

# Dynamical Systems and Deep Learning - Exercises 2

1. Suppose we only have two neurons in the Hopfield network. Assume we have (i)  $w_{12} = w_{21} = 1$  or (ii)  $w_{12} = w_{21} = -1$ .

In the case of asynchronous updating, show that for (i) there are two attracting fixed points namely  $[1, 1]$  and  $[-1, -1]$  and that all orbits converge to one of these, whereas for (ii), the attracting fixed points are  $[-1, 1]$  and  $[1, -1]$  and all orbits converge to one of these.

In the case of synchronous updating, show that for both (i) and (ii), the fixed points do not attract nearby points and there are orbits which oscillate forever.

2. Consider the energy function

$$E = -\frac{1}{2} \sum_{i,j=1}^N w_{ij} x_i x_j$$

in the Hopfield network. We update  $x_m$  to  $x'_m$  and denote the new energy by  $E'$ .

(i) Show that  $E' - E = \sum_{i \neq m} w_{mi} x_i (x_m - x'_m)$ .

(ii) Show that when  $x_m$  flips to  $-x_m$ , we have:  $E' - E = 2h_m x_m$ .

3. Check the following two assertions about  $C_i^k$  as defined in the notes:

- If  $C_i^k$  is negative, then the crosstalk term has the same sign as the desired  $x_i^k$  and thus this value will not change.
- If, however,  $C_i^k$  is positive and greater than 1, then the sign of  $h_i$  will change, i.e.,  $x_i^k$  will change, which means that node  $i$  would become unstable.

(\* ) 4. Suppose we have a Hopfield network with  $N$  nodes that has stored  $p$  random patterns  $\vec{x}^k$ , for  $k = 1, \dots, p$ . Consider for any three values  $1 \leq k_1, k_2, k_3 \leq p$ , the mixture state

$$\vec{x}^{\text{mix}} = \text{sgn}(\vec{x}^{k_1} + \vec{x}^{k_2} + \vec{x}^{k_3})$$

(i) Show that on average  $x_i^{\text{mix}}$  has the same sign as  $x_i^{k_1}$  three times out of four.

- (ii) Deduce that the average Hamming distance of  $\vec{x}^{\text{mix}}$  from  $\vec{x}^{k_1}$  is  $N/4$ .  
(iii) Show also that  $\sum_{i=1}^N x_i^{k_1} x_i^{\text{mix}} = N/2$  on average.  
(iv) Compute  $h_i^{\text{mix}}$  as on page 14 of the notes to derive the average value:

$$h_i^{\text{mix}} = \frac{1}{N} \sum_{j,\ell} x_i^\ell x_j^\ell x_j^{\text{mix}} = \frac{1}{2} x_i^{k_1} + \frac{1}{2} x_i^{k_2} + \frac{1}{2} x_i^{k_3} + \text{cross terms}$$

- (v) Conclude that  $\vec{x}^{\text{mix}}$  is indeed an attractor of the network in most cases.