A TYPE ASSIGNMENT FOR THE STRONGLY NORMALIZABLE \( \lambda \)-TERMS

Garrel Pottinger

Department of History and Philosophy
Carnegie-Mellon University
Pittsburgh, Pennsylvania 15213, USA.

Dedicated to H.B. Curry on the occasion of his 80th birthday

1. INTRODUCTION

This paper provides an assignment of type symbols to the \( \lambda \)-terms which are \( \beta \)-strongly normalizable. The assignment can be viewed as an extension of the formulas-as-types approach to the study of intuitionist logic and, consequently, may be said to have its ultimate origins in the remarks of Curry and Feys (1958, pp. 313-315).

The type symbols employed are formulas built up from grouping indicators, propositional parameters, \( \supset \), and the new connective \( \& \). The intuitive meaning of \( \& \) can be explained by saying that to assert \( A \& B \) is to assert that one has a reason for asserting \( A \) which is also a reason for asserting \( B \). Taken together with the usual intuitionist understanding of \( \supset \), this reading of \( \& \) provides a nice motivation for all but one of the rules used in the first system defined below to establish the promised assignment of type symbols. The exception is a rule designed to allow for as full a treatment of \( \tau \)-conversion as possible.

In Curry and Feys (1958, pp. 313-315) it is point-
ed out that the functional characters assigned to closed terms in the basic theory of functionality are in effect the theorems of intuitionist propositional logic in which the only logical constant which occurs is \( \& \). Given this, it is natural in the present setting to define \( \Lambda \) to be a theorem iff \( \psi \) is the type symbol of a closed term and then consider the properties of \( \psi \) in the light of this definition. Although the bulk of this paper is devoted to questions which arise from considering the systems presented here as providing assignment of type symbols to \( \lambda \)-terms, the consequences of this definition are explored briefly in section 6. In particular, it will be shown that the behavior of \( \exists \) and \( \& \) is quite different from the behavior of \( \forall \) and \( \& \).

This is to be expected, since, according to the usual intuitionist understanding of \( \& \), to assert \( \Lambda \& \Lambda \) is to assert that one has a pair of reasons, the first of which is a reason for asserting \( \Lambda \) and the second of which is a reason for asserting \( \Lambda \).

Evidently, this is quite different from the reading for \( \& \) given above. The point of section 6 is to show how this intuitive difference manifests itself formally.

The results proved here should be compared with the recent work on extended type assignments for \( \lambda \)-terms contained in Coppo and Dezani-Ciancaglini (1978), Salle (1978), Coppo, Dezani-Ciancaglini, and Salle (1979), and Coppo and Dezani-Ciancaglini (1980). They are especially similar to the results presented in Coppo and Dezani-Ciancaglini (1980) but go beyond the latter in that they cover the \( \lambda \& \)-terms and include a treatment of \( \tau \)-conversion. This similarity will be described precisely after the necessary for

mal machinery has been introduced.

In what follows systems will be thought of concatenatively, but, in accordance with Curry's policy, no expression of these systems will be written down. Only \( \mathcal{U} \)-language expressions will appear in this paper. Curry's punctuational conventions will be adopted, and notations used without explanation are to be understood according to Curry's definitions of them. \( '=' \) expresses identity.

2. THE SYSTEMS \( S_1 \) AND \( S_2 \)

Statements of these systems have the form \( t \vdash A \), where \( t \) is a \( \lambda \& \)-term and \( A \) is a formula built up in the way described above. \( t \vdash A \) may be interpreted as saying that \( t \) is a reason for asserting \( A \). \( \Pi, \Omega, R \ldots \) are to be lists of the form \( x_1 \vdash A_1 \ldots x_n \vdash A_n \) (\( n \geq 0 \)), where for all \( i, j (1 \leq i < j \leq n) \), \( x_i \neq x_j \).\n
Sequents of \( S_1 \) and \( S_2 \) have the form \( \Pi \vdash t \vdash A \). \( \Pi \) is the antecedent and \( t \vdash A \) is the succedent of \( \Pi \vdash t \vdash A \). (Note that, according to these definitions, a variable may not occur twice in the antecedent of a sequent. This restriction applies everywhere in what follows, and it is important. For example, it is easy to see that the rule \( \Xi \), which will be stated momentarily, would be unsound without it.)

