A TYPE ASSIGNMENT FOR THE STRONGLY NORMALIZABILE λ-TERMS

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Dedicated to H.B. Curry on the occasion of his 80th birthday

1 INTRODUCTION

This paper provides an assignment of type symbols to the λK-terms which are $\beta\eta$-strongly normalizable. The assignment can be viewed as an extension of the formulas-as-types approach to the study of intuitionistic logic and, consequently, may be said to have its ultimate origins in the remarks of Curry and Feys (1958, pp. 313-315).

The type symbols employed are formulas built up from grouping indicators, propositional parameters, $\rightarrow$, and the new connective $\cap$.[1] The intuitive meaning of $\cap$ can be explained by saying that to assert $A \cap B$ is to assert that one has a reason for asserting A which is also a reason for asserting B. Taken together with the usual intuitionist understanding of $\rightarrow$, this reading of $\cap$ provides a nice motivation for all but one of the rules used in the first system defined below to establish the promised assignment of type symbols. The exception is a rule designed to allow for as full a treatment of η-conversion as possible.

In Curry and Feys (1958, pp. 313-315) it is pointed out that the functional characters assigned to closed terms in the basic theory of functionality are in effect the theorems of intuitionist propositional logic in which the only logical constant which occurs is $\rightarrow$. Given this, it is natural in the present setting to define A to be a theorem iff it is the type symbol of a closed term and then consider the properties of $\cap$ in the light of this definition. Although the bulk of this paper is devited[2] to questions which arise from considering the systems presented here as providing as[3] assignment of type symbols to λ-terms, the consequences of this definition are explored briefly in section 6. In particular, it will be shown that the behavior of $\rightarrow$ and $\cap$ is quite different from the behavior of $\rightarrow$ and &. This is to be expected, since, according to the usual intuitionist understanding of &, to assert $A\&B$ is to assert that one has a pair of reasons, the first of which is a reason for asserting A and the second of which is a reason for asserting B. Evidently, this is quite different from the reading for $\cap$ given above. The point of section 6 is to show how this intuitive difference manifests itself formally,

The results proved here should be compared with the recent work on extended type assignments for λ-terms contained in Coppo and Dezani-Ciancaglini (1978), Sallé (1978), Coppo, Dezani-Ciancaglini, and Sallé (1979), and Coppo and Dezani-Ciancaglini (1980). They are especially similar to the results presented in Coppo and Dezani-Ciancaglini (1980) but go beyond the latter in that they cover the λK-terms and include a treatment of η-

[1] The original symbols used for $\rightarrow$ and $\cap$ were $\supset$ and $\widehat{\&}$.

[2] Should be 'devoted'.

[3] 'as' very likely should be 'an'.

conversion. This similarity will be described precisely after the necessary for mal[4] machinery has been introduced.

In what follows systems will be thought of concatenatively, but, in accordance with Curry's policy, no expression of these systems will be written down. Only U-language expressions will appear in this paper. Curry's punctuational conventions will be adopted, and notations used without explanation are to be understood according to Curry's definitions of them. '=' expresses identity.

2 THE SYSTEMS S_1 AND S_2

Statements of these systems have the form $t : A$[5], where t is a λK-term and A is a formula built up in the way described above. $t : A$ may be interpreted as saying that t is a reason for asserting A.[1] P, Q, $R\ldots$ are to be lists of the form $x_1{:}A_1,\ldots,x_n{:}A_n$ $(n > 0)$, where for all i, j $(1 \le i \le j \le n)$, $x_i \ne x_j$. Sequents of S_1 and S_2 have the form $P \vdash t : A$. P is the antecedent and $t : A$ is the succedent of of[6] $P \vdash t : A$. (Note that, according to these definitions, a variable may not occur twice in the antecedent of a sequent. This restriction applies everywhere in what follows, and it is important. For example, it is easy to see that the rule $\rightarrow$I, which will be stated momentarily, would be unsound without it.)

Intuitively, $x_1{:}A_1,\ldots x_n{:}A_n \vdash t : A$ is supposed to mean that if $x_1,\ldots,x_n$ were replaced by reasons for asserting $A_1,\ldots,A_n$, respectively, in t, then the result would be a reason for asserting A.

Derivations of S_1 are finite, ordered trees of sequents built up according to the following specifications.

