233 Computational Techniques

Problem Sheet for Tutorial 2

Problem 1
Which of the following pairs of vectors are orthogonal:

(a) $[1, 2]$ and $[-1, 1]$,

(b) $[2, 5, 1]$ and $[-3, 1, 1]$,

(c) $[3, 5, 3, -4]$ and $[4, -2, 2, 2]$.

Problem 2
For

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 4 \\ -3 & 2 & 5 \end{bmatrix}, \quad \boldsymbol{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix},$$

decide which of the following products are defined, and compute them:
(a) $\boldsymbol{Au}$, (b) $\boldsymbol{Av}$, (c) $\boldsymbol{A}^T\boldsymbol{v}$, ($d$) $\boldsymbol{u}^T\boldsymbol{v}$, ($e$) $\boldsymbol{uv}^T$.

Problem 3
From the pair of vectors in problem 1(b), construct an orthonormal set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ such that two of them are multiples of the given pair.

Problem 4
Matrix representation of linear maps: Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map and let $\mathbf{e}_1, \mathbf{e}_2$ be a basis for $\mathbb{R}^2$. Suppose

$$f(\mathbf{e}_1) = 5\mathbf{e}_1 - 6\mathbf{e}_2 \quad f(\mathbf{e}_2) = \mathbf{e}_2 + 3\mathbf{e}_1.$$

- Find the matrix $\boldsymbol{A}$ representing f with respect to the basis $\mathbf{e}_1$, $\mathbf{e}_2$.
- If $\boldsymbol{v} \in \mathbb{R}^2$ is given by $\boldsymbol{v} = 2\mathbf{e}_1 - \mathbf{e}_2$. Find $f(\boldsymbol{v})$ and check that the matrix $\boldsymbol{A}$ representing f correctly computes the coordinates of $f(\mathbf{v})$ with respect to the basis $\mathbf{e}_1$, $\mathbf{e}_2$.

Problem 5
Matrix multiplication is not commutative: that is, $\boldsymbol{AB} \neq \boldsymbol{BA}$ in general. As an illustration, prove that a square 2×2 matrix $\boldsymbol{A}$ satisfying $\boldsymbol{AX} = \boldsymbol{XA}$ for every 2×2 matrix $\boldsymbol{X}$ must be a multiple of the unit matrix $\boldsymbol{I}_2$. In other words, prove the following:

$$\boldsymbol{A} \in \mathbb{R}^{2\times 2} \text{ and } \boldsymbol{AX} = \boldsymbol{XA} \text{ for all } \boldsymbol{X} \in \mathbb{R}^{2\times 2} \iff \exists \lambda \in \mathbb{R} \text{ such that } \boldsymbol{A} = \lambda \boldsymbol{I}_2 .$$

(This is true for square matrices of any size!) *Hint*: Compare $\boldsymbol{AX}$ and $\boldsymbol{XA}$ for matrices X which have one entry equal to 1 and all others zero; for instance for

$$\boldsymbol{E}_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{E}_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Note: The formulation was changed slightly in order to clarify the problem.

Solution

Problem 2
Two vectors are orthogonal if their dot product is zero. The dot products are $1\times(-1)+2\times 1 = 1$ for (a), $2\times(-3)+5\times 1+1\times 1 = 0$ for (b) and $3\times 4+5\times(-2)+3\times 2+(-4)\times 2 = 0$ for (c); so the pairs (b) and (c) are orthogonal, the pair (a) is not.

Problem 3
(a)

$$\boldsymbol{Au} = \begin{bmatrix} 1 & 0 & 4 \\ -3 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}.$$

(b) $\boldsymbol{Av}$ is not defined: the column dimension of $\boldsymbol{A}$ is 3, while the dimension of $\boldsymbol{v}$ is only 2.
(c)

$$\boldsymbol{A}^T\boldsymbol{v} = \begin{bmatrix} 1 & -3 \\ 0 & 2 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -7 \\ 6 \\ 23 \end{bmatrix}.$$

(d) $\boldsymbol{u}^T\boldsymbol{v}$ is not defined: $\boldsymbol{u}$ and $\boldsymbol{v}$ do not have the same dimension.
(e)

$$\boldsymbol{uv}^T = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} [2, 3] = \begin{bmatrix} 2 & 3 \\ 4 & 6 \\ -2 & -3 \end{bmatrix}$$

is the *outer product* of $\boldsymbol{u}$ and $\boldsymbol{v}$. It can also be understood as a product of two matrices with dimensions 3×1 and 1×2 respectively.

Problem 4
$[2, 5, 1] =: \boldsymbol{u}_1$ and $[-3, 1, 1] =: \boldsymbol{u}_2$ are already orthogonal. So the easiest thing to do is to find a third vector $\boldsymbol{u}_3$ which is orthogonal to both of them, and then to *normalize* each of the three vectors, i.e. to divide each of them by its Euclidean norm, resulting in a vector of norm 1. (If $\boldsymbol{u} \neq 0$, then its norm is nonzero, and $\boldsymbol{v} := \boldsymbol{u}/\|\boldsymbol{u}\|_2$ has Euclidean norm $\|\boldsymbol{v}\|_2 = 1$.)
In three dimensions, the first step can be done by taking the *vector product*[1] of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, since the vector product is always orthogonal to both vectors from which it is formed. So

$$\boldsymbol{u}_3 = \boldsymbol{u}_1 \times \boldsymbol{u}_2 = \begin{bmatrix} 5\times 1 - 1\times 1 \\ 1\times(-3) - 2\times 1 \\ 2\times 1 - 5\times(-3)) \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ 17 \end{bmatrix}$$

[1]The vector product of two vectors $\boldsymbol{a} = [a_1, a_2, a_3]$ and $\boldsymbol{b} = [b_1, b_2, b_3]$ is defined as the vector $[a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1]$.

