\lceil R\rfloor^{x}\rangle \to(\mu)\ \langle \mu\delta.\langle \mu\gamma.\lceil Q\rfloor^{x} \,|\, \tilde{\mu}y.\lceil P\rfloor^{x}\{\delta/\beta\}\rangle \,|\, \tilde{\mu}z.\lceil R\rfloor^{x}\rangle =_{\alpha}$
$\langle \mu\beta.\langle \mu\gamma.\lceil Q\rfloor^{x} \,|\, \tilde{\mu}y.\lceil P\rfloor^{x}\rangle \,|\, \tilde{\mu}z.\lceil R\rfloor^{x}\rangle \triangleq \lceil(Q\widehat{\gamma}\dagger\widehat{y}P)\,\widehat{\beta}\dagger\widehat{z}R\rfloor^{x}$

(sub_{R}): $\lceil P\widehat{\alpha}\dagger\widehat{x}Q\rfloor^{x} \triangleq \langle \mu\alpha.\lceil P\rfloor^{x} \,|\, \tilde{\mu}x.\lceil Q\rfloor^{x}\rangle \to(\mu)\ \lceil P\rfloor^{x}\{\tilde{\mu}x.\lceil Q\rfloor^{x}/\alpha\} =_{\mu\tilde{\mu}}(8.6)\ \lceil P\{\alpha\nearrow\widehat{x}Q\}\rfloor^{x}$

(sub_{L}): $\lceil P\widehat{\alpha}\dagger\widehat{x}Q\rfloor^{x} \triangleq \langle \mu\alpha.\lceil P\rfloor^{x} \,|\, \tilde{\mu}x.\lceil Q\rfloor^{x}\rangle \to(\tilde{\mu})\ \lceil Q\rfloor^{x}\{\mu\alpha.\lceil P\rfloor^{x}/x\} =_{\mu\tilde{\mu}}(8.7)\ \lceil\{P\widehat{\alpha}\nwarrow x\}Q\rfloor^{x}$ □

Notice that in the first four cases we can swap the (μ) and $(\tilde{\mu})$ reduction steps.

Since reduction is used in the simulation of substitution (Lem. 8.6 and 8.7), we cannot show a similar result for the CBN and CBV reduction strategies. Also, the (μ)-reduction step in the second $(exp\text{-}imp)$ case (over β) takes place in the environment, which would not be allowed in either CBN or CBV, which strengthens our choice to exclude the second alternative of rule $(exp\text{-}imp)$ for both those strategies on $\mathcal{X}_{\text{IS}}$.

This means that, even when changing the active cuts of $\mathcal{X}$ into substitution, and the strong relationship we have established between $\overline{\lambda}\mu\tilde{\mu}$ and $\mathcal{X}_{\text{IS}}$, these calculi are fundamentally different. The absence of implicit variables and names gives $\mathcal{X}_{\text{IS}}$ a more direct control over cut-elimination, and mapping $\mathcal{X}_{\text{IS}}$'s substitution onto $\overline{\lambda}\mu\tilde{\mu}$ creates additional (μ) and $(\tilde{\mu})$ redexes.

Conclusion and Future Work

This paper has presented mappings from $\mathrm{s}\lambda\mu$ to $\overline{\lambda}\mu\tilde{\mu}$, and from $\overline{\lambda}\mu\tilde{\mu}$ to $\mathcal{X}_{\mathrm{IS}}$, which preserve the CBN and CBV reduction strategies. Furthermore, these mappings are strict in the sense that if $M \to N$, then $[\![M]\!]$ and $[\![N]\!]$ are joinable with t such that $[\![M]\!] \to^{+}_{\overline{\lambda}} t$; otherwise put, no reductions are 'lost' in the mapping. It follows from these results that there are mappings from λ-calculus into $\overline{\lambda}\mu\tilde{\mu}$ and $\mathcal{X}$ which preserve CBN and CBV.

Other reduction disciplines: Our focus has been on preserving the CBN and CBV reduction strategies, as these are the most commonly considered. It should be interesting to see if these mappings preserve other evaluation disciplines, such as call-by-need or even call-by-co-need. Of course, this would first require defining call-by-need and co-need for $\mathcal{X}$, although these definitions already exist for $\mathrm{s}\lambda\mu$ and $\overline{\lambda}\mu\tilde{\mu}$. Such definitions may even follow from our translation into $\mathcal{X}$.