Axioms

$$\frac{}{P, x{:}A, Q \vdash x : A}$$

Rules

$\rightarrow$E
$$\frac{P \vdash t : A \rightarrow B \qquad P \vdash u : A}{P \vdash tu : B}$$

$\rightarrow$I
$$\frac{P, x{:}A, Q \vdash t : B}{P, Q \vdash \lambda y.t[y/x] : A \rightarrow B} \text{ (provided } y \ne x \text{ only if } y \text{ is not free in } t\text{)}$$

$\cap$E
$$\frac{P \vdash t : A \cap B}{P \vdash t : A} \qquad \frac{P \vdash t : A \cap B}{P \vdash t : B}$$

$\cap$I
$$\frac{P \vdash t : A \qquad P \vdash t : B}{P \vdash t : A \cap B}$$

η
$$\frac{P \vdash \lambda x.tx : A}{P \vdash t : A} \text{ (provided } x \text{ is not free in } t\text{)}$$

It should be clear that the axioms and rules of S_1 other than η accord with the intended meaning of sequents. η is a rule of type inclusion which allows for the treatment of η-conversion. It will turn out that η-conversion must be restricted, despite the presence of η.

To try to treat η-conversion as fully as possible is obviously reasonable from the point of view of combinatory logic, but it is also clear that this motive is independent of the

[4] 'for mal' should be 'formal'.

[5] Original notation $t \models A$.

[6] One 'of' too many.

motivation given above for the rules of S_1 other than η. This independence can be worked out formally. Let S_1-η be the system obtained from S_1 by deleting η. It is not difficult to see that S_1-η assigns type symbols to the same set of terms as S_1 (though it does not assign the same type symbols), that deleting η does not disturb the treatment of β-conversion given in section 4, and that the set of formulas which are theorems according to the definition given in section 1 is not changed by passing from S_1 to S_1-η. It also happens that the assignment of type symbols provided by S_1-η is is essentially the same as that given in Coppo and Dezani-Ciancaglini (1980) – if one simply rewrites the notation '$[\sigma_1,\ldots,\sigma_n]$' used there as '$A_1 \cap \ldots \cap A_n$' and rewrites '$F[\sigma_1,\ldots,\sigma_n]\tau$' as '$(A_1 \cap \ldots \cap A_n) \to B$', then it is almost trivial to prove that the two assignments are the same.[2] From this it follows that these authors could have extended their treatment of β-conversion to the λK-terms by proceeding in the manner of section 4, below.

Although S_1 expresses the motivation given above in a very clear way, the presence of the rules for $\cap$ and η make it hard to prove things about S_1.[3] It will now be shown that these rules can be avoided by enlarging the stock of axioms and altering the form of $\to$I. The resulting system will be called 'S_2'. First, the auxiliary system CL must be defined.

$\Gamma,\Delta,\Theta,\Gamma_1,\ldots$ are to be finite sequences of formulas. <u>Sequents</u> of CL have the form $\Gamma \Vdash A$.

Axioms

$$\overline{\Gamma, A, \Delta \Vdash A}$$

Rules

$\cap$E $\quad \dfrac{\Gamma \Vdash A \cap B}{\Gamma \Vdash A} \quad \dfrac{\Gamma \Vdash A \cap B}{\Gamma \Vdash B}$

$\cap$I $\quad \dfrac{\Gamma \Vdash A \quad \Gamma \Vdash B}{\Gamma \Vdash A \cap B}$

$\to\cap$ $\quad \dfrac{\Gamma \Vdash A \to B \quad \Gamma \Vdash A \to C}{\Gamma \Vdash A \to B \cap C}$

$\Vdash\to$ $\quad \dfrac{C \Vdash A \quad \Gamma \Vdash A \to B}{\Gamma \Vdash C \to B}$

$\to\Vdash$ $\quad \dfrac{\Gamma \Vdash A \to B \quad B \Vdash C}{\Gamma \Vdash A \to C}$

Simple arguments by induction on the length of CL derivations show that the following rules are admissible in CL.

