# 233 Computational Techniques

Problem Sheet for Tutorial 3

**Problem 1**
In 2 dimensions, the $\ell_p$ norm of a vector $\boldsymbol{x} = (x_1, x_2)$ is given by

$$\|\boldsymbol{x}\|_p = (|x_1|^p + |x_2|^p)^{1/p} \quad \text{for} \quad 1 \le p < \infty \,, \qquad \|\boldsymbol{x}\|_\infty = \max\{|x_1|, |x_2|\} \,.$$

Sketch the surfaces of constant $\ell_p$ norm of 1,

$$C_p := \{\boldsymbol{x} \in \mathbb{R}^2 : \|\boldsymbol{x}\|_p = 1\}$$

for $p = 1, 2, \infty$ in a rectangular coordinate system.

**Problem 2**

(i) Using the definition of the angle between two vectors, prove the *cosine theorem* of trigonometry:

$$\|\boldsymbol{u} - \boldsymbol{v}\|_2^2 = \|\boldsymbol{u}\|_2^2 + \|\boldsymbol{v}\|_2^2 - 2\|\boldsymbol{u}\|_2\|\boldsymbol{v}\|_2 \cos\phi \tag{1}$$

for all $\boldsymbol{u}, \boldsymbol{v} \neq \boldsymbol{0}$, where $\phi$ is the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$. Which theorem is the special case $\phi = \pi/2$?

(ii) From (1) and the fact that the sum of angles in a triangle is equal to $\pi$, deduce

$$\textit{(a)} \quad \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \,, \qquad \textit{(b)} \quad \cos\left(\frac{\pi}{4}\right) = \frac{1}{2}\sqrt{2} \,.$$

**Problem 3**
Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two matrices

$$\boldsymbol{A} = \begin{bmatrix} -3 & 0 & 4 \\ 1 & 2 & 3 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} -9 & 2 & 3 \\ -4 & 8 & 6 \\ 1 & 5 & 7 \end{bmatrix}.$$

Determine $\|\boldsymbol{A}\|_1$, $\|\boldsymbol{A}\|_\infty$ and $\|\boldsymbol{B}\|_1$, $\|\boldsymbol{B}\|_\infty$.

**Problem 4**
Which of the following sets of vectors are linearly independent:
*(a)* $[1, 5]$, $[2, 3]$;
*(b)* $[2, 1, -3]$, $[-1, 1, -6]$, $[1, 1, -4]$;
*(c)* $[1, 0, 3]$, $[-1, 1, 2]$, $[2, 0, -5]$ ?

**Problem 5**
For

$$\boldsymbol{A} = \begin{bmatrix} 2 & 0 & 1 \\ 3 & -1 & 2 \end{bmatrix},$$

find
*(a)* the nullspace of $\boldsymbol{A}$,
*(b)* the nullspace of $\boldsymbol{A}^T$,
*(c)* the range of $\boldsymbol{A}$,
*(d)* the range of $\boldsymbol{A}^T$.
*(e)* Check that null$\boldsymbol{A}^T$ is orthogonal to range$\boldsymbol{A}$, and that null$\boldsymbol{A}$ is orthogonal to range$\boldsymbol{A}^T$.
*(f)* For $\boldsymbol{x} = [1, 1, 1]^T$, find the two vectors $\boldsymbol{x}_R \in$ range$\boldsymbol{A}^T$ and $\boldsymbol{x}_N \in$ null$\boldsymbol{A}$ which satisfy $\boldsymbol{x} = \boldsymbol{x}_R + \boldsymbol{x}_N$. Check that $\boldsymbol{x}_R$ and $\boldsymbol{x}_N$ are orthogonal!

**Problem 6**
Prove:
*(a)* If $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is invertible, then its right and left inverses are equal; that is, if $\boldsymbol{AB} = \boldsymbol{I}$ and $\boldsymbol{CA} = \boldsymbol{I}$ then $\boldsymbol{B} = \boldsymbol{C}$.
*(b)* If $\boldsymbol{A}$ has an inverse, then the columns of $\boldsymbol{A}$ are linearly independent.
*(c)* If $\boldsymbol{A}$ and $\boldsymbol{B}$ are both nonsingular, then $\boldsymbol{AB}$ is nonsingular, and $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$.
*(d)* Suppose $\alpha \in \mathbb{R}$ and $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ such that $\alpha\boldsymbol{u}^T\boldsymbol{v} \neq 1$. Then $\boldsymbol{E} = \boldsymbol{I}_n - \alpha\boldsymbol{u}\boldsymbol{v}^T$ is nonsingular, and its inverse is $\boldsymbol{I}_n - \beta\boldsymbol{u}\boldsymbol{v}^T$, where

$$\beta = \frac{\alpha}{\alpha\boldsymbol{u}^T\boldsymbol{v} - 1}.$$