μ-reductions: It may be surprising that a translation from $\lambda\mu$ to $\overline{\lambda}\mu\tilde{\mu}$ such as the one we defined here exists; as mentioned in Sect. 5, the nature of μ reductions in both is distinct. Although μ-reductions in $\mathrm{s}\lambda\mu$ are often understood to capture their context, they need not capture their entire context. Consider the term $M\,(\mu\alpha.C)\,(\mu\beta.C')$, and the reduction sequence,

$$\begin{array}{rcl} M\,(\mu\alpha.C)\,(\mu\beta.C') & \to & (\mu\gamma.C\{M\cdot\gamma/\alpha\})\,(\mu\beta.C') \\ & \to & (\mu\delta.C'\{(\mu\gamma.C[\{M\cdot\gamma/\alpha\}])\cdot\delta/\beta\}) \end{array}$$

The context of the subterm $\mu\alpha.C$ is $(M\lceil\,\rfloor)\,(\mu\beta.C')$, yet only M is captured by $\mu\alpha.C$. The μ-reductions in $\lambda\mu$ are performed term-by-term, which allows for the continuation to be captured piecewise instead of at-once; this is in contrast with μ in $\overline{\lambda}\mu\tilde{\mu}$, which necessarily substitutes its entire continuation in a command.

Furthermore, there is no direct analogue of the (μ_{L}) reduction in $\overline{\lambda}\mu\tilde{\mu}$; the term $\mu\alpha.\langle t \,|\, \mu\beta.c\cdot\alpha\rangle$ does not have a (head) redex, whereas its $\mathrm{s}\lambda\mu$ analogue $[\alpha]M(\mu\beta.C)$ is reducible;

$$[\alpha]M(\mu\beta.\mathrm{C}) \;\to\; [\alpha]\mu\gamma.\{M\cdot\gamma/\beta\}\mathrm{C} \;\to\; \{M\cdot\gamma/\beta\}\mathrm{C}\{\alpha/\gamma\} \;=\; \{M\cdot\alpha/\beta\}\mathrm{C}$$

(notice that γ does not appear in C).

The translation of Def. 7.1 circumvents this by introducing extra redexes around the applicands so that its image reduces:

$$\begin{array}{rcl} [\![[\alpha]M\,(\mu\beta.C)]\!]^{\mathrm{s}} & = & \langle\mu\gamma.\langle[\![M]\!]^{\mathrm{s}} \,|\, \tilde{\mu}x.\langle\mu\beta.[\![C]\!]^{\mathrm{s}} \,|\, \tilde{\mu}y.\langle x|y\cdot\gamma\rangle\rangle\rangle \,|\, \alpha\rangle \\ & \to^{*} & \langle\mu\beta.[\![C]\!]^{\mathrm{s}} \,|\, \tilde{\mu}y.\langle[\![M]\!]^{\mathrm{s}}|y\cdot\alpha\rangle\rangle, \end{array}$$

which then allows for the (μ_{L}) reduction to be simulated.

An alternative solution could be to add a (μ_{L}) reduction to $\overline{\lambda}\mu\tilde{\mu}$,

$$\langle t \,|\, (\mu\beta.c)\cdot e\rangle \;\to\; c\{\tilde{\mu}x.\langle t|x\cdot e\rangle/\beta\}$$

This would then allow for (μ_{L}) to be directly reflected, rather than just simulated.

Relating Semantics per-discipline: The translations could provide a unified way to relate the by-name and by-value semantics of each calculi. For example, the operational semantics of $\lambda\mu$ in an abstract machine (KAM) could be directly translated into $\overline{\lambda}\mu\tilde{\mu}$ (and thus $\mathcal{X}$), making $\overline{\lambda}\mu\tilde{\mu}$ and $\mathcal{X}$ capable of simulating CBN and CBV $\lambda\mu$-machines in a uniform way. This could furthermore be used to inspect the behaviour of classical realizability models of $\lambda\mu$ through $\overline{\lambda}\mu\tilde{\mu}$ and $\mathcal{X}$.

In a similar vein, one obtains CPS translations from $\lambda\mu$ into the λ-calculus by combining $[\![\cdot]\!]$ with the CPS translations of $\overline{\lambda}\mu\tilde{\mu}$ given by Curien and Herbelin [12]. The resulting translations agree up to equality with the CPS translations of $\lambda\mu$ given in [12], however that given by our mapping has extra redexes that are 'clerical' in nature. These precisely come from the extra $\tilde{\mu}$ redex given in $[\![MN]\!]$ when compared with $(MN)^v$ and $(MN)^n$. This would give the operational semantics obtained as suggested subtly different intensional properties.

Type Systems: It was relatively simple to show the simple type systems of s$\lambda\mu$ and $\overline{\lambda}\mu\tilde{\mu}$ are respected by the translations. The case for polymorphic types would be more subtle; in the presence of control, restrictions are needed to determine when a term can have a polymorphic type [17, 19]. The translation must be ensured to preserve the appropriate restrictions.

The existence of a mapping of s$\lambda\mu$ into $\overline{\lambda}\mu\tilde{\mu}$ means the former probably cannot be given a sound and complete notion of intersection type assignment [2]. Nonetheless, $\overline{\lambda}\mu\tilde{\mu}$ and $\mathcal{X}$ is known to enjoy such a system once restricted to CBN or CBV [2, 4]. Intersection types for CBN $\lambda\mu$ have been explored before [3], and the mapping preserving CBV suggests one is also possible for $\lambda\mu$v.

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