Orderings

Definitions

Let R be a binary relation on a set A.

R is a **pre-order** iff R is reflexive and transitive.

R is **anti-symmetric** iff for all $a, b \in A$

$$a \, R \, b \wedge b \, R \, a \Rightarrow a = b$$

R is a **partial order** iff R is reflexive, transitive and anti-symmetric.

R is **irreflexive** iff $\forall a \in A. \neg(a \, R \, a)$.

R is a **strict partial order** iff R is irreflexive and transitive.

A partial order R is a **total order** iff $\forall a, b \in A. \, (a \, R \, b \vee b \, R \, a)$.

Examples

1. The numerical orders $\leq$ on $\mathcal{N}$, $\mathcal{Z}$ and $\mathcal{R}$ are total orders. The orders $<$ are strict partial orders.
2. Division on $\mathcal{N} \setminus \{0\}$ is a partial order: for all $n, m \in \mathcal{N}$

   $$n \leq m \text{ iff } n \text{ divides } m.$$

3. For any set A, the power set of A ordered by subset inclusion is a partial order.
4. $(A, \leq_A)$ is a partial order and $B \subseteq A$. Then $(B, \leq_B)$ is a partial order, where $\leq_B$ denotes the restriction of $\leq_A$ to B.

Ordering of Products

For any two partially ordered sets $(A, \leq_A)$ and $(B, \leq_B)$, there are two important orders on the product set $A \times B$:

- **product order**

$$(a_1, b_1) \leq_P (a_2, b_2) \text{ iff } (a_1 \leq_A a_2) \wedge (b_1 \leq_B b_2)$$

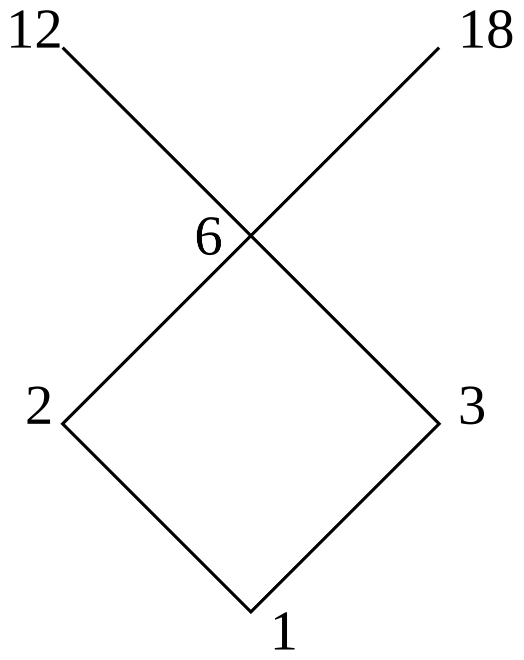
- **lexicographic order**

$$(a_1, b_1) \leq_L (a_2, b_2) \text{ iff } (a_1 <_A a_2) \vee (a_1 = a_2 \wedge b_1 \leq_B b_2).$$

If $(A, \leq)$ and $(B, \leq)$ are total orders, then the lexicographic order on $A \times B$ is total. In general, the product order is partial.

