233 Computational Techniques

Problem Sheet for Tutorial 3

Problem 1
In 2 dimensions, the ℓ_p norm of a vector $\boldsymbol{x} = (x_1, x_2)$ is given by

$$\|\boldsymbol{x}\|_p = (|x_1|^p + |x_2|^p)^{1/p} \quad \text{for} \quad 1 \le p < \infty \,, \qquad \|\boldsymbol{x}\|_\infty = \max\{|x_1|, |x_2|\} \,.$$

Sketch the surfaces of constant ℓ_p norm of 1,

$$C_p := \{\boldsymbol{x} \in \mathbb{R}^2 : \|\boldsymbol{x}\|_p = 1\}$$

for $p = 1, 2, \infty$ in a rectangular coordinate system.

Problem 2

(i) Using the definition of the angle between two vectors, prove the *cosine theorem* of trigonometry:

$$\|\boldsymbol{u} - \boldsymbol{v}\|_2^2 = \|\boldsymbol{u}\|_2^2 + \|\boldsymbol{v}\|_2^2 - 2\|\boldsymbol{u}\|_2\|\boldsymbol{v}\|_2 \cos\phi \tag{1}$$

for all $\boldsymbol{u}, \boldsymbol{v} \neq \boldsymbol{0}$, where ϕ is the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$. Which theorem is the special case $\phi = \pi/2$?

(ii) From (1) and the fact that the sum of angles in a triangle is equal to π, deduce

$$\textit{(a)} \quad \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \,, \qquad \textit{(b)} \quad \cos\left(\frac{\pi}{4}\right) = \frac{1}{2}\sqrt{2} \,.$$

Problem 3
Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two matrices

$$\boldsymbol{A} = \begin{bmatrix} -3 & 0 & 4 \\ 1 & 2 & 3 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} -9 & 2 & 3 \\ -4 & 8 & 6 \\ 1 & 5 & 7 \end{bmatrix}.$$

Determine $\|\boldsymbol{A}\|_1$, $\|\boldsymbol{A}\|_\infty$ and $\|\boldsymbol{B}\|_1$, $\|\boldsymbol{B}\|_\infty$.

Problem 4
Which of the following sets of vectors are linearly independent:
(a) $[1, 5]$, $[2, 3]$;
(b) $[2, 1, -3]$, $[-1, 1, -6]$, $[1, 1, -4]$;
(c) $[1, 0, 3]$, $[-1, 1, 2]$, $[2, 0, -5]$?

Problem 5
For

$$\boldsymbol{A} = \begin{bmatrix} 2 & 0 & 1 \\ 3 & -1 & 2 \end{bmatrix} ,$$

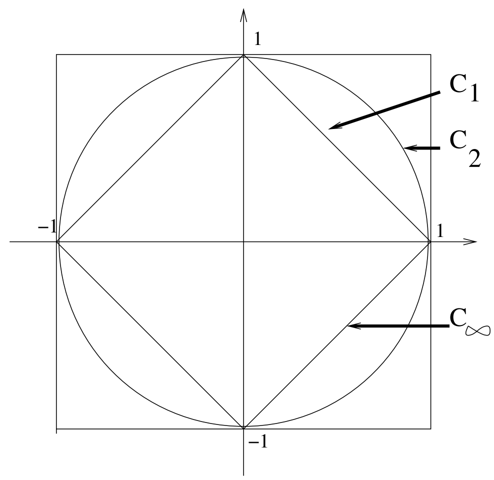
find
(a) the nullspace of $\boldsymbol{A}$,
(b) the nullspace of $\boldsymbol{A}^T$,
(c) the range of $\boldsymbol{A}$,
(d) the range of $\boldsymbol{A}^T$.
(e) Check that null$\boldsymbol{A}^T$ is orthogonal to range$\boldsymbol{A}$, and that null$\boldsymbol{A}$ is orthogonal to range$\boldsymbol{A}^T$.
(f) For $\boldsymbol{x} = [1, 1, 1]^T$, find the two vectors $\boldsymbol{x}_R \in$ range$\boldsymbol{A}^T$ and $\boldsymbol{x}_N \in$ null$\boldsymbol{A}$ which satisfy $\boldsymbol{x} = \boldsymbol{x}_R + \boldsymbol{x}_N$. Check that $\boldsymbol{x}_R$ and $\boldsymbol{x}_N$ are orthogonal!