Intuitively, \( x_1 \vdash A_1 \ldots x_n \vdash A_n \vdash t \vdash A \) is supposed to mean that if \( x_1, \ldots, x_n \) were replaced by reasons for asserting \( A_1, \ldots, A_n \), respectively, in \( t \), then the result would be a reason for asserting \( A \).

Derivations of \( S_1 \) are finite, ordered trees of sequents built up according to the following specifications.
A TYPE ASSIGNMENT

assign the same type symbols), that deleting \( \eta \) does not disturb the treatment of \( \beta \)-conversion given in section 4, and that the set of formulas which are theorems according to the definition given in section 1 is not changed by passing from \( S_1 \) to \( S_1-\eta \). It also happens that the assignment of type symbols provided by \( S_1-\eta \) is essentially the same as that given in Coppo and Dezani-Cianca (1980) -- if one simply rewrites the notation \( '[\sigma_1, \ldots, \sigma_n]' \) used there as \( 'A_1 \hat{\ldots} \hat{\ldots} A_n' \) and rewrites \( 'F[\sigma_1, \ldots, \sigma_n]' \) as \( 'A_1 \hat{\ldots} \hat{\ldots} A_n \circ B' \), then it is almost trivial to prove that the two assignments are the same. From this it follows that these authors could have extended their treatment of \( \beta \)-conversion to the \( \lambda K \)-terms by proceeding in the manner of section 4, below.

Although \( S_1 \) expresses the motivation given above in a very clear way, the presence of the rules for \( \hat{\ldots} \) and \( \eta \) make it hard to prove things about \( S_1 \). It will now be shown that these rules can be avoided by enlarging the stock of axioms and altering the form of \( \Delta \). The resulting system will be called \( 'S_2' \). First, the auxiliary system CL must be defined.

\( \Gamma, \Delta, \Theta, \Gamma, \ldots \) are to be finite sequences of formulas. Sequents of CL have the form \( \Gamma \vdash A \).

Axioms

\[ \Gamma, A, A \Delta \vdash A \]

Rules

\[ \begin{align*}
\Delta & \vdash A \circ B \\
\Gamma & \vdash A \circ B \\
\Gamma & \vdash A \\
\Gamma & \vdash B \\
\Gamma & \vdash A \circ B
\end{align*} \]
\[ \frac{\Gamma \vdash A \supset B \quad \Gamma \vdash A \supset C}{\Gamma \vdash A \supset B \supset C} \]
\[ \frac{\Gamma \vdash C \supset B}{\Gamma, A \vdash C \supset B} \]
\[ \frac{\Gamma \vdash A \supset B \quad \Gamma \vdash B \supset C}{\Gamma \vdash A \supset C} \]

Simple arguments by induction on the length of \( CL \) derivations show that the following rules are admissible in \( CL \).

\[ \frac{\Gamma, \Delta \vdash B}{\Gamma, A, \Delta \vdash B} \quad \frac{\Gamma, B, A, \Delta \vdash C}{\Gamma, A, A, \Delta \vdash C} \quad \frac{\Gamma, A, \Delta \vdash B}{\Gamma \vdash A, \Delta, \Theta \vdash B} \]

**Cut**

\[ \Gamma, \Delta, \Theta \vdash B \]

**X,Y,Z,X_1,...** are to be sets of formulas. Let \( \Gamma^* \) be the set of formulas occurring in \( \Gamma \), and define \( cl(X) = \{ A : \text{for some } \Gamma, \Gamma^* \subseteq X \text{ and } \Gamma \vdash A \text{ is derivable in } CL \} \). Where \( \Sigma \) is the set of formulas, it is easy to see that \( cl \) is a closure operation on \( \Sigma \). Cut \( \vdash \) yields the conclusion that \( cl(cl(X)) = cl(X) \), and the other conditions are immediate from the definitions of \( CL \) and \( cl \).