K$\Vdash$ $\quad \dfrac{\Gamma, \Delta \Vdash B}{\Gamma, A, \Delta \Vdash B}$

C$\Vdash$ $\quad \dfrac{\Gamma, A, B, \Delta \Vdash C}{\Gamma, B, A, \Delta \Vdash C}$

W$\Vdash$ $\quad \dfrac{\Gamma, A, A.\Delta \Vdash B}{\Gamma, A, \Delta \Vdash B}$

Cut$\Vdash$ $\quad \dfrac{\Gamma \Vdash A \quad \Delta, A, \Theta \Vdash B}{\Gamma, \Delta, \quad \Vdash B}$ [7]

[7] Θ is missing from the conclusion.

X, Y, Z, X_1 ... are to be sets of formulas. Let $\Gamma*$ be the set of formulas occurring in Γ, and define

$$\mathrm{cl}\,(X) = \{\, A : \text{ for some } \Gamma, \Gamma* \subseteq X \text{ and } \Gamma \Vdash A \text{ is derivable in CL} \,\}.$$

Where $\mathcal{F}$ is the set of formulas, it is easy to see that cl is a closure operation on $\wp\mathcal{F}$.[4] Cut$\Vdash$ yields the conclusion that $\mathrm{cl}\,(\mathrm{cl}\,(X)) = \mathrm{cl}\,(X)$, and the other conditions are immediate from the definitions of CL and cl.

Where X is a non-empty set of formulas, $P \vdash t : X$ is to be a sequent having P as its antecedent and having some member of X as the formula on the right side of ':'[8] in its succedent. S_2 is defined by the following specifications.

Axioms

$$\overline{P, x{:}A, Q \vdash x : \mathrm{cl}\,(\{\, A \,\})}$$

Rules

$\to$E As in S_1

$\to$I
$$\frac{P, x{:}A_1, Q \vdash t : B_1 \quad \ldots \quad P, x{:}A_n, Q \vdash t : B_n}{P, Q \vdash \lambda y.t[y/x] : \mathrm{cl}\,(\{\, A_1 \to B_1, \ldots, A_n \to B_n \,\})} \text{ (provided } y \neq x \text{ only if } y \text{ is not free in } t)$$

3 THE EQUIVALENCE OF S_1 AND S_2

LEMMA 3.1 If $P \vdash t : A_1, \ldots P \vdash t_n : A_n$ are derivable in S_2 and $A \in \mathrm{cl}\,(\{\, A_1, \ldots, A_n \,\})$, then $P \vdash t : A$ is derivable in S_2.

Proof. Induction on the complexity of t. If t is a variable or t begins with λ, the required argument is trivial. Otherwise, for all i $(1 < i < n)$, $P \vdash A_i$[9] arises via an inference of the form

$$\frac{P \vdash t_1 : B_i \to A_i \quad P \vdash u_1 : B_i}{P \vdash t_1 u_1 : A_i}\ (\to\text{E})$$

It is easy to show that $B_1 \cap \ldots \cap B_n \to A_1 \cap \ldots \cap A_n \in \mathrm{cl}\,(\{\, B_1 \to A_1, \ldots, B_n \to A_n \,\})$ and that $A \in \mathrm{cl}\,(\{\, A_1 \cap \ldots \cap A_n \,\})$. Hence, $B_1 \cap \ldots \cap B_n \to A \in \mathrm{cl}\,(\{\, B_1 \to A_1, \ldots, B_n \to A_n \,\})$. Also, $B_1 \cap \ldots \cap B_n \in \mathrm{cl}\,(\{\, B_1, \ldots, B_n \,\})$. By Hyp. Ind. $P \vdash t : B_1 \cap \ldots \cap B_n \to A$ and $P \vdash u_1 : B_1 \cap \ldots \cap B_n$ are derivable in S_2, so $P \vdash t_1 u_1 : A$ is derivable in S_2 by $\to$E.

LEMMA 3.2 η is admissible in S_2.

Proof. Suppose $P \vdash \lambda x.tx : A$ is derivable in S_2, and suppose x is not free in t. $P \vdash \lambda x.tx : A$ is derived by an inference of the form

$$\frac{Q, y{:}B_1, R \vdash ty : C_1 \quad \ldots \quad Q, y{:}B_n, R \vdash ty : C_n}{Q, R \vdash \lambda x.tx : \mathrm{cl}\,(\{\, B_1 \to C_1, \ldots, B_n \to C_n \,\})}\ (\to\text{I})$$

In turn, for each $i (l \leq i \leq n)$, $Q, y{:}B_i, R \vdash ty : C_i$ arises through an inference of the form

$$\frac{Q, y{:}B_i, R \vdash t : D_i \to C_i \quad Q, y{:}B_i, R \vdash y : D_i}{Q, y{:}B_i, R \vdash ty : C_i}\ (\to\text{E})$$

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[8] The original has $\models$ in stead of ':'.