Problem 6
Prove:
(a) If $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is invertible, then its right and left inverses are equal; that is, if $\boldsymbol{AB} = \boldsymbol{I}$ and $\boldsymbol{CA} = \boldsymbol{I}$ then $\boldsymbol{B} = \boldsymbol{C}$.
(b) If $\boldsymbol{A}$ has an inverse, then the columns of $\boldsymbol{A}$ are linearly independent.
(c) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are both nonsingular, then $\boldsymbol{AB}$ is nonsingular, and $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$.
(d) Suppose $\alpha \in \mathbb{R}$ and $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ such that $\alpha \boldsymbol{u}^T \boldsymbol{v} \neq 1$. Then $\boldsymbol{E} = \boldsymbol{I}_n - \alpha \boldsymbol{u}\boldsymbol{v}^T$ is nonsingular, and its inverse is $\boldsymbol{I}_n - \beta \boldsymbol{u}\boldsymbol{v}^T$, where

$$\beta = \frac{\alpha}{\alpha \boldsymbol{u}^T \boldsymbol{v} - 1} .$$

Solution

Problem 1



Problem 2

(i) The left hand side of (1) is

$$(\boldsymbol{u}-\boldsymbol{v})^T(\boldsymbol{u}-\boldsymbol{v}) = \boldsymbol{u}^T\boldsymbol{u} - \boldsymbol{u}^T\boldsymbol{v} - \boldsymbol{v}^T\boldsymbol{u} + \boldsymbol{v}^T\boldsymbol{v} = \boldsymbol{u}^T\boldsymbol{u} + \boldsymbol{v}^T\boldsymbol{v} - 2\boldsymbol{u}^T\boldsymbol{v} \ .$$

Here $\boldsymbol{u}^T\boldsymbol{u} = \|\boldsymbol{u}\|_2^2$ and $\boldsymbol{v}^T\boldsymbol{v} = \|\boldsymbol{v}\|_2^2$. The remaining term is

$$-2\boldsymbol{u}^T\boldsymbol{v} = -2\|\boldsymbol{u}\|_2\|\boldsymbol{v}\|_2 \frac{\boldsymbol{u}^T\boldsymbol{v}}{\|\boldsymbol{u}\|_2\|\boldsymbol{v}\|_2} = -2\|\boldsymbol{u}\|_2\|\boldsymbol{v}\|_2 \cos\phi \ ,$$

by the definition of the angle between two vectors which was given in the lecture notes. The cosine theorem allows to compute the angles of a triangle provided the lengths of the sides are known. If $\boldsymbol{u}$ and $\boldsymbol{v}$ are two sides of the triangle, then their lengths are $\|\boldsymbol{u}\|_2$ and $\|\boldsymbol{v}\|_2$, and the length of the third side is $\|\boldsymbol{u}-\boldsymbol{v}\|_2$). If $\phi = \pi/2$, then the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ is a right angle, and the cosine theorem then says that the squared lengths of the sides adjoining the right angle add up to the squared length of the side opposite. This is *Pythagoras' theorem.*

(ii) Take $\boldsymbol{u}$ and $\boldsymbol{v}$ in (1) as defining two sides of a triangle enclosing the desired angle. For *(a)*, take an equilateral triangle, for *(b)* one with one right angle and the other two angles $\pi/4$.

Problem 3
Reminder:

$$\|\boldsymbol{A}\|_1 = \max_j \|\boldsymbol{a}_j\|_1 \quad \text{the maximum absolute column sum,}$$

$$\|\boldsymbol{A}\|_\infty = \max_i \|\boldsymbol{a}^i\|_1 \quad \text{the maximum absolute row sum.}$$

Therefore, for $\boldsymbol{A}$: $\|\boldsymbol{A}\|_1 = \max\{3+1,\ 2,\ 4+3\} = \max\{4,2,7\} = 7$
and $\|\boldsymbol{A}\|_\infty = \max\{3+4,\ 1+2+3\} = \max\{7,6\} = 7$.
Accidentally, the two norms are equal.

For $\boldsymbol{B}$: $\|\boldsymbol{B}\|_1 = \max\{14,15,16\} = 16$ and $\|\boldsymbol{B}\|_\infty = \max\{14,18,13\} = 18$.