Alternatively, let $\boldsymbol{u}_3^T = [a, b, c]$. Then the orthogonality conditions for the set $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ are

$$0 = \boldsymbol{u}_3^T \boldsymbol{u}_1 = 2a + 5b + c \quad \text{and} \quad 0 = \boldsymbol{u}_3^T \boldsymbol{u}_2 = -3a + b + c\,.$$

We can rearrange the second equation as $c = 3a - b$ and use this to eliminate c from the first equation: $0 = 2a + 5b + 3a - b = 5a - 4b$, or $b = -(5/4)a$. We can now express c in terms of a alone as $c = 3a + (5/4)a = (17/4)a$. So we get $\boldsymbol{u}_3^T = [a, -(5/4)a, (17/4)a] = a[1, -5/4, 17/4]$ and we can check that this vector is really orthogonal to both $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ for *any* choice of a. For instance, for $a = 4$, we obtain $\boldsymbol{u}_3^T = [4, -5, 17]$ as before.
The norms of the three vectors are

$$\begin{aligned} \|\boldsymbol{u}_1\| &= \sqrt{2^2 + 5^2 + 1^2} = \sqrt{30}\,, \quad \|\boldsymbol{u}_2\| = \sqrt{(-3)^2 + 1^2 + 1^2} = \sqrt{11}\,, \\ \|\boldsymbol{u}_3\| &= \sqrt{4^2 + (-5)^2 + 17^2} = \sqrt{330}\,, \end{aligned}$$

and so the resulting orthonormal set is

$$\boldsymbol{v}_1^T = [2, 5, 1]/\sqrt{30}\,, \quad \boldsymbol{v}_2^T = [-3, 1, 1]/\sqrt{11}\,, \quad \boldsymbol{v}_3^T = [4, -5, 17]/\sqrt{330}\,.$$

By the way, the $\boldsymbol{v}_i$ are only determined up to sign – orthogonality is a bilinear relation, and the negative of a vector has the same norm as the original vector. (So, for instance, if your third vector is $[-4, 5, -17]/\sqrt{330}$, that's also correct.)

Problem 5

(i)

$$\boldsymbol{A} = \begin{bmatrix} 5 & 3 \\ -6 & 1 \end{bmatrix}$$

(ii)

$$f(\boldsymbol{v}) = 7\mathbf{e}_1 - 13\mathbf{e}_2 = (7, -13)^T.$$

$$\boldsymbol{A}\boldsymbol{v} = \begin{bmatrix} 5 & 3 \\ -6 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ -13 \end{bmatrix}$$

Problem 6

Part "⇒": Let

$$\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then for $\boldsymbol{X} = \boldsymbol{E}_{12}$,

$$\boldsymbol{A}\boldsymbol{E}_{12} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix}, \quad \boldsymbol{E}_{12}\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix},$$

and so $\boldsymbol{A}\boldsymbol{E}_{12} = \boldsymbol{E}_{12}\boldsymbol{A}$ if and only if $a = d$ and $c = 0$. Similarly for $\boldsymbol{X} = \boldsymbol{E}_{21}$:

$$\boldsymbol{A}\boldsymbol{E}_{21} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & 0 \\ d & 0 \end{bmatrix}, \quad \boldsymbol{E}_{21}\boldsymbol{A} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix},$$

and so $\boldsymbol{A}\boldsymbol{E}_{21} = \boldsymbol{E}_{21}\boldsymbol{A}$ if and only if $a = d$ and $b = 0$. So from the hypothesis that $\boldsymbol{A}\boldsymbol{X} = \boldsymbol{X}\boldsymbol{A}$ for all $\boldsymbol{X}$, it follows that $a = d$ and $b = c = 0$, that is, $\boldsymbol{A}$ must be of the form

$$\boldsymbol{A} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = \lambda \boldsymbol{I}_2 \quad \text{for } \lambda = a.$$

Part "⇐": The unit matrix satisfies $\boldsymbol{X}\boldsymbol{I}_m = \boldsymbol{I}_m\boldsymbol{X} = \boldsymbol{X}$ for every matrix $\boldsymbol{X}$ and in every dimension m; so if $\boldsymbol{A} = \lambda\boldsymbol{I}_2$ for $\lambda \in \mathbb{R}$, then $\boldsymbol{A}\boldsymbol{X} = (\lambda\boldsymbol{I}_2)\boldsymbol{X} = \lambda(\boldsymbol{I}_2\boldsymbol{X}) = \lambda\boldsymbol{X}$ and $\boldsymbol{X}\boldsymbol{A} = \boldsymbol{X}(\lambda\boldsymbol{I}_2) = (\boldsymbol{X}\lambda)\boldsymbol{I}_2 = \boldsymbol{X}\lambda = \lambda\boldsymbol{X}$.