[9] A_i should be $t : A_i$.

Since y is not free in t, it can be shown by induction on the length of S_2 derivations that, for all $i(l \leq i \leq n)$, $Q,R \vdash t : D_i \to C_i$ is derivable in S_2. Also, for all $i(l \leq i \leq n)$, $Q,y{:}B_i,R \vdash y : D_i$ is an axiom of S_2, and, hence, $D_i \in \mathrm{cl}(\{B_i\})$. Lemma 3.1 implies that, for all $i(l \leq i \leq n)$, $Q,R \vdash t : B_i \to C_i$ is derivable in S .[10] The desired conclusion follows from this and lemma 3.1.

Corollary 3.3 If $P \vdash t : A$ is derivable in S_1, then $P \vdash t : A$ is derivable in S_2.

Proof. Immediate from lemmas 3.1 and 3.2.

By induction on the length of S_1 and S_2 derivations it can be shown that the following rules are admissible in S_1 and S_2.

$$\mathrm{K}\vdash \quad \frac{P,Q \vdash t : B}{P,x{:}A,Q \vdash t : B}$$

$$\mathrm{C}\vdash \quad \frac{P,x{:}A,y{:}B,Q \vdash t : C}{P,y{:}B,x{:}A,Q \vdash t : C}$$

$$\mathrm{W}\vdash \quad \frac{P,x{:}A,y{:}A,Q \vdash t : B}{\Gamma,x{:}A,Q \vdash t[x/y] : B}$$

$$\mathrm{Cut}\vdash \quad \frac{P \vdash u : A \quad P,x{:}A,Q \vdash t : B}{P,Q \vdash t[u/x] : B}$$

Lemma 3.4 If $B \in \mathrm{cl}(\{A\})$, then $P,x{:}A,Q \vdash x : B$ is derivable in S_1.

Proof. By induction on the length of CL derivations ending with $A \Vdash B$. (K$\Vdash$ and W$\Vdash$ imply that there is no loss of generality.) Let $\underset{\sim}{D}$ be the given derivation. If $\underset{\sim}{D}$ is an axiom or ends with $\cap$E or $\cap$I, the required argument is trivial. In the $\to\cap$ case one proceeds via Hyp. Ind., K$\vdash$, $\to$E, $\cap$I, $\to$I, and η. Hyp. Ind., K$\vdash$, $\to$E, $\to$I and η suffice in the $\Vdash\to$ case, and Hyp. Ind., K$\vdash$, $\to$E, Cut$\vdash$, $\to$I, and η yield the desired conclusion in the $\to\Vdash$ case.

Lemma 3.5 If $P \vdash t : A_1, \ldots, P \vdash t : A_n$ are derivable in S_1 and $A \in \mathrm{cl}(\{A_1,\ldots,,A\})$, then $P \vdash t : A$ is derivable in S_1.

Proof. $P \vdash t : A_1 \cap \ldots \cap A_n$ can be derived in S_1 by means of $\cap$I, and lemma 3.4 implies that $P,x{:}A_1 \cap \ldots \cap A_n \vdash x : A$ is derivable in S_1. Cut$\vdash$ yields the desired conclusion.

Theorem 3.6 $P \vdash t : A$ is derivable in S_1 iff $P \vdash t : A$ is derivable in S_2.

Proof. Immediate from corollary 3.3 and lemmas 3.4 and 3.5.

From now on 'derivable' will often be written instead of 'derivable in S_1 and 'derivable in S_2'. Also, if Q is a result of permuting elements of P, P and Q may be identified in view of C$\vdash$ and W$\vdash$. This will be done in what follows.

4 REDUCTION AND CONVERSION

Define:

$X_{t,P} = \{A : P \vdash t : A \text{ is derivable}\}$.

