Market making with an inverse Kelly strategy

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Abstract. Kelly's celebrated staking system calculates the optimal fraction of wealth to bet on each of a series of favourable bets. We previously extended Kelly's ideas to allow for the much wider range of bets available today, in particular laying as well as backing, and including previously placed bets. Here we extend this further by suggesting that it may be inverted to provide a market making tool. For an event we set the odds to be those which Kelly would have chosen for a fixed fraction. We show how the fraction used here may be thought of as a measure of how tightly the prices will be quoted. As previously matched bets are included in the calculation the quoted prices will be able to respond to errors in the model used; the market making system is independent from the model. Prices are able to respond to trades made in related markets. In order to provide data for a future simulation we ran a number of tests of this method on a live exchange, using different fractions and different sizes for several different markets in the lower English football leagues. We present some preliminary findings from this data.

1. Introduction

Kelly's fractional staking has been much discussed in literature in the 50 years since first publication. Many are convinced: Breiman (1961); Thorp (1969); Markowitz(1976) and Bell and Cover (1980). Others are not: Samuelson (1971) and Merton and Samuelson (1974). Much of the published work over the recent decades has concentrated upon financial markets, for example, Cover (1991) and Bell and Cover (1980) . For many years after Kelly's work was published there was little innovation by bookmakers. Recently in some countries there has been a relaxation of some of the strict laws which control gambling, and this has brought about change.

The introduction of betting exchanges, in particular, has been responsible for much of this. Exchanges are are peer to peer betting platforms, (individuals place bets with each other rather than a bookmaker). The exchange is responsible for settling bets and taking and making payments, but not for the risk. Now when considering our portfolio a there are new choices available. Two of the most interesting are the ability to place lay bets (i.e. betting that something will not happen) and taking bets off before the race has finished or even started.

We, Noon et al (2012), have previously extended Kelly's staking to include these new possibilities and now we turn our attention to a new possibility. We are able to place bets which won't immediately match, but if they do they will trade at a better price than that previously available. In particular, we consider the case when we are the first to place a price in a particular market and we intend to place not just one price, but all possible prices. This activity is often referred to as market making.

2. Market Making

On the popular exchanges major events such as horse racing at the leading courses and top flight football have substantial activity. A considerable time before the start of such an event there are usually market prices with little over-round (beyond the exchange commission). The markets in lower profile events such as football matches several divisions from the top are less well served. On a recent Wednesday the over-round for betting on the correct score for the following Saturday's League One matches was about 12%, and there is no realistic price in any market for matches on the Tuesday. (English League One is the third level, after the Premiership and the Championship.) Anyone planning to place a bet would need to choose a price, rather than merely deciding to trade, or not to trade with an existing price. We suggest choosing a notional sum of money, C, and then selecting the decimal odds o such that Kelly's fractional stake f gives the desired bet size b. Then if p is the probability of the backed event from Kelly we have:

$$f = p - \frac{(1-p)}{(o-1)},$$

rearranging this gives:

$$o = 1 + \frac{1-p}{p-f}. \tag{1}$$

From this we see that we need $p > f$. If offering lay bets as well as back bets there is another important consideration. Each of the entries is considered on its own, but it is possible that more than one will trade at once. For example, if you are offering back and lay prices on a tennis match it is possible that someone may match your back on one player and your lay bet on the other player. With some exchanges this is in fact likely; Betfair's web client automatically does this calculation and a client will see size which is generated by combining a direct transaction with appropriate combinations of other bets. So b should be chosen to allow for this.

To be practical this method combines existing bets into the calculation and at the same time we consider multiple markets. So if an event has n possible outcomes and the probability of outcome i is p_i, and if there are m possible bets which depend solely upon those outcomes, then bet j has odds o_j and traditionally we are choosing the fraction of our wealth x_j to wager on bet j. For each of the j we can take into account previously made bets with size s_j and odds e_j. M is a matrix with $M_{ij} = 1$ if bet j is considered winning if outcome i occurs and $M_{ij} = 0$ otherwise. Previously we would have wanted to maximise A where:

$$A = \sum_i p_i \log\left(1 + \sum_j M_{ij}\left(o_j x_j + e_j s_j - x_j + s_j\right)\right).$$

In this case we find for each j in turn the value of o_j such that maximising A gives $x_j = f$. Again we must ensure that $f < p_i$ for all i. Other than that constraint we are free to choose f as necessary to control the desired over-round. From equation 1 we can see that $1/o = (p - f) / (1 - f)$, so the lay under-round if we place m back bets is

$$\frac{1-mf}{1-f}.$$

The choice of C becomes a decision about the confidence of the model used. If making markets on one side only (all backs, or all lays) we might choose it such that $f = b/C$, or similarly when offering bets on both sides $f=2b/C$, where b is the size of the bet placed in the market. In this case the prices will move so as to unwind the first bet placed at our fair value, or the middle of our previous market. With higher confidence in a model a higher value of C would seem suitable as prices would change more slowly with trading activity.

3. Practical Test – Preliminary Results

In previous work we have been able to create a test using historical data. Suitable assumptions, such as a test size much smaller than the market so as not to impact it needed to be made. The significant point in this technique is that prices aren't available at the time the bets are placed into the exchange. Whether the bets will be matched immediately, improved upon, or ignored is not immediately obvious. In the near future we will build a simulation tool to assist with this work, but we will need some data for calibration. One of the advantages of studying betting markets rather than financial markets is that it is possible to make tests in real markets with a considerably small capital outlay; the minimum bet on Betfair is £2.

