Decidability of consistency of function and derivative information for a triangle and a convex quadrilateral

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Abstract

Given a triangle in the plane, a planar convex compact set and an upper and and a lower bound, we derive a linear programming algorithm which checks if there exists a real-valued Lipschitz map defined on the triangle and bounded by the lower and upper bounds, whose Clarke subgradient lies within the convex compact set. We show that the problem is in fact equivalent to finding a piecewise linear surface with the above property. We extend the result to a convex quadrilateral in the plane. In addition, we obtain some partial results for this problem in higher dimensions.

1 Introduction

The Mean Value Theorem (MVT) in dimension one has a simple (in fact trivial) converse: Given a non-empty compact interval C of the real line (regarded as the derivative information for a function) and two points $(x_1, h_1), (x_2, h_2) \in \mathbb{R}^2$ in the plane with $x_1 < x_2$, the following conditions are equivalent:

- There is a continuous map $f : [x_1, x_2] \to \mathbb{R}$ which is C^1 in (x_1, x_2) with $f' \in C$ such that $f(x_i) = h_i$ for $i = 1, 2$.
- $(h_2 - h_1)/(x_2 - x_1) \in C$.

Moreover when $(h_2 - h_1)/(x_2 - x_1) \in C$, there is a minimal function satisfying this property namely the affine map L through the two points (x_1, h_1) and (x_2, h_2) which has the property that for any other function f satisfying this property we have $\min f \leq \min L$ and $\max L \leq \max f$.

In this technical report we aim to extend the above converse of the MVT to higher dimensions as follows. The original motivation for this problem arose from solving the problem of consistency of the function and derivative information for a locally Lipschitz real-valued map defined on a finite dimensional Euclidean space, where the derivative information is given by the Clarke subgradient. It was shown in [1] that this consistency is decidable if the Clarke gradient at each point is approximated by the smallest axis aligned hyper-rectangle it contains, but the general problem when the Clarke gradient is given by a non-empty compact convex subset was left as open.

2 The case of a triangle

Suppose we have a non-empty convex and compact polygon B of the plane as the derivative information. Assume three distinct points $v_i = (x_i, y_i)$ with $i = 1, 2, 3$ and T_{123}, or simply T, is the closed region defined by these three points. Suppose we have $h_i \in \mathbb{R}$ for $i = 1, 2, 3$. We aim to establish if there is a Lipschitz witness $z = w(x, y)$ with $w : T \to \mathbb{R}$, that goes through the three points (v_i, h_i), for $i = 1, 2, 3$, whose derivative everywhere is contained in B. Let $z = P(x, y) = \alpha x + \beta y + \gamma$ with $P : T \to \mathbb{R}$ be the plane that goes through three points (v_i, h_i), for $i = 1, 2, 3$ with $\nabla P = (\alpha, \beta) = b$. Then, w satisfies our requirements iff $w - P$ goes through $(v_i, 0)$, for $i = 1, 2, 3$, with its derivative consistent with $B - b$. So we can equivalently consider the latter problem of finding a Lipschitz maps that goes through $(x_i, y_i, 0)$, for $i = 1, 2, 3$, with its derivative contained everywhere in $B' := B - b$.

Let $e_{ij} : v_j - v_i$, with ij in the cyclic order $1, 2, 3$ and let $e_{ij}^{\perp} \in \mathbb{R}^2$ be unit vector orthogonal to e_{ij} in the direction into the triangle T.

By the mean value theorem (MVT) for Lipschitz maps, for a witness to exist, it is necessary that for all distinct pairs $i, j = 1, 2, 3$, we have: $0 \in B' \cdot e_{ij}$.

Assume therefore that these conditions, called the MVT conditions, hold. Thus, there exist $b_{ij} \in B'$ for cyclic ordered pairs ij such that $b_{ij} \cdot e_{ij} = 0$, i.e., $b_{ij} = k e_{ij}^{\perp}$ for some $k \in \mathbb{R}$. Let $[b_{ij}^-, b_{ij}^+]$ be the interval along the direction $e_{ij}^{\perp}$ that is contained in B'. Let P_{ij}^- be the plane with $\nabla P_{ij}^- = b_{ij}^-$ that contains e_{ij} for each cyclic order ij.

Proposition 2.1. *Suppose the MVT conditions hold. Then, there is Lipschitz map $w^* : T \to \mathbb{R}$ that goes through the three points $(x_i, y_i, 0)$ with $i = 1, 2, 3$ and whose derivative is contained in B', such that for every other witness w with these properties we have:* $\min w \leq w^* \leq \max w$.