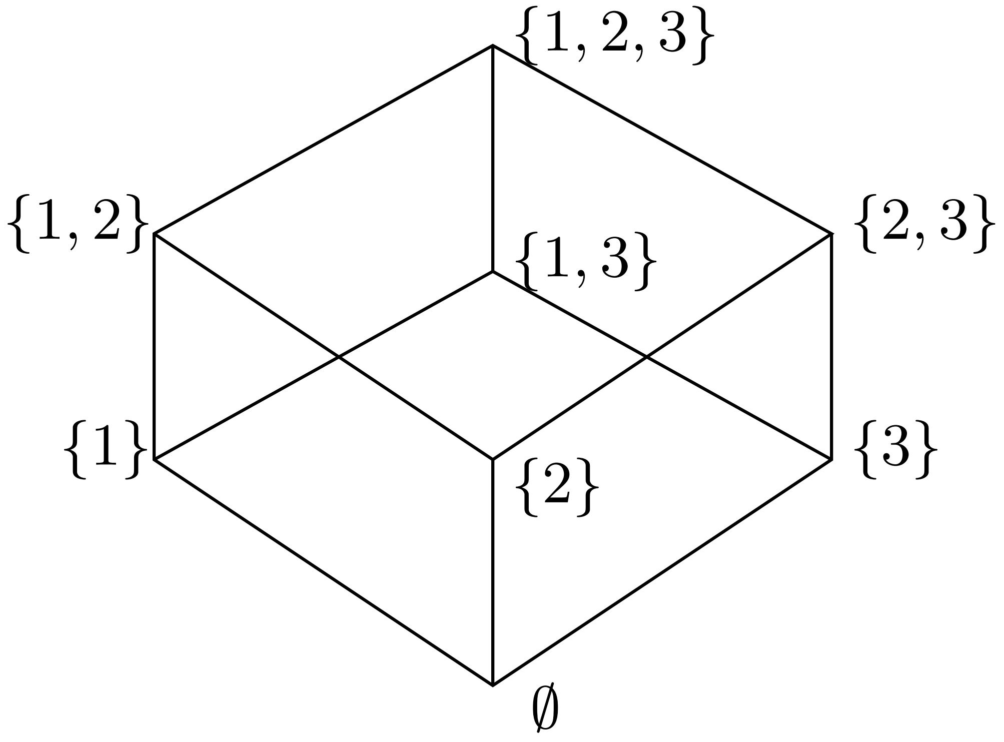
Hasse Diagram 1

The Hasse diagram for the relation ‘is a divisor of’ for the set $\{1, 2, 3, 6, 12, 18\}$ is



Hasse Diagram 2

The Hasse diagram for the binary relation $\subseteq$ on $\mathcal{P}(\{1, 2, 3\})$ is



Analysing Partial Orders

Let $(A, \leq)$ be a partial order.

1. An element $a \in A$ is **minimal** iff $\forall b \in A. \, (b \leq a \Rightarrow b = a)$.
2. An element $a \in A$ is **least** iff $\forall b \in A. \, a \leq b$.
3. An element $a \in A$ is **maximal** iff $\forall b \in A. \, (a \leq b \Rightarrow a = b)$.
4. An element $a \in A$ is **greatest** iff $\forall b \in A. \, b \leq a$.

Topological Sorting

Let $(A, \leq)$ denote a finite **partial** order. We can construct a **total** order $\leq_T$ on A such that $\forall a, b \in A.(a \leq b \Rightarrow a \leq_T b)$.

Proof Choose a minimal element $a_1 \in A$.

Such an element exists since A is finite.

Note that $(A \backslash \{a_1\}, \leq)$ is also a partial order.

If it is non-empty, choose a minimal element $a_2 \in A \backslash \{a_1\}$.

Continue this process, until there are no more elements left.

Since A is finite, this process must terminate.

The total order is given by the sequence $a_1, a_2, a_3, \ldots$.

Example

Consider the function $\mathsf{Ack} : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$ defined by

$$\begin{aligned}
\mathsf{Ack}(0, y) &= y + 1 \\
\mathsf{Ack}(x+1, 0) &= \mathsf{Ack}(x, 1) \\
\mathsf{Ack}(x+1, y+1) &= \mathsf{Ack}(x, \mathsf{Ack}(x+1, y))
\end{aligned}$$

We will prove that this function always terminates using a **well-founded partial order**.

Well-founded Partial Orders

A partial order $(A, \leq)$ is **well-founded** if and only if it has no infinite decreasing chain of elements:

that is, for **every** infinite sequence $a_1, a_2, a_3, \ldots$ of elements in A with $a_1 \geq a_2 \geq a_3 \geq \ldots$, there exists $m \in \mathcal{N}$ such that $a_n = a_m$ for every $n \geq m$.

Proposition

If two partial orders $(A, \leq)$ and $(B, \leq)$ are well-founded, then the lexicographical order on $A \times B$ is also well-founded.

Proof Suppose $(a_1, b_1) \geq_L (a_2, b_2) \geq_L (a_3, b_3) \geq_L \ldots$.

Then $a_1 \geq_A a_2 \geq_A a_3 \geq_A$ ... by the definition of $\geq_L$.

Since $(A, \leq_A)$ is well-founded, there exists $m \in \mathcal{N}$ such that $a_n = a_m$ for every $n \geq m$.

We also have $b_m \geq_B b_{m+1} \geq_B b_{m+3} \geq_B \ldots$.

This sequence must also end up being constant because $(B, \leq_B)$ is well-founded. Thus, the original sequence is ultimately constant.

Back to the Ack Function

Consider the strict lexicographical order on $\mathcal{N} \times \mathcal{N}$ by

$(x, y) < (x', y')$ if and only if $x < x'$ or ($x = x'$ and $y < y'$)

Notice that

$$\begin{aligned}
(x+1,0) &> (x,1) \\
(x+1,y+1) &> (x,\mathsf{Ack}(x+1,y)) \\
(x+1,y+1) &> (x+1,y)
\end{aligned}$$

Evaluating the Ack function takes us down the order. The order is well-founded. Hence, the Ack program always gives an answer.