Efficient computation of passage time densities and distributions in Markov chains using Laguerre method

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The Laguerre method for the numerical inversion of Laplace transforms is a well known approach to the approximation of probability density functions (PDFs) and cumulative distribution functions (CDFs) of first passage times in Markov chains. Results are presented that relate the Laguerre generating functions and Laguerre coefficients of a PDF with those of the corresponding complementary CDF. This enables the ability to compute the PDF or CDF from the Laplace transform of either at the cost of computing only one set of Laguerre coefficients.

Introduction: The Laplace transform and its close relative, the Fourier transform, are widely used in electronic and process control engineering applications. They are usually applied to functions to simplify complex time-domain calculations by manipulating these functions in the frequency-domain. Numerical transform inversion methods are often then applied to obtain the time-domain function from the derived transform.

An important application of numerical Laplace transform inversion arises in the calculation of first passage times in continuous-time Markov chain-based performance models of transaction processing and computer communication systems. Here the time-domain function is computed by numerically inverting a derived Laplace transform of a passage-time PDF or CDF [1]. The computational cost of this process is determined by the number of Laplace transform evaluations performed, since each evaluation involves the construction and solution of a set of sparse complex linear equations the rank of which is given by the number of states in the Markov chain (typically several million for real-life applications). The Laguerre method is often the numerical inversion method of choice in such applications for two reasons. First, a finite number of Laplace transform evaluations suffices to compute the value of the function for all values of the time variable t (whereas for the other methods such as the Euler method, this burden is proportional to the number of t-values computed). Secondly, since the sojourn time in states is exponentially distributed, first passage time densities are convolutions of exponential densities. Passage time distributions are therefore short-tail distributions with semi-exponential tails, a class of functions to which the Laguerre method is well-suited [2].

The integration method is used for calculating the Laguerre coefficients when the Laplace transform $f^*(s)$ of the time-domain function $f(t)$ is known. It is based on the Laguerre series representation

$$f(t) = \sum_{n=0}^{\infty} q_n l_n(t), \quad t \geq 0 \tag{1}$$

where $l_n(t)$ is the Laguerre function [1]. The Laguerre coefficients q_n are approximated by the trapezoidal rule [1] which gives

$$q_n \approx \frac{1}{2nr^n}\left\{Q(r) + (-1)^n Q(-r) + 2\sum_{j=1}^{n}(-1)^j \mathrm{Re}(Q(re^{\pi ji/n}))\right\} \tag{2}$$

The Laguerre generating function $Q(z)$ is obtained from $f^*(s)$ using the binary transformation $z = (2s-1)/(2s+1)$ as follows:

$$Q(z) \equiv \sum_{0}^{\infty} q_n z^n = (1-z)^{-1} f^*\left(\frac{(1+z)}{2(1-z)}\right) \tag{3}$$

Cumulative distribution functions of first passage times are important measures in performance evaluation studies. A passage-time CDF indicates the probability with which one of a set of target states in the Markov chain is reached within a certain time duration when starting from a specified start state. From a CDF it is straightforward to derive any number of moments of passage time as well as passage time percentiles. As a route to the CDF, the complementary CDF (CCDF) $F^c(t) \equiv 1 - F(t)$ is preferred since it has a smooth structure more suited to numerical inversion ($F^c(t)$ is a non-negative decreasing function with $F^c(t) \to 0$ as $t \to \infty$). Convergence of the Laguerre coefficients depends on the smoothness of the function and this can be improved by scaling and exponential dampening. It is also known that the Laguerre coefficients of a CCDF will have an equal or faster convergence rate than its PDF [3].

This Letter proves a relation between the generating functions of a CCDF and PDF (correcting an error in [3]) and this development leads to a recurrence relation involving the Laguerre coefficients of PDFs and CCDFs. Numerical results are then presented for a passage-time case study.

Relation between $Q'(z)$ and $Q(z)$: For the purpose of relating the Laguerre generating function of a CCDF with that of its corresponding PDF, let the Laplace transform of the CCDF be represented as

$$F^c(s) = \frac{1}{s}(1 - f^*(s)) \tag{4}$$

Similarly to (1) and (3),

$$F^c(t) = \sum_{n=0}^{\infty} q'_n l_n(t) \quad \text{and} \quad Q'(z) = (1-z)^{-1} F^c(s)|_{s=\frac{(1+z)}{2(1-z)}} \tag{5}$$

Substituting (4) in the expression for $Q'(z)$ in (5) yields

$$\begin{aligned} Q'(z) &= (1-z)^{-1} \cdot \frac{1}{s}(1-f^*(s))|_{s=\frac{(1+z)}{2(1-z)}} \\ &= \frac{2}{(1+z)}\left(1 - f^*\frac{(1+z)}{2(1-z)}\right) \\ &= \frac{2}{(1+z)}(1-z) \\ &\quad \left((1-z)^{-1} - \underbrace{(1-z)^{-1} f^*\left(\frac{(1+z)}{2(1-z)}\right)}_{=Q(z)}\right) \\ &= -\frac{2(1-z)}{(1+z)}(Q(z) - (1-z)^{-1}) \end{aligned} \tag{6}$$

This expression for obtaining the Laguerre generating function of a CCDF from the generating function of the PDF corrects an error in the one presented without proof in [3].

Relation between q'_n and q_n: Laguerre coefficients can be computed either by the integration method as described above or by the moments method if the moments of $f(t)$ are known instead of the Laplace transform of $f(t)$. The former approach is considered here since the Laplace transform of a passage time PDF can be obtained as described in [1]. We proceed to derive a direct numerical relation between the Laguerre coefficients of a PDF and CCDF when the integration method is used. Note that this result is distinct from the relation between the PDF and CCDF coefficients derived in [2] in the context of the moments method.