$X_t = \cup X_{t,P}$

$t_1 \;\mathrm{RED}_{1\beta}\; t_2$ iff there exists t, x, and u s.t. x is not free in t only if $X_u \neq 0$,[5] and t_2 is a result of replacing an occurrence of $(\lambda x.t)u$ in t_1 by an occurrence of $t[u/x]$.

$\mathcal{F}_{\to} = \{A \to B : A,B \in \mathcal{F}\}$

[10] Subscript missing here, should be S_2.

t_1 RED$_{1\eta}$ t_2 iff there exists t and x s.t. $X_t \subseteq \mathrm{cl}(\mathcal{F}_{\rightarrow})$, x is not free in t, and t_2 is a result of replacing an occurrence of $\lambda x.tx$ in t_1 by an occurrence of t.

t_1 RED$_{1\beta\eta}$ t_2 iff t_1 RED$_{1\beta}$ t_2 or t_1 RED$_{1\eta}$ t_2.

$=_\alpha$ is the usual relation of α-conversion.

t_1 RED$_\beta$ $t_2[t_1$ RED$_{\beta\eta}$ $t_2]$ iff there exist $v_1 \ldots, v_n$ $(1 \leq n)$ s.t. $v_1 = t_1$, $v_n = t_2$, and, for all $i < n$, $v_1 =_\alpha v_{i+l}$[11] or v_i RED$_{1\beta}$ v_{i+l} $[v_i$ RED$_{1\beta\eta}$ $v_{i+l}]$.

t_1 CONV$_\beta$ t_2 $[t_2$ CONV$_{\beta\eta}$ $t_2]$ iff there exists $v_1 \ldots, v_n$ $(1 \leq n)$ s.t. $v_1 = t_1$, $v_n = t_2$, and, for all $i < n$, t_1 RED$_\beta$ t_2 or t_2 RED$_\beta$ t_1 (t_1 RED$_{\beta\eta}$ t_2 RED$_{\beta\eta}$ t_1[12]).

TERM $= \{ t : X_t \neq 0 \}$.

It will now be shown that if t CONV$_{\beta\eta}$ u, then $X_t = X_u$, and a fortiori, that TERM is closed under CONV$_{\beta\eta}$.

A CL derivation $\underset{\sim}{D}$ is normal iff no sequent occurrence in $\underset{\sim}{D}$ is both the conclusion of a $\cap$I and the premiss of a $\cap$E. It can be shown by induction on the length of CL derivations that if $\Gamma \Vdash A$ is derivable in CL, then there is a normal CL derivation which ends with $\Gamma \Vdash A$. If $\underset{\sim}{D}$ is a normal CL derivation which ends with $A_1 \rightarrow B_1, \ldots. A_n \rightarrow B_n \Vdash A \cap B$, induction on the length of $\underset{\sim}{D}$ yields the conclusion that the last inference of $\underset{\sim}{D}$ is a $\cap$I. It follows that the last inference of a normal CL derivation ending with $A_1 \rightarrow B_1, \ldots. A_n \rightarrow B_n \Vdash A \cap B$ is not a $\cap$E.

LEMMA 4.1 It $A \rightarrow B \in \mathrm{cl}(\{ A_1 \rightarrow B_1, \ldots, A_n \rightarrow B_n \})$, then there exist $C_1 \rightarrow D_1, \ldots, C_m \rightarrow D_m \in \{ A_1 \rightarrow B_1, \ldots, A_n \rightarrow B_n \}$ s.t. $A \rightarrow B \in \mathrm{cl}(\{ C_1 \rightarrow D_1, \ldots, C_m \rightarrow D_m \})$, $C_1 \ldots, C_m \in \mathrm{cl}(\{ A \})$, and $B \in \mathrm{cl}(\{ D_1, \ldots, D_m \})$.

Proof. By induction on the length of normal CL derivations ending with $A_1 \rightarrow B_1, \ldots,$ $A_n \rightarrow B_n \Vdash A \rightarrow B$. (K$\Vdash$, C$\Vdash$, and W$\Vdash$ imply that there is no loss of generality.)

LEMMA 4.2 If $P, x{:}A, Q \vdash t : B$ is derivable and $A \in \mathrm{cl}(\{ C \})$, then $P, x{:}C, Q \vdash t : B$ is derivable.

