A Tighter Bound on the Area Occupied by a Fractal Image

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Abstract

We derive a bounding circle for a fractal image specified by an iterated function system. The radius of the bounding circle is smaller than those from previously published material. The bounding circle is important in fractal design and plotting software as it enables a fractal image to be scaled correctly to fit the screen of a digital computer.

1 Introduction

Iterated function system (IFS) fractal images, as popularised by Barnsley[1, 2], are constructed from sheared, reduced, rotated and displaced copies of themselves. Various algorithms exist for plotting fractal images from their affine transformations[2, 3, 4, 5, 6, 7]. However, in order to plot a fractal image on the screen of a digital computer, all of these algorithms require advance knowledge of a *bounding area* inside which the image is known to lie. Upper bounds for the radius of a circle enclosing an IFS fractal image are derived in [8, 9]. In this letter, we derive a tighter bound than in previously published material.

2 A bounding circle for the image of an IFS

We follow [9] in choosing a point (xc, yc) and deriving the radius of a bounding circle centred on (xc, yc). A circle B of radius r centred on (xc, yc) has co-ordinates

$$(xc + r\cos\theta, yc + r\sin\theta),\ 0 \le \theta < 2\pi$$

An affine transformation defined by $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$ maps B to the ellipse

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$$(a(xc + r\cos q) + b(yc + r\sin q) + e, c(xc + r\cos q) + d(yc + r\sin q) + f),\ 0 \le q < 2p$$

B is a bounding circle for a fractal image if each transformation in its IFS maps B to an ellipse that lies inside B. Figure 1 shows the fractal image *dragon,* a bounding circle for *dragon,* and the ellipses to which the bounding circle is mapped by the transformations.

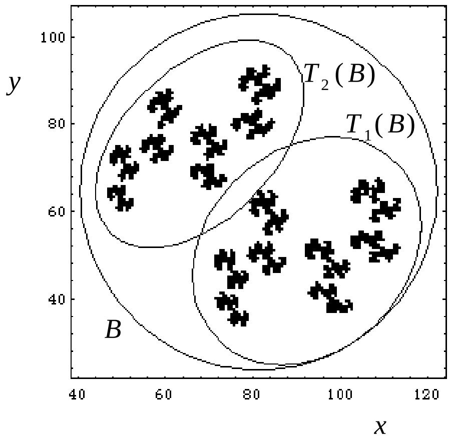

Figure 1 The fractal image *dragon*, defined by

$$T_1 + \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0.5 & -0.4 \\ 0.5 & 0.4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 77.5 \\ -15.5 \end{pmatrix} \text{ and}$$

$$T_2 + \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0.5 & -0.3 \\ 0.5 & 0.3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 46.5 \\ 15.5 \end{pmatrix},$$

a bounding circle B of radius 40.8 centred on $(81.3, 64.5)$, and the ellipses $T_1(B)$ and $T_2(B)$.

The distance between two points (x,y) and (x',y') is given by $\sqrt{(x-x')^2 + (y-y')^2}$. The ellipse E from a transformation lies inside B if all of the points on E lie inside B, i.e. if $\forall q,\ 0 \le q < 2p$,

$$(a(xc + r\cos q) + b(yc + r\sin q) + e - xc)^2 +$$
$$(c(xc + r\cos q) + d(yc + r\sin q) + f - yc)^2 \le r^2$$

We can use this inequality to get r as a function of q, xc and yc. Clearly r will be minimum when the two sides of the inequality are equal for at least one value of q. Figure 2 shows graphs of r vs. q for the transformations defining *dragon* with $(xc, yc) = (81.3, 64.5)$.

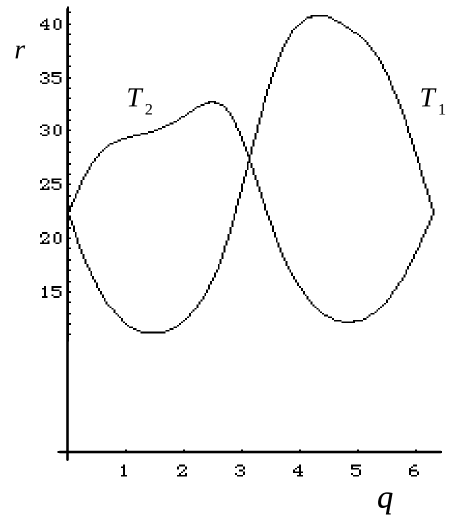

Figure 2 r vs. q for the transformations of *dragon* with $(xc, yc) = (81.3, 64.5)$.

The maximum value of r on Figure 2 is the radius of a bounding circle for *dragon* centred on $(81.3, 64.5)$. Maximising with respect to q, we get r = 40.8.

3 Discussion

For any given centre, we find that our algorithm (*A*) always gives a radius less than or equal to the radius given by the previous best formula[8, 9] (*PF*), as shown in Figure 3.

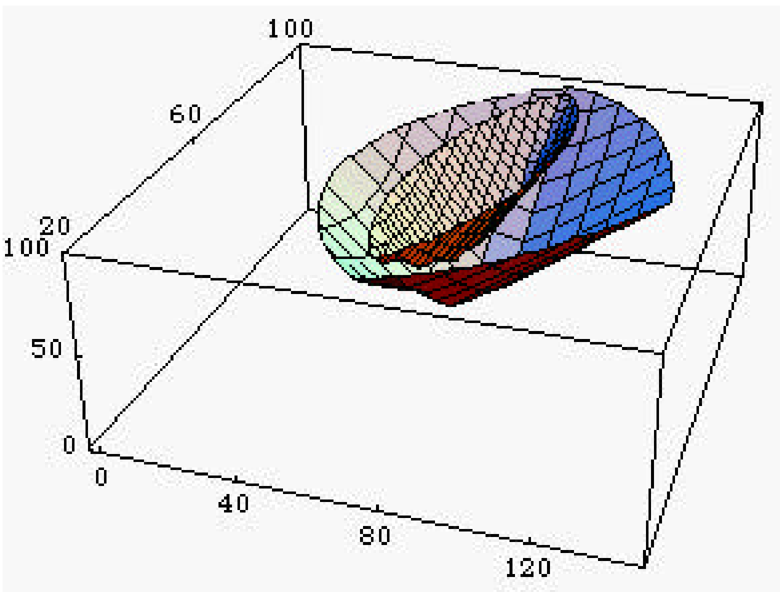

Figure 3 r vs. (xc, yc) for *dragon*, for *A* and *PF* where $r \le 100$. The higher surface is *PF* and the lower surface is *A*. It is clear that $PF(xc, yc)$ $A(xc, yc)$.

For example, for *dragon*, the smallest radius for *PF* is 59.7 at the centre $(81.3, 64.5)$, compared with a radius of 40.8 given by *A* at this centre. The smallest radius for *A* is 35.2 at the centre $(83.8, 62.0)$, compared with a radius of 67.6 given by *PF* at this centre. The results for *dragon* are typical of IFS fractal images.

The fundamental difference between the two approaches is that *PF* assumes that the orientation of each ellipse is in its worst-case, i.e. that its major axis lies along a diameter of B. *A* calculates the actual orientation of each ellipse.

The algorithm for finding the best centre for *PF*[9] will also work for *A* if the equation for r is expressed symbolically in terms of xc and yc.

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