Consider (6) and substitute for $Q'(z)$ with $\sum q'_n z^n$ and for $Q(z)$ with $\sum q_n z^n$. This gives

$$\begin{aligned} (q'_0 + q'_1 z + q'_2 z^2 + \cdots) &= -\frac{2(1-z)}{(1+z)} \\ &\quad \times \left((q_0 + q_1 z + q_2 z^2 + \cdots) - (1-z)^{-1}\right) \end{aligned} \tag{7}$$

Multiplying both sides by $(1+z)$ and simplifying,

$$\begin{aligned} & q'_0 + (q'_0 + q'_1)z + (q'_1 + q'_2)z^2 + \ldots \\ &= -2(q_0 - 1) - 2(q_1 - q_0)z - 2(q_2 - q_1)z^2 - \ldots) \end{aligned} \tag{8}$$

Comparing coefficients with similar powers of z gives

$$q_0 = 1 - \frac{1}{2}q'_0 \tag{9}$$

and

$$\begin{aligned} q_1 &= -\frac{1}{2}(q'_0 + q'_1) + q_0 \\ q_2 &= -\frac{1}{2}(q'_1 + q'_2) + q_1 \\ &\vdots \end{aligned} \tag{10}$$

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Signal processing

Hence the nth equation is,

$$q_n = -\frac{1}{2}(q'_{n-1} + q'_n) + q_{n-1} \qquad (11)$$

Using (9) and (11) we can now relate the Laguerre coefficients of the PDF with the Laguerre coefficients of the CCDF. This is a convenient result especially for implementation purposes because only one set of coefficients, either $\{q_n\}$ or $\{q'_n\}$, needs to be computed.

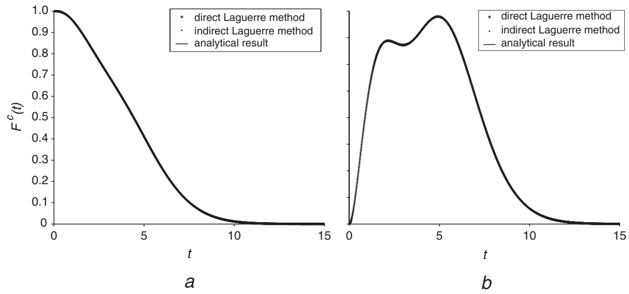

Fig. 1 *Numerical approximation and analytical CCDF and PDF of cycle time*

a CCDF
b PDF

Numerical results: Consider a bimodal cycle-time PDF composed of two Erlang densities with shape and rate parameters $n_1 = 3$, $\lambda_1 = 1$ and $n_2 = 12$, $\lambda_2 = 2$:

$$f(t) = \left(\frac{t^2 e^{-t}}{2!} + \frac{2^{12} t^{11} e^{-2t}}{11!}\right) \Big/ 2 \qquad (12)$$

The corresponding Laplace transform is given by

$$f^*(s) = \left(\left(\frac{1}{1+s}\right)^3 + \left(\frac{2}{2+s}\right)^{12}\right) \Big/ 2 \qquad (13)$$

We calculate the cycle-time PDF $f(t)$ and complementary CDF $F^c(t)$ from their respective Laplace transforms using two different methods for computing the Laguerre coefficients: the direct approach of (2) and the indirect approach of (11), whereby the Laguerre coefficients of the PDF (resp. CCDF) are derived from those of the CCDF (resp. PDF). Figs 1*a* and *b* compare the corresponding numerical results with analytical curves for both cases; agreement is excellent. Fig. 2 shows the relative error between numerical and analytical results for various values of t when calculating the CCDF (see Fig. 2*a*) and PDF (see Fig. 2*b*) using both the direct approach and the indirect approach. The relative error trends increase at large time arguments corresponding to very small absolute values of $f(t)$, due to aliasing (caused by approximating the Laguerre coefficients using the trapezoidal rule) and finite precision arithmetic errors.

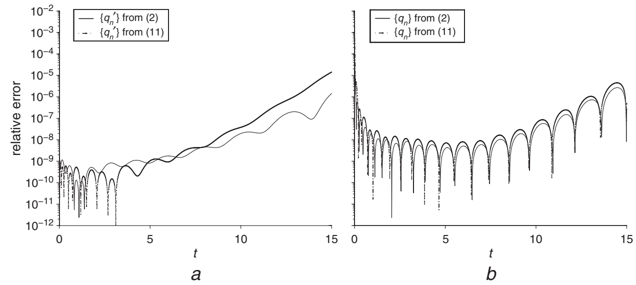

Fig. 2 *Relative error of numerical calculation of CCDF and PDF when Laguerre coefficients are computed directly using (2) and indirectly using (11)*

a CCDF
b PDF

Conclusions: This Letter presents an efficient approach to approximating passage time densities and distributions in Markov chains using the Laguerre method. Resulting expressions relate the Laguerre generating functions of the PDF and CCDF and also the set of Laguerre coefficients of the PDF and CCDF. The recurrence relation for the Laguerre coefficients is well suited for practical implementation, halving the computational burden when both PDF and CCDF are calculated.

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References

1 Harrison, P., and Knottenbelt, W.: 'Passage time distributions in large Markov chains'. Proc. ACM SIGMETRICS 2002, Marina Del Rey, CA, USA, June 2002, pp. 77–85
2 Abate, J., and Whitt, W.: 'Infinite-series representations of Laplace transforms of probability density functions for numerical inversion', *J. Oper. Res. Soc. Jpn.*, 1999, **42**, (3), pp. 268–285
3 Abate, J., Choudhury, G., and Whitt, W.: 'On the Laguerre method for numerically inverting Laplace transforms', *INFORMS J. Comput.*, 1996, **8**, (4), pp. 413–427