Where \( X \) is a non-empty set of formulas, \( \Gamma \vdash t \vdash X \) is to be a sequent having \( \Gamma \) as its antecedent and having some member of \( X \) as the formula on the right side of \( \vdash \) in its succedent. \( S_2 \) is defined by the following specifications.

**Axioms**

\[ \Gamma, x \vdash A, Q \vdash x = cl([A]) \]

\[ \frac{\Gamma \vdash A \supset B_1 \ldots \supset B_n \supset A_n}{\Gamma, x \vdash A, Q \vdash t \vdash B_1 \ldots \vdash P, x \vdash A, Q \vdash t \vdash B_n} \]

Provided \( y \neq x \) only if \( y \) is not free in \( t \).

3. THE EQUIVALENCE OF \( S_1 \) AND \( S_2 \)

**Lemma 3.1** If \( \Gamma \vdash t \vdash A_1, \ldots, \Gamma \vdash t \vdash A_n \) are derivable in \( S_2 \) and \( A \in cl([A_1, \ldots, A_n]) \), then \( \Gamma \vdash t \vdash A \) is derivable in \( S_2 \).

**Proof.** Induction on the complexity of \( t \). If \( t \) is a variable or \( t \) begins with \( \lambda \), the required argument is trivial. Otherwise, for all \( i (1 \leq i \leq n) \), \( \Gamma \vdash A_i \) arises via an inference of the form

\[ \Gamma \vdash t_1 \vdash B_i \supset A_i \quad \Gamma \vdash u_1 \vdash B_i \]

\[ \Gamma \vdash t_1 u_1 \vdash A_i \]

It is easy to show that \( B_1 \supset \ldots \supset B_n \supset A_1 \supset \ldots \supset A_n \in cl([B_1 \supset A_1, \ldots, B_n \supset A_n]) \). Hence, \( A \in cl([B_1 \supset A_1, \ldots, B_n \supset A_n]) \). Also, \( B_1 \supset \ldots \supset B_n \in cl([B_1, \ldots, B_n]) \). By Hyp. Ind. \( \Gamma \vdash t_1 \vdash B_i \supset \ldots \supset B_n \supset A \) and \( \Gamma \vdash u_1 \vdash B_i \supset \ldots \supset B_n \) are derivable in \( S_2 \), so \( \Gamma \vdash t_1 u_1 \vdash A \) is derivable in \( S_2 \) by \( \Xi \).

**Lemma 3.2** \( \eta \) is admissible in \( S_2 \).

**Proof.** Suppose \( \Gamma \vdash \lambda x.t x \vdash A \) is derivable in \( S_2 \), and suppose \( x \) is not free in \( t \). \( \Gamma \vdash \lambda x.t x \vdash A \) is derived by an inference of the form
Q, y \vdash B_1, R \vdash t \vdash C_1 \quad \ldots \quad Q, y \vdash B_n, R \vdash t \vdash C_n \quad \Rightarrow \quad Q, R \vdash \lambda x. t x \vdash cl([B_1 \supset C_1, \ldots, B_n \supset C_n])

In turn, for each \( i(1 \leq i \leq n) \), \( Q, y \vdash B_i, R \vdash t \vdash D_i \supset C_i \) arises through an inference of the form

\[
Q, y \vdash B_i, R \vdash t \vdash D_i \supset C_i \quad Q, y \vdash B_i, R \vdash y \vdash D_i
\]

Since \( y \) is not free in \( t \), it can be shown by induction on the length of \( S_2 \) derivations that, for all \( i(1 \leq i \leq n) \), \( Q, R \vdash t \vdash D_i \supset C_i \) is derivable in \( S_2 \).

Also, for all \( i(1 \leq i \leq n) \), \( Q, y \vdash B_i, R \vdash y \vdash D_i \) is an axiom of \( S_2 \), and, hence, \( D_i \in cl([B_i]) \). Lemma 3.1 implies that, for all \( i(1 \leq i \leq n) \), \( Q, R \vdash t \vdash B_i \supset C_i \) is derivable in \( S \). The desired conclusion follows from this and Lemma 3.1.

**COROLLARY 3.3.** If \( P \vdash t \vdash A \) is derivable in \( S_1 \), then \( P \vdash t \vdash A \) is derivable in \( S_2 \).