Problem 4
The sets *(a)* and *(c)* are independent, *(b)* is not. There are two ways of doing this: by working from the definition of linear dependence, and by using determinants.
From the definition (*(a)* only): A set $\{\boldsymbol{a}_1, \dots, \boldsymbol{a}_n\}$ of vectors is linearly dependent if
(i) there are real numbers $x_1, \dots, x_n$, not all zero, such that $x_1\boldsymbol{a}_1 + \dots + x_n\boldsymbol{a}_n = \boldsymbol{0}$,
or equivalently if
(ii) there is a real vector $\boldsymbol{x} = [x_1, \dots, x_n]^T \neq \boldsymbol{0}$ such that $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$, where $\boldsymbol{A} = [\boldsymbol{a}_1, \dots, \boldsymbol{a}_n]$ is the matrix which has $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$ as its columns.
If the only numbers x_i satisfying (i) are $x_1 = \dots = x_n = 0$ (the only vector $\boldsymbol{x}$ satisfying (ii) is the zero vector), then the set $\{\boldsymbol{a}_1, \dots, \boldsymbol{a}_n\}$ is linearly independent.
So let x_1, x_2 be such that

$$x_1 \begin{bmatrix} 1 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} . \tag{2}$$

From the first component one finds $x_1 = -2x_2$, and the second component gives $0 = 5x_1 + 3x_2 = -7x_2$ (substituting for x_1 according to the first equation). From the second equation, it follows that $x_2 = 0$, and from the first that $x_1 = 0$ as well. So the only pair x_1, x_2 satisfying (1) is $x_1 = x_2 = 0$. According to (i), the vectors $[1,5]^T$ and $[2,3]^T$ are linearly independent.
Using determinants: Recall the following facts about determinants:
- A square matrix $\boldsymbol{A}$ is singular if and only if its determinant is zero.
- The determinant of a 1×1 matrix (i.e. a real number) is the number itself.
- The determinant of a 2×2 matrix is

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc .$$

- The determinant of a 3×3 matrix can be computed in various ways; one of them is

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} = a \cdot \det \begin{bmatrix} e & f \\ h & k \end{bmatrix} - b \cdot \det \begin{bmatrix} d & f \\ g & k \end{bmatrix} + c \cdot \det \begin{bmatrix} d & e \\ g & h \end{bmatrix} .$$

(Note the pattern: • The signs are alternating. • Each term in the sum is the product of an element $*$ of the first row and the determinant of the 2×2 matrix which remains if the row and column of $*$ are deleted.)
The first fact leads to the following criterion for linear dependence:
(iii) n vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ in n-dimensional space are linearly dependent if and only if the matrix $[\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n]$ has zero determinant.
(*Proof:* $\det[\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n] = 0 \iff [\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n]$ singular $\iff$ *(ii)* above.)
Application:
(a) The determinant of $\begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix}$ is $1 \times 3 - 2 \times 5 = -7 \neq 0$, so $[1,5]$ and $[2,3]$ are linearly independent. (So are $[1,2]$ and $[5,3]$!)
(b) The same in 3 dimensions:

$$\begin{aligned} \det \begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & 1 \\ -3 & -6 & -4 \end{bmatrix} &= 2 \cdot \det \begin{bmatrix} 1 & 1 \\ -6 & -4 \end{bmatrix} - (-1) \cdot \det \begin{bmatrix} 1 & 1 \\ -3 & -4 \end{bmatrix} + 1 \cdot \det \begin{bmatrix} 1 & 1 \\ -3 & -6 \end{bmatrix} \\ &= 2 \times 2 - (-1) \times (-1) + 1 \times (-3) = 0 \,, \end{aligned}$$

so $[2,1,-3]$, $[-1,1,-6]$ and $[1,1,-4]$ are linearly dependent.
(c) Here the determinant of the resulting matrix is -11, so the vectors $[1,0,3]$, $[-1,1,2]$ and $[2,0,-5]$ are linearly independent.