Proof. If $b \in B$ (i.e. $0 \in B'$), then the plane $w^* = P$ satisfies our conditions; see the thick dashed triangle in the figure. Suppose, therefore, that $b \notin B$. By considering witnesses of the form $w - P$ we can equivalently consider the reduced problem with $h_i = 0$ and $B' = B - b$. Then $0 \notin B'$. In particular, the convex hull of the three segments $[b_{ij}^-, b_{ij}^+]$ along $e_{ij}^{\perp}$ does not contain 0. This means that there exist two pairs $i_1 i_2$ and $i_3 i_1$ such that precisely one of the following two conditions hold:

(i) Both $b_{i_1 i_2}^-$ and $b_{i_3 i_1}^-$ have positive components along $e_{i_1 i_2}^{\perp}$ and $e_{i_3 i_1}^{\perp}$ respectively while $b_{i_2 i_3}^+$ has negative component on $e_{i_2 i_3}^{\perp}$.

(i) Both $b_{i_1 i_2}^-$ and $b_{i_3 i_1}^-$ have negative components along $e_{i_1 i_2}^{\perp}$ and $e_{i_3 i_1}^{\perp}$ respectively while $b_{i_2 i_3}^+$ has positive component on $e_{i_2 i_3}^{\perp}$.

We consider the first case; the second is similar. Thus, assume there exist two pairs $i_1 i_2$ and $i_3 i_1$ such that $b_{i_1 i_2}^-$ and $b_{i_3 i_1}^-$ have positive components along $e_{i_1 i_2}^{\perp}$ and $e_{i_3 i_1}^{\perp}$ respectively while $b_{i_2 i_3}^+$ has negative component on $e_{i_2 i_3}^{\perp}$.

Let $c_0 := b_{i_3 i_1}^-, c_1, \ldots, c_{k-1}, c_k := b_{i_1 i_2}^-$ be the vertices of B' from $b_{i_3 i_1}^-$ to $b_{i_1 i_2}^-$ on the same side of the origin with respect to the line ℓ that goes through $b_{i_3 i_1}^-$ and $b_{i_1 i_2}^-$. For any $c \in \mathbb{R}^2$ let $z = P_c(x, y) = c_1 x + c_2 y + \gamma_c$ be the plane through (v_{i_1}, h_{i_1}) with $\nabla P_c = c$. Therefore, using our previous notation, we have: $P_{c_0} = P_{i_3 i_1}$ and $P_{c_k} = P_{i_1 i_2}$. Let $w^* := \min\{P_{c_j}^- : 0 \leq j \leq k\}$.

Now let w be any Lipschitz map through the three points $v_i = (x_i, y_i, 0)$ for $i = 1, 2, 3$ and consistent with B'. Then, w is differentiable a.e., and is equal to the integral of its derivative. Consider any path $p : [0, 1] \to T : t \mapsto v_{i_1} + t(r - v_{i_1})$ from vertex v_{i_1} to a point r on the opposite edge E of T. Then

$\inf B' \cdot (r - v_{i_1}) = c_j \cdot (r - v_{i_1})$ for some j with $0 \le j \le k$ and $w^* \upharpoonright [v_{i_1}, r] = P_{c_j}$, where $[v_{i_1}, r]$ is the line segment from v_{i_1} to r in T. Thus we get:

$$w(r) - w(v_{i_1}) = \int_0^1 w'(v_{i_1} + t(r - v_{i_1})) \cdot (r - v_{i_1})\, dt \ge \int_0^1 c_j \cdot (r - v_{i_1})\, dt =$$

$$\int_0^1 (w^*)'(v_{i_1} + t(r - v_{i_1})) \cdot (r - v_{i_1})\, dt = w^*(r) - w^*(v_{i_1})$$

On the other hand,$w(v_{i_1}) = w^*(v_{i_1}) = h_{i_1} = 0$. But $\max w^* = \max_{r \in E} w^*(r)$. Thus we get

$$\max w \ge \max_{r \in E} w(r) \ge \max_{r \in E} w^*(r) = \max w^*$$

and hence $\max w \ge \max w^*$. Since $\min w \le \min\{h_i : i = 1, 2, 3\} = \min w^*$, the result follows.

□

Now consider the original problem with three points (x_i, y_i, h_i), for $i = 1, 2, 3$, and derivative information B. Suppose lower and upper limits $c^- \le c^+$ are given. Let $P : T \to \mathbb{R}$ be the plane that goes through three points (x_i, y_i, h_i), for $i = 1, 2, 3$ with $\nabla P = b$ and put $B' = B - b$. Consider w^* constructed above.

Theorem 2.2. *There is a witness to consistency if and only if the following two conditions hold:*

- *For all distinct pairs $i, j = 1, 2, 3$, we have: $0 \in B' \cdot e_{ij}$.*
- *$c^- \le w^* + P \le c^+$.*

This shows that consistency is semi-decidable, i.e., for any given $h = (h_1, h_2, h_3) \in \mathbb{R}^3$ we can decide if there is a witness for consistency with heights h_i at vertex v_i for $i = 1, 2, 3$. Returning to the original problem we have the following. Let rational points v_i for $i = 1, 2, 3$, forming a triangle in the plane, a rational convex polygon B and rational numbers $c^- \le c^+$ be given.