**Proof.** Immediate from lemmas 3.1 and 3.2.

By induction on the length of \( S_1 \) and \( S_2 \) derivations it can be shown that the following rules are admissible in \( S_1 \) and \( S_2 \):

\[
\begin{align*}
K & : & P, Q \vdash t \vdash B & \\
C & : & P, x \vdash A, Q \vdash t \vdash B & \\
W & : & P, x \vdash A, y \vdash B, Q \vdash t \vdash C & \\
\text{Cut} & : & P, x \vdash A, y \vdash B, Q \vdash t \vdash C & \\
\end{align*}
\]

**LEMMA 3.4.** If \( B \in cl([A]) \), then \( P, x \vdash A, Q \vdash t \vdash B \) is derivable in \( S_1 \).

**Proof.** By induction on the length of \( CL \) derivations ending with \( A \vdash B \). \( K \vdash \) and \( W \vdash \) imply that there is no loss of generality.) Let \( \Delta \) be the given derivation. If \( \Delta \) is an axiom or ends with \( \Delta A \) or \( \Delta A \), the required argument is trivial. In the \( \Gamma \vdash \Delta \) case one proceeds via Hyp. Ind., \( \Gamma \vdash \) \( \Delta x, \Delta \), \( \Delta \), and \( \eta \). Hyp. Ind., \( \Gamma \vdash \Delta x, \Delta \), \( \Delta \), and \( \eta \) suffice in the \( \vdash \Gamma \) case, and Hyp. Ind., \( \Gamma \vdash \Delta x, \Delta \), \( \Delta \), and \( \eta \) yield the desired conclusion in the \( \Gamma \vdash \Delta \) case.

**LEMMA 3.5.** If \( P \vdash t \vdash A_1, \ldots, P \vdash t \vdash A_n \) are derivable in \( S_1 \) and \( A \in cl([A_1, \ldots, A_n]) \), then \( P \vdash t \vdash A \) is derivable in \( S_1 \).

**Proof.** \( P \vdash t \vdash A_1 \vdash \ldots \vdash A_n \) can be derived in \( S_1 \) by means of \( \Delta \), and Lemma 3.4 implies that \( P, x \vdash A_1 \vdash \ldots \vdash A_n \vdash t \vdash A \) is derivable in \( S_1 \). Cut \( \vdash \) yields the desired conclusion.

**THEOREM 3.6.** \( P \vdash t \vdash A \) is derivable in \( S_1 \) iff \( P \vdash t \vdash A \) is derivable in \( S_2 \).

**Proof.** Immediate from corollary 3.3 and lemmas 3.4 and 3.5.

From now on 'derivable' will often be written instead of 'derivable in \( S_1 \)' and 'derivable in \( S_2 \)'. Also, if \( Q \) is a result of permuting elements of \( P, P \) and \( Q \) may be identified in view of \( C \vdash \) and \( W \vdash \). This will be done in what follows.

4. REDUCTION AND CONVERSION

Define:

\[
X_t \vdash A (P \vdash t \vdash A \text{ is derivable})
\]

\[
X_t = UX_t, P
\]
LEMMA 4.1. If \( A \vDash B \in \text{cl}(\{A_1 \supset B_1, \ldots, A_n \supset B_n\}) \), then there exist \( C_1 \supset D_1, \ldots, C_m \supset D_m \in \{A_1 \supset B_1, \ldots, A_n \supset B_n\} \) s.t. \( A \vDash B \in \text{cl}(\{C_1 \supset D_1, \ldots, C_m \supset D_m\}) \).

Proof. By induction on the length of normal CL derivations ending with \( A_1 \supset B_1, \ldots, A_n \supset B_n \models A \supset B \). \((K) \vdash \cdot, \text{C} \vdash \cdot, \text{and W} \vdash \cdot\) imply that there is no loss of generality.

LEMMA 4.2. If \( P,x \models A,Q \models t = B \) is derivable and \( A \in \text{cl}(\{C\}) \), then \( P,x \models C,Q \models t = B \) is derivable.