Using a model based upon Dixon and Coles (1997), fitted to the previous matches of the current season and the two prior seasons. but with a value for tau of zero we estimated the probabilities of final scores for English Football League One and Two (the third and fourth level respectively). When a match was added to

Betfair the market making tool randomly assigned it a category. One category was to ignore the market, placing no bets, offering these markets as controls. Other categories had various choices for *b*, *C* and *f*. A final group, category 8, reduced *f* as confidence in the current prices increased. For those active matches the tool quoted markets in Match Odds, Over/Under 2.5 and Over/Under 3.5. We had hoped to quote in Correct Score markets but the cost of doing so was deemed too high for the initial test. The amount of capital needed to quote back and lay bets of size b in each of the m runners of a market is $b(m + l - 2)$, where l is the highest lay odds placed. The market making tool ran, with some interruptions, from 11th February to 28th March 2013, a period of 215 football matches. Over this time we placed 6 415 bets.

In several cases the software was interrupted and in some others there were human interventions which introduced errors. Sometimes the thread detecting new markets in Betfair was too slow and by the time it placed prices in the market there were already prices there. Occasionally these matched immediately. One of the parameters our future simulation will need to fit is the inter-arrival times for the bets. These almost instant matching bets skew the data significantly. Some of these we can easily detect and have excluded reducing our total bet count to 5722 bets. Others are less obvious. We are still in the early stages of cleaning the data and need to combine several log files to sort out those occasions when we are less confident with the data. Here are some preliminary results.

Table 1: Showing the spread of data across the two leagues we considered, and the average volume of matched bets.

	Matches Ignored	**Category 1-7**	**Category 8**	**Total Matches**
League One	45	49	13	107
League Two	46	47	15	108
Market Size (£)	39 682	40 689	84 669	

These are not uniformly distributed across this period. In the first half of the time period there is a higher proportion of ignored matches than there is later in the period. The market size is the average of the total volume of matched bets in each of our market categories. We started quoting in a low proportion of matches and increased this as there were no significant losses. As previously stated, quoting in the Correct Score market was particularly expensive and was, therefore, assigned a low probability. This happened only once, matching 19 lay bets not including the winning score.

Table 2: The number of back and lay bets for each type of market considered.

	Match Odds	**Over/Under 2.5**	**Over/Under 3.5**	**Correct Score**
Back	1065	577	216	0
Lay	2864	749	232	19

It is immediately obvious for markets with more than two outcomes that customers prefer to make back bets (so we make lay bets). Perhaps for binary markets customers still favour back bets, but the Betfair web client converts our back bets into lay bets on the other outcome.

We had speculated that quoting prices in a market might increase the total activity in that market (beyond our own activity). From the above table it can be seen that in general it doesn't, at least not significantly. The exception is Category 8. This type of market maker quoted tight prices (f=0.015) after an initial settling down phase. Performing an F-test on the data shows that the variances are significantly different with a high degree of confidence. We are investigating further.

We had also expected that bets which matched quickly might be more likely to be losing bets. To estimate this we calculated for each market a money weighted average odds, normalised so that the reciprocal gives a market implied probability. We use this to calculate an expected profit on each bet. We anticipated a positive correlation between this and the time between placing a bet and it being matched. Surprisingly the calculated correlation is -0.07. It is possible that when the data has been further cleaned this will change, but it is surprising. It is possible we need to consider how far in advance of the match we are placing bets. If we place bets too far ahead it may be that no-one notices for many hours, or days, and this will probably include the bet with the highest negative expected value.

We look forward to investigating this data further, to try to reveal some hidden information.

References

Bell, R. M. and Cover , T. M. (1980) Competitive Optimality of Logarithmic Investment. *Mathematics of Operations Research, INFORMS* **5**, 161-166.

Breiman, L. (1961) Optimal Gambling Systems for Favorable Games. *Proceedings Fourth Berkeley Symposium*, **1**, 65-78.

Cover T. M. (1991) Universal Portfolios. *Mathematical Finance*, **1**, 1-29.

Dixon, M. J. and Coles, S. G. (1997)

Modelling Association Football Scores and Inefficiencies in the Football Betting Market. *Journal of the Royal Statistical Society. Series C*, **46**, 265-280

Markowitz, H. M. (1976) Investment for the Long Run: New Evidence for an Old Rule. *The Journal of Finance*, **31**, 1273-1286.

Merton, R. C. and Samuelson, P. A. (1974) Fallacy of the log-normal approximation to optimal portfolio decision-making over many periods. *Journal of Financial Economics*, **1**, 67 – 94.

Noon, E., Knottenbelt, W. J. and Kuhn, D. (2012) Kelly's fractional staking updated for betting exchanges.

IMA Journal of Management Mathematics 2012; doi: 10.1093/imaman/dps015.

Samuelson, P. A. (1971) The "Fallacy" of Maximizing the Geometric Mean in Long Sequences of Investing or Gambling. *Proceedings National Academy Sciences USA*, **68**, No. 10, pp 2493-2496.

Thorp, E. O. (1969) Optimal Gambling Systems for Favorable Games. *Review of the International Statistical Institute*, **37**, 273-293.