Theorem 2.3. *It is decidable that there exist h_i ($i = 1, 2, 3$) for which there exists a Lipschitz witness going through (v_i, h_i) (for $i = 1, 2, 3$) consistent with B and the bound c^- and c^+ in the closed region bounded by the triangle.*

Proof. We first check if there exists $h \in \mathbb{R}^3$ such that the plane $z = P(x, y) = \alpha x + \beta y + \gamma$ going through the three points (v_i, h_i) for $i = 1, 2, 3$ has gradient $(\alpha, \beta) \in B$. If this condition holds then P is clearly a witness and we are finished. Otherwise, we know from the construction in this section that either condition (i) or condition (ii) holds. If, for example, condition (i) holds, then there is a witness w if and only if $w^* + P$ is a witness where w^* is the piecewise linear surface constructed above using $B' = B - \nabla P$ and $z = P(x, y)$ is the plane through the three points (v_i, h_i) with $h_i = w(v_i)$ for $i = 1, 2, 3$. We now show that for each consecutive pair of vertices c_j and c_{j+1} in the construction of w^* above, the line of intersection of the two planes P_{c_j} and $P_{c_{j+1}}$ (equivalently the line of intersection of the two planes $P'_j := P_{c_j} + P$ and $P'_{j+1} := P_{c_{j+1}} + P$) is perpendicular to the line segment $c_j c_{j+1}$. Recall that all planes P_{c_j} pass through the point $(v, 0)$ with $v := v_{i_1}$. Thus we have

$$P_j(v) = P_{j+1}(v) = 0$$

For $u = (x, y) \in T$ we thus have:

$$P'_j(u) = P'_{j+1}(u) \iff P_{c_j}(u) = P_{c_{j+1}}(u) \iff P_{c_j}(u) - P_{c_j}(v) = P_{c_{j+1}}(u) - P_{j+1}(v)$$

$$\iff c_j \cdot (u - v) = c_{j+1} \cdot (u - v)$$

From the last equality we get $(u - v) \cdot (c_j - c_{j+1}) = 0$ as required. Now let the line in T passing through v_{i_1} perpendicular to $c_{j-1} - c_j$ intersect the edge $v_{i_3} v_{i_2}$ at point d_j for $j = 1, \ldots, k$. Thus, the edge $i_3 i_2$ is partitioned by the points $d_0 := v_{i_3}, d_1, \ldots, d_k, d_{k+1} := v_{i_2}$ and T is triangulated into $k + 1$ sub-triangles on each of which w^* is linear. Let $t_j = w^*(d_j)$ for $0 \le j \le k + 1$. Thus, the existence of a witness $w^* + P$ is equivalent to the existence of $t_j \in \mathbb{R}$ for $0 \le j \le k + 1$ such that

- $c^- \le t_j \le c^+$ for $0 \le j \le k + 1$, and.
- The plane going through the three points (d_j, t_j), (d_{j+1}, t_{j+1}) and $(v_{1_1}, 0)$ has its gradient in B' for $0 \le j \le k$.

□

For extension to quadrilaterals we adopt the following notation for the minimal object of a triangle given a non-empty compact and convex polygon $B \subset \mathbb{R}^2$. Let $p_i := (v_i, h_i)$. Then our construction shows that the minimal object is generated by two planes $P_{b_{i_1 i_2}}$ and $P_{b_{i_3 i_1}}$ that go through the edges $p_{i_1} p_{i_2}$ and $p_{i_3} p_{i_1}$ such that $b_{i_1 i_2}$ and $b_{i_3 i_1}$ are extreme points of B satisfying $b_{ij} \cdot (v_j - v_i) = (h_j - h_i)$ such that $|b_{ij} \cdot e^{\perp}_{ij}|$ has its minimum value for $i = i_1, j = i_2$ and $i = i_3, j = i_1$. We write $M^{+}_{123}\{p_{i_1} p_{i_2}, p_{i_3} p_{i_1}\}$, respectively $M^{-}_{123}\{p_{i_1} p_{i_2}, p_{i_3} p_{i_1}\}$, to indicate that the minimal object for the triangle T_{123} with vertices v_1, v_2, v_3 goes through the edges $p_{i_1} p_{i_2}$ and $p_{i_3} p_{i_1}$ and is above, respectively below, the plane $P : T_{123} \to \mathbb{R}$. We let

$$M_{123}\{p_{i_1} p_{i_2}, p_{i_3} p_{i_1}\} := M^{+}_{123}\{p_{i_1} p_{i_2}, p_{i_3} p_{i_1}\} \veebar M^{-}_{123}\{p_{i_1} p_{i_2}, p_{i_3} p_{i_1}\},$$

where $\veebar$ stands for exclusive or.