Proof. By induction on the length of \( S_2 \) derivations.

LEMMA 4.3. If \( P \models t_1 = A \) is derivable and \( t_1 \models A \supset B \), then \( P \models t_2 = A \) is derivable.

Proof. Induction on the complexity of \( t_1 \). Hyp. Ind. suffices if a proper part of \( t_1 \) is replaced. Otherwise, lemmas 4.1, 4.2, 3.1, and Cut\( \vdash \cdot \) yield the desired conclusion.

LEMMA 4.4. If \( \mathcal{Q} \) is a normal \( S_2 \) derivation ending with \( P \models [u/x]t = A \), then \( X \models v = u \) and \( \mathcal{Q} \models v = \mathcal{Q} \) " occurs in \( \mathcal{Q} \), and \( X \subseteq \text{cl}(\{C\}) \), then \( P,y \models C \models [y/x]t = A \) is derivable in \( S_2 \).

Proof. Induction on the complexity of \( t \).

LEMMA 4.5. If \( P \models t = A \) is derivable and \( x \) is free in \( t \), then \( P \) has the form \( Q,x \models B,R \).

Proof. Induction on the length of \( S_2 \) derivations.

LEMMA 4.6. If \( \mathcal{Q} \) is an \( S_2 \) derivation which ends with \( P,x \models A,Q \models t = B \) and \( P_1,x \models C,Q_1 \models t_1 = B_1 \) occurs in \( \mathcal{Q} \), then \( A = C \).
PROOF. Induction on the length of $\mathcal{Q}$.

LEMMA 4.7. If $x$ is free in $t$, $\mathcal{Q}$ is an $S_2$ derivation which ends with $P \vdash [u/x]t \models A$, and $X = \{B_i \mid \text{for some } Q \text{ and } v, v = u \text{ and } Q \models v \models B \text{ occurs in } \mathcal{Q} \}$, then $X \neq \emptyset$.

PROOF. Induction on the length of $\mathcal{Q}$.

For $P = x_1 = A_1, \ldots, x_n = A_n, y_1 = B_1, \ldots, y_m = B_m$ and $Q = x_1 = C_1, \ldots, x_n = C_n, z_1 = D_1, \ldots, z_k = D_k$, where $y_1, \ldots, y_m$ are distinct from $z_1, \ldots, z_k$, let $P + Q = x_1 = A_1 \land C_1, \ldots, x_n = A_n \land C_n, y_1 = B_1, \ldots, y_m = B_m, z_1 = D_1, \ldots, z_k = D_k$.

LEMMA 4.8. If $t_1 \text{ RED}_1 t_2$ and $P \vdash t_2 \models A$ is derivable, then there is an $R$ s.t. $R \vdash t_1 \models A$ is derivable.

PROOF. Induction on the complexity of $t_2$. If a proper part of $t_2$ is replaced, Hyp. Ind. suffices.

Suppose $t_2 = [u/x]t$. If $x$ is not free in $t$, then $X \neq \emptyset$. Let $Q \models u \models B$ be derivable. By $K\vdash$ and $\exists \exists P \models \lambda x t \models B \models A$ is derivable. By lemma 4.2 and $K\vdash P + Q \models u \models B$ and $P + Q \models \lambda x t \models B \models A$ are derivable, so $P + Q \vdash (\lambda x t)u \models A$ is derivable by $\exists \exists$.

If $x$ is free in $t$, let $\mathcal{Q}$ be an $S_2$ derivation ending with $P \vdash [u/x]t \models A$ and let $X$ be as in lemma 4.7. Lemma 4.7 implies that $X \neq \emptyset$. By lemma 4.4, $P, y = B_1 \land \ldots \land y = B_n \vdash [y/x]t \models A$ is derivable, where $X = \{B_1, \ldots, B_n \}$. Hence, $P \vdash \lambda x t = B_1 \land \ldots \land B_n \models A$ is derivable by $\exists \exists$. Let $Q_1 \vdash v_1 = B_1 \land \ldots \land v_n = B_n$ be the sequents of the form $Q \models v = B$ s.t. $v = u \models A$ and $Q \models v \models B$ occurs in $\mathcal{Q}$. Since for all $i (1 \leq i \leq n)$, $v_i = u$ it can be shown by induction on the complexity of $v_1, \ldots, v_n$ that $Q_1 \vdash u \models B, \ldots, Q_n \vdash u \models B$ are derivable. By suppressing elements of $Q_1, \ldots, Q_n$, which involve variables not free in $u$ and applying lemmas 4.5 and 4.6, $K\vdash$, and $C\vdash$, it follows that $P \vdash u \models B_1, \ldots, P \vdash u \models B_n$ are derivable. By lemma 3.1, $P \vdash u \models B_1 \land \ldots \land B_n$ is derivable, so $P \vdash (\lambda x t)u \models A$ is derivable by $\exists \exists$.