Proof. By induction on the length of S$_2$ derivations,

LEMMA 4.3 If $P \vdash t_1 : A$ is derivable and t_1 RED$_{1\beta}$ t_2, then $P \vdash t_2 : A$ is derivable.

Proof. Induction on the complexity of t_1. Hyp. Ind. suffices if a proper part of t_1 is replaced. Otherwise, lemmas 4.1, 4.2, 3.1, and Cut$\vdash$ yield the desired conclusion.

LEMMA 4.4 If $\underset{\sim}{D}$ is an S$_2$ derivation ending with $P \vdash t[u/x] : A$, $X = \{ B :$ for some Q and v, $v =_\alpha u$ and $Q \vdash v : B$ occurs in $\underset{\sim}{D} \}$, and $X \subseteq \mathrm{cl}(\{ C \})$, then $P, y{:}C \vdash t[y/x] : A$ is derivable in S$_2$.

Proof. Induction on the complexity of t.

LEMMA 4.5 If $P \vdash t : A$ is derivable and x is free in t, then P has the form $Q, x{:}B, R$.

Proof. Induction on the length of S$_2$ derivations

LEMMA 4.6 If $\underset{\sim}{D}$ is an S$_2$ derivation which ends with $P, x{:}A, Q \vdash t : B$ and $P_1, x{:}C, Q_1 \vdash t_1 : B_1$ occurs in $\underset{\sim}{D}$, then $A = C$.

Proof. Induction on the length of $\underset{\sim}{D}$.

LEMMA 4.7 If x is free in t, $\underset{\sim}{D}$ is an S$_2$ derivation which ends with $P \vdash t[u/x] : A$, and $X = \{ B :$ for some Q and v, $v =_\alpha u$ and $Q \vdash v : B$ occurs in $\underset{\sim}{D} \}$, then $X \neq 0$.

[11] v_1 should be v_i.

[12] It should say 't_1 RED$_{\beta\eta}$ t_2 or t_2 RED$_{\beta\eta}$ t_1 rather than 't_1 RED$_{\beta\eta}$ t_2 RED$_{\beta\eta}$ t_1'.

Proof. Induction on the length of $\underset{\sim}{D}$.

For $P = x_1{:}A_1,\ldots,x_n{:}A_n,y_1{:}B_1,\ldots,y_m{:}B_m$ and $Q = x_1{:}C_1,\ldots,x_n{:}C_n,z_1{:}D_1,\ldots,z_k{:}D_k$, where $y_1,\ldots,y_m$ are distinct from $z_1,\ldots,z_k$, let $P+Q = x_1{:}A_1 \cap C_1,\ldots,x_n{:}A_n \cap C_n$, $y_1{:}B_1,\ldots,y_n{:}B_n$, $z_1{:}D_1,\ldots,z_k{:}D_k$.[13]

Lemma 4.8 If t_1 $\text{RED}_{1\beta}$ t_2 and $P \vdash t_2 : A$ is derivable, then there is an R s.t. $R \vdash t_1 : A$ is derivable.

Proof. Induction on the complexity of t_2. If a proper part of t_2 is replaced, Hyp. Ind. suffices. Suppose $t_2 = t[u/x]$.

If x is not free in t, then $X_u \neq 0$. Let $Q \vdash u : B$ be derivable. By K$\vdash$ and $\rightarrow$I $P \vdash \lambda x.t : B \rightarrow A$ is derivable. By lemma 4.2 and K$\vdash$ $P+Q \vdash u : B$ and $P+Q \vdash \lambda x.t : B \rightarrow A$ are derivable , so $P+Q \vdash (\lambda x.t)u : A$ is derivable by $\rightarrow$E.