3 Extension to convex quadrilaterals

Consider a convex quadrilateral Q with vertices v_i, $i = 1, \ldots, 4$, enumerated clockwise. Let $B \subset \mathbb{R}^2$ be a compact convex polygon, representing the derivative information. Denote the edges of the quadrilateral by $i(i + 1)$ mod 4 for $i = 1, 2, 3, 4$. We denote the triangle with vertices v_i, v_j, v_k by T_{ijk} where i, j, k are in cyclic order $1, 2, 3, 4$. Let $e^{\perp}_{i(i+1)}$ be the unit vector in the orthogonal complement of $i(i + 1)$ directed into the quadrilateral. We will also let $e^{\perp}_{13}$ and $e^{\perp}_{24}$ be the orthogonal complements of the diameters $v_1 v_3$ and $v_2 v_4$ respectively, say, in the directions into the two triangles T_{123} and T_{234} respectively.

Assume we are given $h_i \in \mathbb{R}$, $i = 1, \ldots, 4$, such that the MVT conditions hold for the points $p_i := (v_i, h_i)$ for all the pairs of distinct vertices. Suppose $P : \mathbb{R}^2 \to \mathbb{R}$ is the plane that goes through

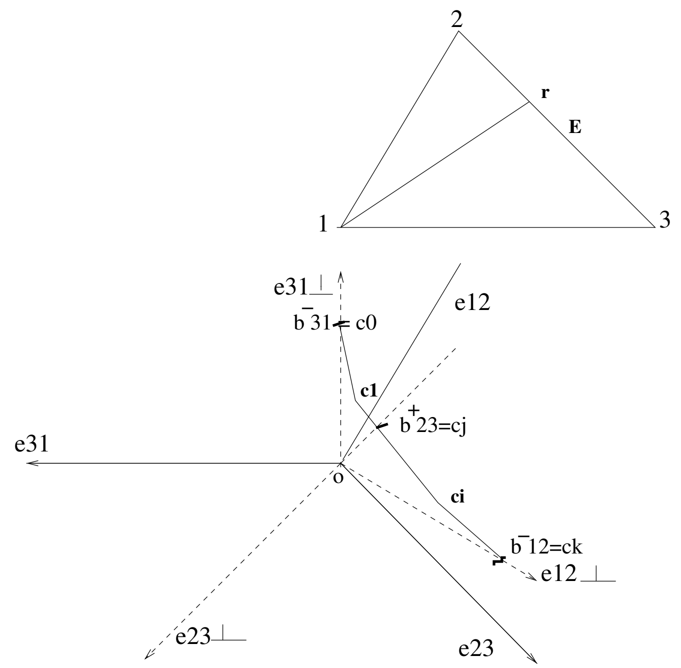

Figure 1.

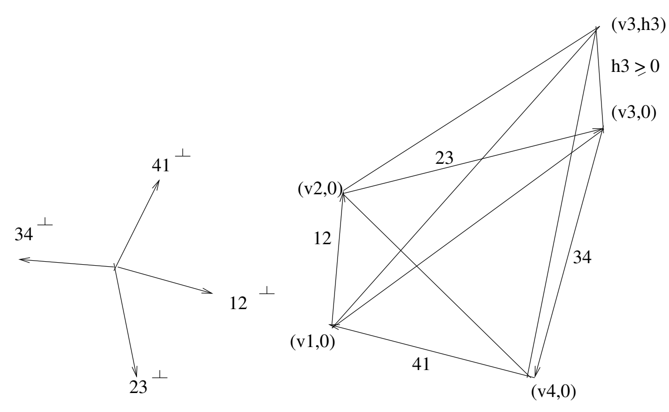

Figure 2.

$(v_4, h_4), (v_1, h_1)$ and (v_2, h_2). By considering $w - P$ for each witness w, we can again assume that $h_4 = h_1 = h_2 = 0$. We can also assume without loss of generality that $h_3 \geq 0$. See Figure 2.

Denote the plane going through the three distinct points p_i, p_j, p_k by $P_{ijk} : Q \to \mathbb{R}$. Each line segment $p_i p_{i+1}$ is an edge of two of these four triangles, namely $P_{i(i+1)k_1}$ and $P_{i(i+1)k_2}$ for $k_1, k_2 \neq i, j$ and $k_1 \neq k_2$. We have either $P_{i(i+1)k_1} \geq P_{i(i+1)k_2}$ or $P_{i(i+1)k_1} \leq P_{i(i+1)k_2}$. In fact, by our assumptions, we know that

$$P_{124} \leq P_{123}, \qquad P_{124} \leq P_{134}, \qquad P_{234} \leq P_{123}, \qquad P_{234} \leq P_{134} \tag{1}$$

For each distinct pair v_i, v_j, by the MVT conditions, there exists $b \in B$ such that $b \cdot (v_j - v_i) = (h_j - h_i)$. Let $b_{ij} \in B$ be the extreme point of B with $b_{ij} \cdot (v_j - v_i) = (h_j - h_i)$ such that $|b_{ij} \cdot e_{ij}^{\perp}|$ has its minimum value. Let $P_{ij} : Q \to \mathbb{R}$ be the linear map through the line segment $p_i p_j$ with $\nabla P_{ij} = b_{ij}$.