COROLLARY 4.9. If $t_1 \text{ RED}_1 t_2$, then $X_{t_1} = X_{t_2}$.

PROOF. Immediate from lemmas 4.3 and 4.8.

LEMMA 4.10. If $t_1 \text{ RED}_1 t_2$, then $X_{t_1}, P = X_{t_2}, P'$.

PROOF. Induction on the complexity of $t_1$. If a proper part of $t_1$ is replaced, Hyp. Ind. suffices.

Suppose $t_1 = \lambda x t x$. Then $t_2 = t$.

$X_{\lambda x t x}, P \subseteq X_{t, P}$ by $\exists \exists$. Suppose $A \in X_{t, P}$. It will be shown by induction on the complexity of $A$ that $A \in X_{\lambda x t x}, P$. Since $X_{t, P} \subseteq \text{cl}(\mathcal{Z})$, it can be shown by induction on the length of $\mathcal{Z}$ derivations that $A$ is not a propositional parameter.

If $A = A_1 \models A_2$, the desired conclusion follows via $K\vdash$, $\exists \exists$, and $\exists \exists$. If $A = A_1 \land A_2$, apply lemma 3.1, Hyp. Ind., and lemma 3.1 again in order to complete the argument.

COROLLARY 4.11. If $t_1 \text{ RED}_1 t_2$, then $X_{t_1} = X_{t_2}$.

PROOF. Immediate from lemma 4.10.

THEOREM 4.12. If $t_1 \text{ RED}_1 t_2$, then $X_{t_1} = X_{t_2}$.

PROOF. As was remarked in the proof of lemma 4.8, $\exists \exists$ causes no trouble, so the theorem follows from corollaries 4.9 and 4.11.
5. TERM IS THE SET OF $\beta\eta$-STRONGLY NORMALIZABLE TERMS

$x$ is the head of $x$, $\lambda x t$ is the head of $\lambda x t$.

The head of $tu$ is the head of $t$.

**Lemma 5.1.** If $t$ is $\beta$-normal, then the head of $t$ is a variable or $t$.

**Proof.** Induction on the complexity of $t$.

**Lemma 5.2.** If the head of $t$ is a variable and $t \in \text{TERM}$, then $X_t = 3$.

**Proof.** Induction on the complexity of $t$.

**Theorem 5.3.** If $t$ is $\beta$-normal, then $t \in \text{TERM}$.

**Proof.** Induction on the complexity of $t$, using lemmas 5.1 and 5.2 as required.

**Theorem 5.4.** If $t$ is $\beta\eta$-strongly normalizable (in the usual sense), then $t \in \text{TERM}$.

**Proof.** The $\beta\eta$-strongly normalizable terms are the same as the $\beta$-strongly normalizable terms. Proceed by induction on the maximum number of $\beta$-contractions in a reduction of $t$ to a $\beta$-normal term, using theorem 5.3 and theorem 4.12 as required.

In order to prove the converse of theorem 5.4, it suffices to show that every member of TERM is $\beta$-strongly normalizable. A method for proving this will now be explained.

$s,s_1,\ldots$ are to be sequents of $S_1$ and $S_2$. $s_1$ $\beta$-reduces to $s_2$ iff, for some $P$, $t$, $u$, and $A$,

$s_1 = P |- t = A$, $s_2 = P |- u = A$, and $t$ $\beta$-reduces to $u$ (in the ordinary sense). $s$ is $\beta$-strongly normalizable iff every $\beta$-reduction of $s$ contains only finitely many $\beta$-contractions.