Problem 5
(a) The null space of $\boldsymbol{A}$ is the set of all vectors $\boldsymbol{x}$ such that $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$. If $\boldsymbol{x}$ is such a vector, then

$$\boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} 2 & 0 & 1 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_3 \\ 3x_1 - x_2 + 2x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} ,$$

and so $x_3 = -2x_1$ and $x_2 = 3x_1 + 2x_3 = -x_1$. In vector notation, $\boldsymbol{x}^T = [x_1, -x_1, -2x_1] = x_1[1,-1,-2]$, that is, the null space of $\boldsymbol{A}$ is the set of all scalar multiples of the vector $[1,-1,-2]^T$. A concise notation for this statement is

$$\text{null}\boldsymbol{A} = \mathbb{R} \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} .$$

(b) If $\boldsymbol{y} \in \text{null}\boldsymbol{A}^T$, then

$$\boldsymbol{0} = \boldsymbol{A}^T\boldsymbol{y} = \begin{bmatrix} 2 & 3 \\ 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2y_1 + 3y_2 \\ -y_2 \\ y_1 + 2y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} ,$$

and so $y_2 = y_1 = 0$. It follows that $\text{null}\boldsymbol{A}^T = \{\boldsymbol{0}\}$. Another way to see this is to notice that the two columns in $\boldsymbol{A}^T$ are linearly independent; therefore $\boldsymbol{A}^T\boldsymbol{y} = \boldsymbol{0}$ forces $\boldsymbol{y} = \boldsymbol{0}$.

A third way is to do *(c)* first and to use the first of the dimension formulae

$$\begin{aligned}\dim(\text{range}\boldsymbol{A}) + \dim(\text{null}\boldsymbol{A}^T) &= \#\text{ of rows of } \boldsymbol{A}\ ,\\ \dim(\text{range}\boldsymbol{A}^T) + \dim(\text{null}\boldsymbol{A}) &= \#\text{ of columns of } \boldsymbol{A}.\end{aligned}$$

For the range space of $\boldsymbol{A}$ is two-dimensional, and so the null space of $\boldsymbol{A}^T$ is zero-dimensional. The only zero-dimensional vector space is $\{\boldsymbol{0}\}$.
(c) The range of $\boldsymbol{A}$ consists of all vectors in $\mathbb{R}^2$ which can be written in the form $\boldsymbol{Ax}$ for some $\boldsymbol{x} \in \mathbb{R}^3$. As

$$\boldsymbol{Ax} = x_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} ,$$

the range space of $\boldsymbol{A}$ is spanned by the three vectors $[2,3]^T$, $[0,-1]^T$, and $[1,2]^T$. Now it is easy to see that any two of these vectors are linearly independent; so their span is the entire $\mathbb{R}^2$. In other words, $\text{range}\boldsymbol{A} = \mathbb{R}^2$, and $\dim(\text{range}\boldsymbol{A}) = 2$.
(d) The range of $\boldsymbol{A}^T$ is

$$\text{range}\boldsymbol{A}^T = \{\boldsymbol{A}^T\boldsymbol{y} : \boldsymbol{y} \in \mathbb{R}^2\} = \left\{ y_1 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + y_2 \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \ : \quad y_1, y_2 \in \mathbb{R} \right\} .$$

Its dimension is indeed two, since the two generating vectors are linearly independent.
(e) $\text{null}\boldsymbol{A}^T = \{\boldsymbol{0}\}$, and the zero vector is orthogonal to any vector. For $\text{null}\boldsymbol{A}$ and $\text{range}\boldsymbol{A}^T$, it suffices to check that the vectors which generate them are orthogonal. Indeed,

$$[1,-1,-2] \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 0 \qquad \text{and} \qquad [1,-1,-2] \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = 0 .$$

(f) One way to do this is to compute the coefficients λ_i in

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \underbrace{\lambda_1 \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}}_{\boldsymbol{x}_N} + \underbrace{\lambda_2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}}_{\boldsymbol{x}_R} ;$$

$\boldsymbol{x}_N$ and $\boldsymbol{x}_R$ are then given by the bracketed terms.
$\boldsymbol{x}_N = -(1/3)[1,-1,-2] = [-1/3, 1/3, 2/3]$ and $\boldsymbol{x}_R = [1,1,1] - [-1/3,1/3,2/3] = [4/3, 2/3, 1/3]$.
These two vectors are indeed orthogonal: $\boldsymbol{x}_N^T\boldsymbol{x}_R = -(1/3)\times 4/3 + 1/3 \times 2/3 + 2/3 \times 1/3 = 0$.