Then we have one of three possibilities: (i) $P_{ij} \geq P_{ijk_1}$ and $P_{ij} \geq P_{ijk_2}$, (ii) $P_{ij} \leq P_{ijk_1}$ and $P_{ij} \leq P_{ijk_2}$, (iii) $P_{ijk_1} < P_{ij} < P_{ijk_2}$ or $P_{ijk_1} > P_{ij} > P_{ijk_2}$.

We will show that if the MVT condition hold then we always have at least one minimal object for Q. If there are more than one minimal objects, then they will have the same maximum and minimum and thus in practice it is sufficient to find one minimal object for each given data. The following examples provide the prototype of the generic cases for a minimal object.

Assume $P_{12} \leq P_{124}$ (thus $P_{12} \leq P_{123}$) and $P_{41} \leq P_{124}$ (thus $P_{41} \leq P_{134}$). Then let us the following four cases (there are in fact more possibilities but these are sufficient).

(a) $P_{23} \geq P_{123}$ (thus $P_{23} \geq P_{234}$) and $P_{34} \geq P_{134}$ (thus $P_{34} \geq P_{234}$). In this case we have

$$(M_{123}\{p_1p_3, p_1p_3\} \veebar M_{123}\{p_1p_3, p_2p_3\}) \wedge (M_{134}\{p_1p_3, p_1p_4\} \veebar M_{134}\{p_1p_3, p_3p_4\})$$

Thus, the minimal objects in the two triangles T_{123} and T_{134}, as constructed in the previous section, go through the common edge p_1p_3 and thus, by gluing them together, they form a continuous piecewise linear map of type $Q \to \mathbb{R}$ which is a minimal object for Q.

(b) $P_{23} \leq P_{234}$ (thus $P_{23} \leq P_{123}$) and $P_{34} \geq P_{134}$ (thus $P_{34} \geq P_{234}$). Then, b_{41}, b_{12} and b_{23} are extreme points of B in clockwise order. Let c_i for $1 \leq i \leq n_1$ be the vertices of B in clockwise order between b_{41} and b_{12}, and let c_i for $n_1 < i \leq n_2$ be the vertices of B in clockwise order between b_{12} and b_{23}. Let P_{c_i}, for $1 \leq i \leq n_2$, be the planes with $\nabla P_{c_i} = c_i$ such that they go through $p_1 = (v_1, 0)$ for $1 \leq i \leq n_1$ and go through $p_2 = (v_2, 0)$ for $n_1 < i \leq n_2$. Put

$$P = \max\{P_{41}, P_{12}, P_{23}, P_{c_i} : 1 \leq i \leq n_2\} \tag{2}$$

Then P will be a minimal object. See Figure 3.

Note that in this case three of the four planes $P_{i(i+1)}$ (for $i = 1, 2, 3, 4 \bmod 4$) were below the corresponding planes of type $P_{i(i+1)k}$ (for some $k \neq i, i+1$). If three of the planes $P_{i(i+1)}$ (for $i = 1, 2, 3, 4 \mod 4$) were above the corresponding planes of type $P_{i(i+1)k}$ (for some $k \neq i, i+1$), then we would similarly obtain a minimal object by taking the minimum of the corresponding planes in Equation 2.

(c) $P_{23} \geq P_{123}$ (thus $P_{23} \geq P_{234}$) and $P_{34} \leq P_{234}$ (thus $P_{34} \leq P_{134}$). As in 1(b), using b_{34}, b_{41}, b_{12} and vertices of B in clockwise order from b_{34} to b_{41} and from b_{41} to b_{12}.

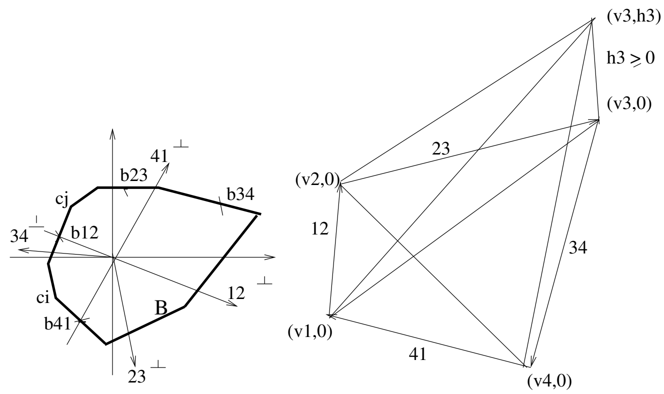

Figure 3.