If x is free in t, let $\underset{\sim}{D}$ be an S_2 derivation ending with $P \vdash t[u/x] : A$ and let X be as in lemma 4.7. Lemma 4.7 implies that $X \neq 0$. By lemma 4.4 $P,y{:}B_1 \cap \ldots \cap B_n \vdash t[y/x] : A$ is derivable, where $X = \{B_1,\ldots,B_n\}$. Hence, $P \vdash \lambda x.t : B_1 \cap \ldots \cap B_n \rightarrow A$ is derivable by $\rightarrow$I. Let $Q_1 \vdash v_1 : B_1,\ldots,Q_n \vdash v_n : B_n$ be the sequents of the form $Q \vdash v : B$ s.t $v =_\alpha u$ and $Q \vdash v : B$ occurs in $\underset{\sim}{D}$. Since for all i $(1 \leq i \leq n)$, $v_i =_\alpha u$ it can be shown by induction on the complexity of $v_1,\ldots,v_n$ that $Q_1 \vdash u : B,\ldots,Q_n \vdash u : B$ are derivable. By suppressing elements of $Q_1,\ldots,Q_n$ which involve variables not free in u and applying lemmas 4.5 and 4.6, K$\vdash$, and C$\vdash$, it follows that $P \vdash u : B_1$, …, $P \vdash u : B_n$ are derivable. By lemma 3.1 $P \vdash u : B_1 \cap \ldots \cap B_n$ is derivable, so $P \vdash (\lambda x.t)u : A$ is derivable by $\rightarrow$E.

Corollary 4.9 If t_1 $\text{RED}_{1\beta}$ t_2, then $X_{t_1} = X_{t_2}$.

Proof. Immediate from lemmas 4.3 and 4.8.

Lemma 4.10 If t_1 $\text{RED}_{1\eta}$ t_2, then $X_{t_1,P} = X_{t_2,P}$.

Proof. Induction on the complexity of t_1. If a proper part of t_1 is replaced, Hyp. Ind. suffices. Suppose $t_1 = \lambda x.tx$. Then $t_2 = t$.

$X_{\lambda x.tx,P} \subseteq X_{t,P}$ by η. Suppose $A \in X_{t,P}$. It will be shown by induction on the complexity of A that $A \in X_{\lambda x.tx,P}$. Since $X_{t,P} \subseteq \text{cl}(\mathcal{F}_\rightarrow)$, it can be shown by induction on the length of CL derivations that A is not a propositional parameter.

If $A = A_1 \rightarrow A_2$, the desired conclusion follows via K$\vdash$, $\rightarrow$E, and $\rightarrow$I. If $A = A_1 \cap A_2$, apply lemma 3.1, Hyp. Ind., and lemma 3.1 again in order to complete the argument.

Corollary 4.11 If t_1 $\text{RED}_{1\eta}$ t_2, then $X_{t_1} = X_{t_2}$.

Proof. Immediate from lemma 4.10.

Theorem 4.12 If t_1 $\text{CONV}_{\beta\eta}$ t_2, then $X_{t_1} = X_{t_2}$.

Proof. As was remarked in the proof of lemma 4.8, $=_\alpha$ causes no trouble, so the theorem follows from corollaries 4.9 and 4.11.

5 TERM IS THE SET OF $\beta\eta$-STRONGLY NORMALIZABLE TERMS

x is the head of x. $\lambda x.t$ is the head of $\lambda x.t$. The head of tu is the head of t.

Lemma 5.1 If t is β-normal, then the head of t is a variable or t.

Proof. Induction on the complexity of t.

[13] In the original, $z_k{:}D_k$ was stated as $z_k \vdash D_k$ rather than $z_k \models D_k$.

Lemma 5.2 If the head of t is a variable and $t \in$ term, then $X_t = \mathcal{F}$.

Proof. Induction on the complexity of t.

Theorem 5.3 if t is β-normal, then $t \in$ term.

Proof. Induction on the complexity of t, using lemmas 5.1 and 5.2 as required.

Theorem 5.4 If t is $\beta\eta$-strongly normalizable (in the usual sense), then $t \in$ term.

Proof. The $\beta\eta$-strongly normalizable terms are the same as the β-strongly normalizable terms. Proceed by induction on the maximum number of β-contractions in a reduction of t to a β-normal term, using theorem 5.3 and theorem 4.12 as required.

In order to prove the converse of theorem 5.4, it suffices to show that every member of term is β-strongly normalizable. A method for proving this will now be explained.

s, $s_1 \ldots$ are to be sequents of S_1 and S_2. s_1 β-reduces to s_2 iff, for some P, t, u, and A, $s_1 = P \vdash t : A$, $s_2 = P \vdash u : A$, and t β-reduces to u (in the ordinary sense). s is β-strongly normalizable iff every β-reduction of s contains only finitely many β-contractions.

