Complex Systems- Exercises 2

1. Find all fixed points of $F : \mathbb{R} \to \mathbb{R}$ with $F(x) = x^3 - 2x$ and determine their nature. Show that F has a period orbit $\{1, -1\}$ of period 2. What is the type of this periodic orbit?

2. Find the explicit form of

(i) the maps f_1, f_2, f_3 in the generation of the Sierpinky triangle, and

(ii) the maps f_1, f_2, f_3, f_4 in the generation of the Koch curve.

3. Find the fixed points of each of the maps f_1, f_2, f_3, f_4 in the generation of the Koch curve. What points of $\{1, 2, 3, 4\}^{\mathbb{N}}$ correspond to these points? Do the same for the Sierpinski triangle and the Cantor set.

4. Consider the Cantor set C and its generating sequence $\langle I_n \rangle_{n \geq 0}$. Find $d_H(I_n, C)$ in $\mathcal{P}(\mathbb{R})$.

5. Find the attractor of the IFS $\{f_1, f_2\}$ in $\mathbb{R}$, where, for $x \in \mathbb{R}$, $f_1(x) = ax$ and $f_2(x) = (1 - a)x + a$, with $0 < a < 1$.

6. Repeat question 5 with f_1, f_2 given by $f_1(x) = 0$ and $f_2(x) = \frac{2}{3}x + \frac{1}{3}$.

7. Describe the attractor of the IFS, $f_0, f_1, f_2 : \mathbb{R} \to \mathbb{R}$ with $f_j \mapsto \frac{x}{4} + \frac{3j}{8}$ $(j = 1, 2, 3)$ and find its similarity dimension.

8.

(i) Show that if $g : \mathbb{R}^m \to \mathbb{R}^m$ has contracting factor $s < 1$, then the closed ball with centre $u \in \mathbb{R}^m$ and of radius $\|u - g(u)\|/(1-s)$ is a trapping region for g, i.e., is mapped by g into itself.

(ii) Given an IFS, $f_1, f_2, \ldots, f_N : \mathbb{R}^m \to \mathbb{R}^m$, find a closed ball centred at u which is mapped by f into itself, where $f : \mathcal{P}(\mathbb{R}^m) \to \mathcal{P}(\mathbb{R}^m) : A \mapsto \bigcup_{i=1}^{N} f_i[A]$.

(iii)* When $m = 2$ and $N = 2$ and the maps $f_i : \mathbb{R}^2 \to \mathbb{R}^2$ $(1 \le i \le 2)$ are both affine, explain how you would find $u \in \mathbb{R}^2$ and $R \ge 0$ such that the closed ball $C(u, R)$ is the smallest trapping disk for f?

9*. Show that if $F : [a, b] \to \mathbb{R}$ is differentiable and its derivative F' is continuous at a fixed point x_0 of F, then x_0 is attracting (repelling) if $|F'(x_0)| < 1$ $(|F'(x_0)| > 1)$.

Hint: Assume $|F'(x_0)| < 1$. The continuity of F' at x_0 implies that there exists some $\delta > 0$ such that $|F'(x)| < k$ for $x_0 - \delta < x < x_0 + \delta$ where $k = (|F'(x_0)| + 1)/2$ (simply put $\epsilon = (1 - |F'(x_0)|)/2$ in the definition of continuity of F' at x_0). Now apply the mean value theorem to $[x_0, x]$ or $[x, x_0]$ where $|x_0 - x| < \delta$ and note that $k < 1$.