(d) $P_{23} \le P_{234}$ (thus $P_{23} \le P_{123}$) and $P_{34} \le P_{234}$ (thus $P_{34} \le P_{134}$). Let $n_1 \le n_2 \le n_3 \le n_4$ be such that the set vertices of B in clockwise order from b_{12} to b_{23} to b_{34} to b_{41} and back to b_{12}, respectively, can be written as c_i for $1 \le i \le n_1$, $n_1 < i \le n_2$, $n_2 < i \le n_3$ and $n_3 < i \le n_4$ respectively. Let P_{c_i}, for $1 \le i \le n_1$, $n_1 < i \le n_2$, $n_2 < i \le n_3$ and $n_3 < i \le n_4$, respectively, be the planes that go through the vertices p_2, p_3, p_4 and p_1. Then, the minimal object is given by:

$$\max\{P_{12}, P_{23}, P_{34}, P_{41}, P_{c_i} : 1 \le i \le n_4\} \tag{3}$$

We note that if the four planes $P_{i(i+1)}$ were above the corresponding planes of type $P_{i(i+1)k}$ for some $k \neq i, i+1$, then we would get a minimal object by taking the minimum of the corresponding planes in Equation 3.

Theorem 3.1. *For a convex quadrilateral Q with vertices v_i ($i = 1, 2, 3, 4$), given B and real values $h_i \in \mathbb{R}$, ($i = 1, 2, 3, 4$), there exists a minimal object that is consistent with B and goes through the points (v_i, h_i), ($i = 1, 2, 3, 4$), if and only if the MVT conditions hold for all pairs of distinct points (v_i, h_i), ($i = 1, 2, 3, 4$).*

Proof. The only part follows immediately from the mean value theorem. Suppose therefore that the MVT conditions hold for all pairs of distinct points (v_i, h_i), ($i = 1, 2, 3, 4$). Let P be the plane through the points (v_i, h_i) for $i = 4, 1, 2$. By considering $w - P$ for any witness w, we can consider the reduced problem as presented earlier with $h_1 = h_2 = h_3 = 0$, the vertices v_i for $i = 1, 2, 3, 4$ in clockwise order and can assume without loss of generality that $h := h_3 \ge 0$.

Consider the two triangles $p_1p_2p_4$ and $p_2p_3p_4$ which have the edge p_2p_4 in common, and consider the four planes P_{12}, P_{23}, P_{34} and P_{41}, respectively, in relation to the planes P_{124}, P_{234}, P_{234} and P_{123}, respectively, i.e. consider the restriction of the following piecewise linear maps to Q:

(i) $P_{12} - P_{124} : Q \to \mathbb{R}$

(ii) $P_{23} - P_{234} : Q \to \mathbb{R}$

(iii) $P_{34} - P_{234} : Q \to \mathbb{R}$

(iv) $P_{41} - P_{124} : Q \to \mathbb{R}$

We have the following cases:

- If three of the maps above are non-negative or non-positive then we obtain a minimal object as in (b) above.
- If all four maps above are non-negative or non-positive then we obtain a minimal object as in (d) above.
- Suppose two of the maps are non-negative and the other two are non-positive:
  - If (i) and (iv) have opposite signs (and thus (ii) and (iii) have also opposite signs), then we obtain a minimal object as in (a), i.e., the gluing of the minimal object in T_{124} and T_{234}.
  - If (i) and (iv) are both non-positive then (ii) and (iii) are both non-negative and, in addition, the following two maps are also non-positive by Relations 1:

    $$P_{12} - P_{123} : Q \to \mathbb{R}, \qquad P_{12} - P_{134} : Q \to \mathbb{R}$$

    There are now two cases which we will consider in relation to the two triangles P_{123} and P_{341}:

    * If at least one of the two maps $P_{23} - P_{123} : Q \to \mathbb{R}$ and $P_{34} - P_{134} : Q \to \mathbb{R}$ is non-positive then we obtain a minimal object as in (b) or (d) with respect to the two triangles P_{123} and P_{341}.
    * If both maps $P_{23} - P_{123} : Q \to \mathbb{R}$ and $P_{34} - P_{134} : Q \to \mathbb{R}$ are non-negative, then we obtain a minimal-object as in (a) in relation to the two triangles P_{123} and P_{341}.
  - If (i) and (iv) are both non-negative then (ii) and (iii) are both non-positive and, in addition, the following two maps are also non-positive by Relations 1:

    $$P_{23} - P_{123} : Q \to \mathbb{R}, \qquad P_{34} - P_{134} : Q \to \mathbb{R}$$

    We again have two cases in relation to the two triangles P_{123} and P_{341}:

    * If at least one of the two maps $P_{12} - P_{123} : Q \to \mathbb{R}$ and $P_{41} - P_{134} : Q \to \mathbb{R}$ is non-positive then we obtain a minimal object as in (b) or (d) in relation to the two triangles P_{123} and P_{341}.
    * If both maps $P_{12} - P_{123} : Q \to \mathbb{R}$ and $P_{41} - P_{134} : Q \to \mathbb{R}$ are non-negative, then we obtain a minimal object as in (a) in relation to the two triangles P_{123} and P_{341}.