Let $s = P \vdash t : A$ be derivable. If A is a propositional parameter and s is β-strongly normalizable, then s is computable. If $A = A_1 \rightarrow A_2$ and, for every computable sequent $Q \vdash u : A$, $P + Q \vdash tu : A_2$ is computable, then s is computable. If $A = A_1 \cap A_2$ and $P \vdash t : A_1$ and $P \vdash t : A_2$ are computable, then s is computable.

Given this definition, it is easy to modify the arguments of Stenlund (1972, pp. 126-131) so as to prove that every derivable sequent is β-strongly normalizable. The converse of theorem 5.4 follows.

Theorem 5.5 term $= \{ t : t$ is $\beta\eta$-strongly normalizable$\}$

Proof. By theorem 5.4 and the method for proving the converse of theorem 5.4 which has just been described.

6 ∩ AS A CONNECTIVE

It was remarked in section 1 that ∩ behaves quite differently from &. This will now be made apparent.

A is a theorem iff, for some t, $\vdash t : A$ is derivable. This amounts to saying that A is a theorem iff A is realized by a closed member of term.

Given theorem 4.12 and 5.5, it is easy to show that the following formulas are not theorems: $p \rightarrow .\, q \rightarrow p \cap q$, $p \rightarrow q \rightarrow .\, p \rightarrow r \rightarrow .\, p \rightarrow q \cap r$, $p \cap q \rightarrow r \rightarrow .\, p \rightarrow .\, q \rightarrow r$. On the other hand, the following sequents are derivable.

$\vdash \lambda x.xx : A \cap (A \rightarrow B) \rightarrow B$
$\vdash \lambda x.\lambda y.xy : (A \rightarrow B) \cap (A \rightarrow C) \rightarrow .\, A \rightarrow B \cap C$
$\vdash \lambda x.\lambda y, xy : A \rightarrow B \cap C \rightarrow .\, (A \rightarrow B) \cap (A \rightarrow C)$
$\vdash \lambda x.\lambda y.xy : A \rightarrow C \rightarrow .\, A \cap B \rightarrow C$
$\vdash \lambda x.\lambda y.x : A \cap B \rightarrow .\, A \rightarrow B$
$\vdash \lambda x.\lambda y, xyy : A \rightarrow (B \rightarrow C) \rightarrow .\, A \cap B \rightarrow C$
$\vdash \lambda x.x : A \cap B \rightarrow A$
$\vdash \lambda x.x : A \rightarrow A \cap A$
$\vdash \lambda x.x : A \cap B \rightarrow B \cap A$
$\vdash \lambda x.x : A \cap (B \cap C) \rightarrow (A \cap B) \cap C$

Since the meaning of ∩ is reasonably clear (to claim that $A \cap B$ is to claim that one has a reason for asserting A which is also a reason for asserting B), it would obviously be of interest to figure out how to add $\cap$ to intuitionist logic and then consider the analysis of intuitionist mathematical reasoning in the light of the resulting system.

FOOTNOTES

1. This is crude, but it will suffice to motivate the rules and axioms of the system S_1. Clearly, it would be nice to be able to replace this sort of talk by a pleasant realisability interpretation. For those who believe that all is syntax the results proved here will in effect do that. It is in fact possible to produce a set theoretically based realizability interpretation for the formal machinery employed in this paper, which should be some comfort to those who do not believe that all is syntax. But that interpretation is far from pleasant, and this paper is too small to contain it.
2. One needs a lemma to the effect that in an S_1-η derivation no sequent need ever be both the conclusion of an $\cap$I and the premiss of an $\cap$E (cf. the remarks preceding lemma 4.1 and the proof of that lemma), but it is easy to prove this by induction on the length of S_1-η derivations.
3. η is the real culprit here. If attention were restricted to S_1-η, then it would suffice to control $\cap$I and $\cap$E in the way explained in note 2.
4. $\wp(\mathcal{F}) =$ the powerset of $\mathcal{F}$.
5. $0 =$ the empty set.

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Note: When I transferred this file to latex, I was tempted to correct the small mistakes and typos that occur in it, but decided not to. What you see here is the original text; I have only changed some notation to the one that is nowadays current, and put in footnotes when needed. I did add the '.' in all λ-abstractions, and changed the substitution from prefix to postfix.
Steffen van Bakel.