Problem 6
(a) $\boldsymbol{B} = \boldsymbol{IB} = (\boldsymbol{CA})\boldsymbol{B} = \boldsymbol{C}(\boldsymbol{AB}) = \boldsymbol{CI} = \boldsymbol{C}$.
(b) If $\boldsymbol{A} = [\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n]$ and $\boldsymbol{x} = [x_1, \ldots, x_n]^T$, then $\boldsymbol{Ax} = x_1\boldsymbol{a}_1 + \ldots + x_n\boldsymbol{a}_n$. Now suppose that the right hand side is zero. Then $\boldsymbol{Ax} = \boldsymbol{0}$, and by pre-multiplying with the inverse

$\boldsymbol{A}^{-1}$, it follows that $\boldsymbol{x} = \boldsymbol{0}$, and so the columns $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ are linearly independent.
(c) To prove that $\boldsymbol{AB}$ is nonsingular, we have to show $\boldsymbol{ABz} = \boldsymbol{0} \Rightarrow \boldsymbol{z} = \boldsymbol{0}$. So suppose that $\boldsymbol{ABz} = \boldsymbol{0}$. Then, as $\boldsymbol{A}$ is nonsingular, it follows that $\boldsymbol{Bz} = \boldsymbol{0}$, and since $\boldsymbol{B}$ is also nonsingular, that $\boldsymbol{z} = \boldsymbol{0}$. For the inverse of $\boldsymbol{AB}$, it suffices to check that

$$(\boldsymbol{AB})(\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}) = \boldsymbol{A}(\boldsymbol{BB}^{-1})\boldsymbol{A}^{-1} = \boldsymbol{AIA}^{-1} = \boldsymbol{AA}^{-1} = \boldsymbol{I} \ ,$$

so $\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$ is indeed the inverse of $\boldsymbol{AB}$.
(d) To prove that $\boldsymbol{E}$ is nonsingular, we have to show that $\boldsymbol{Ez} = \boldsymbol{0} \Rightarrow \boldsymbol{z} = \boldsymbol{0}$. So suppose that $\boldsymbol{Ez} = \boldsymbol{0}$. As $\boldsymbol{E} = \boldsymbol{I} - \alpha \boldsymbol{uv}^T$, this is equivalent to

$$\boldsymbol{z} = \alpha(\boldsymbol{v}^T\boldsymbol{z})\boldsymbol{u} \ . \tag{3}$$

We compute $\boldsymbol{v}^T\boldsymbol{z}$ by taking the scalar product of the last equation with $\boldsymbol{v}^T$; the result is $\boldsymbol{v}^T\boldsymbol{z} = \alpha(\boldsymbol{v}^T\boldsymbol{z})(\boldsymbol{v}^T\boldsymbol{u})$, or $\boldsymbol{v}^T\boldsymbol{z}(1 - \alpha\boldsymbol{u}^T\boldsymbol{v}) = 0$. But it was assumed that $\alpha\boldsymbol{u}^T\boldsymbol{v} \neq 1$; so it follows that $\boldsymbol{v}^T\boldsymbol{z} = 0$. By (2), this also implies $\boldsymbol{z} = \boldsymbol{0}$.
Again, it remains to check that $\boldsymbol{I} - \beta\boldsymbol{uv}^T$ is really the inverse of $\boldsymbol{E}$:

$$\begin{aligned}(\boldsymbol{I} - \alpha\boldsymbol{uv}^T)(\boldsymbol{I} - \beta\boldsymbol{uv}^T) &= \boldsymbol{I} - \beta\boldsymbol{uv}^T - \alpha\boldsymbol{uv}^T + \alpha\beta(\boldsymbol{v}^T\boldsymbol{u})\boldsymbol{uv}^T \\ &= \boldsymbol{I} + \{\alpha\beta(\boldsymbol{u}^T\boldsymbol{v}) - \alpha - \beta\}\boldsymbol{uv}^T \ ;\end{aligned}$$

here the term in the curly bracket is

$$\beta(\alpha\boldsymbol{u}^T\boldsymbol{v} - 1) - \alpha = \alpha - \alpha = 0 \ .$$