$\square$

Let rational points v_i for $i = 1, 2, 3, 4$, forming a convex quadrilateral in the plane, a rational convex polygon B and rational numbers $c^- \leq c^+$ be given.

Corollary 3.2. *It is decidable that there exists $h_i \in \mathbb{R}$, ($i = 1, 2, 3$) for which there exists a Lipschitz map that goes through (v_i, h_i) for $i = 1, 2, 3, 4$, is consistent with B and has lower and upper bounds c^- and c^+ in the closed region bounded by the quadrilateral.*

4 Partial extension to higher dimensions

Suppose we have a non-empty convex and compact polygon $B \subset \mathbb{R}^n$ as the derivative information. Assume we have $n+1$ points $v_i \in \mathbb{R}^n$ with $i = 0, 1 \ldots, n$ that are affinely independent, i.e., $v_i - v_0$ are linearly independent in $\mathbb{R}^n$ for $1 \le i \le n$. Let T be the convex hull of these points. Suppose we have $h_i \in \mathbb{R}$ for $0 \le i \le n$ and two real numbers $c^- \le c^+$. Consider the question whether there is a Lipschitz witness $w : T \to \mathbb{R}$, that goes through the points (v_i, h_i), for $0 \le i \le n$, whose derivative everywhere is contained in B. Let $P : T \to \mathbb{R}$ be the hyper plane that goes through (v_i, h_i), for $0 \le i \le n$ with $b := \nabla P$. Then, w satisfies our requirements iff $c^- \le w \le c^+$ and $w_0 := w - P$ goes through $(v_i, 0)$, for $0 \le i \le n$, with its derivative contained in $B - b$. So we can equivalently consider the latter problem of finding a Lipschitz map w_0 that goes through $(v_i, 0)$, for $0 \le i \le n$, with its derivative contained in $B' := B - b$ and satisfies $c^- \le w_0 + P \le c^+$. If $0 \in B'$ then the constant hyper-plane with value 0 passes through all the points $(v_i, 0)$. Thus, we assume $0 \notin B'$.

Let $e_{ij} := v_j - v_i$, with ij in the lexicographic order. Let $e_{ij}^{\perp}$ be the hyperplane of the orthogonal complement of e_{ij} in $\mathbb{R}^n$ with its normal directed into the simplex T.

By the mean value theorem (MVT) for Lipschitz maps, for a witness to exist, it is necessary that for all distinct pairs $i, j = 0, 1, \ldots, n$, we have: $0 \in B' \cdot e_{ij}$, i.e., $B' \cap e_{ij}^{\perp} \neq \emptyset$. Assume therefore that these conditions, called the MVT conditions, hold. Thus, for each distinct pair of vertices v_i and v_j, there exists $b_{ij}^* \in B'$ with minimum absolute value such that $b_{ij}^* \in e_{ij}^{\perp}$.

In the case, $n = 2$, we have seen that there is a vertex v_{i_1} of the triangle T such that $b_{i_1 i_2}^* = k e_{i_1 i_2}^{\perp}$ and $b_{i_1 i_3}^* = l e_{i_1 i_3}^{\perp}$ with k and l having the same sign. This meant that there is a minimal object which is either increasing or decreasing on T.

The property in $n = 2$ that there exists a vertex such that the extremal values of the derivative induced from the MVT have the same sign in the direction into the simplex does not hold for $n > 2$. As an example consider $n = 3$. Assume T is the standard simplex in $\mathbb{R}^3$ with vertices at the origin and at unit distance from the origin along each coordinate axis, x_i for $i = 1, 2, 3$. Thus, we have: $v_0 = (0, 0, 0)$, $v_1 = (1, 0, 0)$, $v_2 = (0, 1, 0)$ and $v_3 = (0, 0, 1)$. For each co-dimension one face F_{ijk} of T containing distinct vertices v_i, v_j and v_k, let n_{ijk} be the unit normal to F_{ijk} in the direction into T. Now let

$$B' = \text{Conv}\{(0, 1, 1), (-1, 0, -1), (1, -2, 0)\}$$

It is straightforward to check that the MVT conditions hold with respect to B' and T, and that we have the following values for b_{ij}^*:

$$\begin{array}{lll} b_{01}^* = (0, \frac{3}{25}, \frac{4}{25}), & b_{02}^* = (-\frac{5}{41}, 0, \frac{4}{41}), & b_{03}^* = (-\frac{5}{17}, \frac{8}{3}, 0) \\ b_{12}^* = (-\frac{1}{12}, -\frac{1}{12}, \frac{1}{12}), & b_{13}^* = (-\frac{1}{19}, -\frac{1}{19}, -\frac{1}{19}), & b_{23}^* = (-\frac{10}{51}, \frac{1}{51}, \frac{1}{51}) \end{array}$$