Let $s = P |- t = A$ be derivable. If $A$ is a propositional parameter and $s$ is $\beta\eta$-strongly normalizable, then $s$ is computable. If $A = A_1 \supset A_2$ and, for every computable sequent $Q |- u = A_1$, $P \supset Q |- tu = A_2$ is computable, then $s$ is computable.

If $A = A_1 \supset A_2$ and $P |- t = A_1$ and $P |- t = A_2$ are computable, then $s$ is computable.

Given this definition, it is easy to modify the arguments of Stenlund (1972, pp. 126-131) so as to prove that every derivable sequent is $\beta\eta$-strongly normalizable. The converse of theorem 5.4 follows.

**Theorem 5.5.** TERM = \{ $t$: $t$ is $\beta\eta$-strongly normalizable \}

**Proof.** By theorem 5.4 and the method for proving the converse of theorem 5.4 which has just been described.

6. $\&$ AS A CONNECTIVE

It was remarked in section 1 that $\&$ behaves quite differently from $\supset$. This will now be made apparent.

$A$ is a theorem iff, for some $t$, $P |- t = A$ is derivable. This amounts to saying that $A$ is a theorem iff $A$ is realized by a closed member of TERM.

Given theorem 4.12 and 5.5, it is easy to show that the following formulas are not theorems: $p \supset q \supset p \supset q \supset p \supset q \supset r \supset p \supset q \supset r$. On the other hand, the following sequents are derivable:

$\vdash \lambda x. xx = A \supset (A \supset B) \supset B$

$\vdash \lambda x y. xx = A \supset (A \supset B) \supset B \supset B \supset C$

$\vdash \lambda x y. xy = A \supset B \supset C \supset (A \supset B) \supset (A \supset C)$

$\vdash \lambda x y. xy = A \supset (A \supset B) \supset (A \supset B) \supset C \supset C$
\( \lambda \lambda \lambda x : A \rightarrow B \rightarrow C \)

\( \lambda \lambda \lambda y : (B \rightarrow C) : A \rightarrow B \rightarrow C \)

\( \lambda \lambda x : A \rightarrow B \rightarrow A \)

\( \lambda \lambda x : A \rightarrow A \rightarrow A \)

\( \lambda \lambda x : A \rightarrow B \rightarrow B \rightarrow A \)

\( \lambda \lambda x : (B \rightarrow C) : (A \rightarrow B) \rightarrow C \)

Since the meaning of \( \hat{\lambda} \) is reasonably clear (to claim that \( \hat{A} \hat{B} \) is to claim that one has a reason for asserting \( \hat{A} \) which is also a reason for asserting \( \hat{B} \)), it would obviously be of interest to figure out how to add \( \hat{\lambda} \) to intuitionist logic and then consider the analysis of intuitionist mathematical reasoning in the light of the resulting system.

FOOTNOTES

1. This is crude, but it will suffice to motivate the rules and axioms of the system \( S_1 \). Clearly, it would be nice to be able to replace this sort of talk by a pleasant realizability interpretation. For those who believe that all is syntax the results proved here will in effect do that. It is in fact possible to produce a set theoretically based realizability interpretation for the formal machinery employed in this paper, which should be some comfort to those who do not believe that all is syntax, but that interpretation is far from pleasant, and this paper is too small to contain it.

2. One needs a lemma to the effect that in an \( S_1 \rightarrow \eta \) derivation no sequent need ever be both the conclusion of an \( \hat{\lambda} \) and the premise of an \( \hat{\eta} \) (cf. the remarks preceding lemma 4.1 and the proof of that lemma), but it is easy to prove this by induction on the length of \( S_1 \rightarrow \eta \) derivations.

3. \( \eta \) is the real culprit here. If attention were restricted to \( S_1 \rightarrow \eta \), then it would suffice to control \( \hat{\lambda} \) and \( \hat{\eta} \) in the way explained in note 2.

4. \( P^2 = \) the power set of \( \emptyset \).

5. \( \emptyset \) = the empty set.

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