We then obtain the following table for the sign of $b_{ij}^* \cdot n_{ijk}$ from which we find that there is no vertex v_i of T for which the three planes going through its three edges e_{ij} for $j \neq i$ with extremal values of b_{ij}^* are all non-negative or all non-positive over T, i.e., for which $b_{ij}^* \cdot n_{ijk}$ has the same sign for $j \neq i$ and $k \neq i, j$.

	n_{012}	n_{013}	n_{023}	n_{123}
b^*_{01}	$-$	$+$		
b^*_{02}	$-$		$+$	
b^*_{03}		$-$	$+$	
b^*_{12}	$+$		$+$	
b^*_{13}		$+$		$-$
b^*_{23}			$-$	$-$

We can however decide, given T and B', if there is a Lipschitz map w on T with Clarke gradient contained in B whose directional derivative in T, wherever it exists, in the direction from some vertex into the simplex is always non-negative or always non-positive.

To decide this problem, we ask if there is a vertex v_i, with $0 \leq i \leq n$, of the simplex with points $b_{ij} \in B' \cap e^{\perp}_{ij}$ for $j \neq i$ such that

$$b_{ij} \cdot n_{ijk_1 \ldots k_{n-2}} \tag{4}$$

has the same sign for $j \neq i$ and distinct $k_m \neq i, j$ for $1 \leq m \leq n-2$. Since this question is equivalent to a finite number of linear inequalities, it is decidable. If such a vertex v_i exists, then we take the points b^*_{ij} that satisfy these common sign conditions such that each b^*_{ij} is at its minimum absolute value. Assume for definiteness that all the above signs are non-negative. Let U be the set of vertices of B' and put $U_i = U \cap \text{conv}\{0, b^*_{ij} : j \neq i\}$, i.e., the set of vertices of B' contained in the convex hull of the origin and the extremal points b^*_{ij}. For each $u \in U_i$, let P_u be the hyper-plane through v_i with $\nabla P_u = u$. Consider the map $P^+ : T \to \mathbb{R}$ defined by

$$P^+ = \min\{P_{ij}, P_u : j \neq i \,\&\, u \in U_i\}$$

Then, P^+ goes through $(v_k, 0)$ for $0 \leq k \leq n$, its Clarke gradient is contained in B' and its direction derivative in T in any direction $v - v_i$ for $v \in T$ is non-negative.

Proposition 4.1. *Suppose the MVT conditions hold. Then the following two statements are equivalent and decidable:*

(i) There is a real-valued Lipschitz map defined in T with Clarke gradient contained in B' that goes through $(v_k, 0)$ for $0 \leq k \leq n$ such that its directional derivative in T, wherever it exists, in the direction from some vertex into the simplex is always non-negative.

(ii) There is a vertex v_i for some $i = 0, \ldots, n$ such that all scalar products in (4) have non-negative signs.

Moreover, if these equivalent statements hold, then for any map w satisfying the conditions in (i) we have $\max P^+ \leq \max w$*, i.e., P^+ is a minimal object for such maps.*

For the example above, we can check that we have a solution for the two vertices v_1 and v_2. A simple calculation shows that for vertex v_1 we have:

$$b^*_{01} = (0, 0, 1/4), \quad b^*_{12} = (0, 0, 1/4), \quad b^*_{13} = (-1/4, -1/4, -1/4),$$

which give rise to the hyper-planes $P_{01}, P_{12}, P_{13} : T \to \mathbb{R}$ with $P_{01}(x_1, x_2, x_3) = x_3/4$ and $P_{12} = P_{13}$ with

$$P_{12}(x_1, x_2, x_3) \mapsto -\frac{x_1}{4} - \frac{x_2}{4} - \frac{x_2}{4} - \frac{1}{4}$$

Thus, the minimal object for witness maps that are increasing in the direction $v - v_1$ for any $v \in T$ is given

$$P^+ = \min\{P_{01}, P_{12}\},$$

which has its maximum equal to $1/8$ over T.

It can be checked that for the vertex v_2 we obtain the same hyper-planes P_{01} and P_{12} and thus the same minimal object.

We also obtain the following result for the original problem.

Theorem 4.2. *It is decidable that there exist $h_i \in \mathbb{R}$, $0 \leq i \leq n$, for which there exists a real-valued Lipschitz map w defined in T that goes through (v_i, h_i), with $c^- \leq w \leq c^+$ in T, whose Clarke gradient is contained in B and whose directional derivative in T, wherever it exists, in the direction from some vertex into T is always greater than, or always less than, that of the hyper-plane that goes through the points (v_i, h_i) for $0 \leq i \leq n$.*

References

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