Dynamical Systems and Deep Learning- Exercises 2 (solutions)

1. Suppose we only have two neurons in the Hopfield network. Assume we have (i) $w_{12} = w_{21} = 1$ or (ii) $w_{12} = w_{21} = -1$.

In the case of asynchronous updating, show that for (i) there are two attracting fixed points namely $[1, 1]$ and $[-1, -1]$ and that all orbits converge to one of these, whereas for (ii), the attracting fixed points are $[-1, 1]$ and $[1, -1]$ and all orbits converge to one of these.

In the case of synchronous updating, show that for both (i) and (ii), the fixed points do not attract nearby points and there are orbits which oscillate forever.

Solution:

(1) Asynchronous updating: We assume the first node is updated followed by the second node (the alternative rule would be similar):

(i) $w_{12} = w_{21} = 1$: We have using the updating rule:

$$11 \to 11, \text{ thus } 11 \text{ is a fixed point }, \quad -1{-1} \to -1{-1}, \text{ thus } -1-1 \text{ is a fixed point}$$

$$1-1 \to -1-1, \text{ thus } 1-1 \text{ is in the basin of attraction of } -1-1$$

$$-11 \to 11, \text{ thus } -11 \text{ is in the basin of attraction of } 11$$

(ii) $w_{12} = w_{21} = -1$:

$$1{-1} \to 1{-1}, \text{ thus } 1-1 \text{ is a fixed point }, \quad 1{-1} \to 1{-1}, \text{ thus } 1-1 \text{ is a fixed point}$$

$$11 \to -11, \text{ thus } 11 \text{ is in the basin of attraction of } -11$$

$$-1-1 \to 1-1, \text{ thus } -1-1 \text{ is in the basin of attraction of } 1-1$$

(2) Synchronous updating:

(i) $w_{12} = w_{21} = 1$:

$$11 \to 11, \text{ thus } 11 \text{ is a fixed point; } -1{-1} \to -1{-1}, \text{ thus } -1-1 \text{ is a fixed point}$$

$$1{-1} \to -11, \quad -11 \to 1{-1}, \text{ thus } \{-11, 1-1\} \text{ is an orbit of period 2}$$

(ii) $w_{12} = w_{21} = -1$:

$11 \to -1{-1}, \quad -1{-1} \to 11,$ thus $\{11, -1-1\}$ is an orbit of period 2

$1{-1} \to 1{-1},$ thus $1-1$ is a fixed point; $-11 \to -11,$ thus -11 is a fixed point

2. Consider the energy function

$$E = -\frac{1}{2}\sum_{i,j=1}^{N} w_{ij}x_i x_j$$

in the Hopfield network. We update x_m to x'_m and denote the new energy by E'.
(i) Show that $E' - E = \sum_{i\neq m} w_{mi}x_i(x_m - x'_m)$.
(ii) Show that when x_m flips to $-x_m$, we have: $E' - E = 2h_m x_m$.

Solution: (i) We have:

$$E' - E = -\frac{1}{2}\sum_{i,j=1}^{N} w_{ij}x'_i x'_j + \frac{1}{2}\sum_{i,j=1}^{N} w_{ij}x_i x_j,$$

where $x'_i = x_i$ for $i \neq m$. Thus, all terms for which $i \neq m$ and $j \neq m$ will cancel out. All other terms will have $i = m$ or $j = m$. Since w_{ij} is symmetric, we have $w_{im}x_i x_m = w_{mi}x_m x_i$ and also $w_{im}x_i x'_m = w_{mi}x'_m x_i$ for $i \neq m$. Thus, terms that have $i = m$ or $j = m$ come in pairs of equal value and we obtain:

$$E' - E = -\sum_{i\neq m} w_{mi}x_i x'_m + \sum_{i\neq m} w_{mi}x_i x_m$$

$$= \sum_{i\neq m} w_{mi}x_i(x_m - x'_m).$$

(ii) When x_m flips, from (i) we get:

$$E' - E = \sum_{i\neq m} 2w_{mi}x_i x_m = 2h_m x_m.$$

3. Check the following two assertions about C_i^k as defined in the notes:

- If C_i^k is negative, then the crosstalk term has the same sign as the desired x_i^k and thus this value will not change.

  Solution: Let the crosstalk term be written as $C = \frac{1}{N}\sum_{j=1}^{N}\sum_{\ell\neq k} x_i^\ell x_j^\ell x_j^k$. So we have $C_i^k = -x_i^k C$. Now $C_i^k < 0$ implies $x_i^k C > 0$ so x_i^k and C will have the same sign. Thus $x_i^k + C$ has the same sign as x_i^k so the value x_i^k of the node i will not change.

- If, however, C_i^k is positive and greater than 1, then the sign of h_i will change, i.e., x_i^k will change, which means that node i would become unstable.
  Solution: Suppose $C_i^k > 1$. If $x_i^k = 1$ we will have $C < -1$ and thus $x_i^k + C < 0$ and the value of the node i will change to -1. If $x_i^k = -1$ we will have $C > 1$ and thus $x_i^k + C > 0$ and the value of the node i will change to 1. It is easy to see that if $0 \leq C_i^k < 1$ then the value of the node at i will not change. (Check this!)

4. Suppose we have a Hopfield network with N nodes that has stored p random patterns $\vec{x}^k$, for $k = 1, \ldots, p$. Consider for any three values $1 \leq k_1, k_2, k_3 \leq p$, the mixture state

$$\vec{x}^{\text{ mix}} = \text{sgn}(\vec{x}^{k_1} + \vec{x}^{k_2} + \vec{x}^{k_3})$$

(i) Show that on average x_i^{mix} has the same sign as $x_i^{k_1}$ three times out of four.
Solution: They will only fail to have the same sign when both $x_i^{k_2}$ and $x_i^{k_3}$ have the opposite sign of $x_i^{k_1}$ which happens one out of four.
(ii) Deduce that the average Hamming distance of $\vec{x}^{\text{ mix}}$ from $\vec{x}^{k_1}$ is $N/4$.

Solution: On average x_i^{mix} and $x_i^{k_1}$ will differ from each other one out of four times. Since the patterns are random and we have N nodes, the average number of components that are different in $\vec{x}^{\text{ mix}}$ from $\vec{x}^{k_1}$ is $N/4$.
(iii) Show also that $\sum_{i=1}^{N} x_i^{k_1} x_i^{\text{mix}} = N/2$ on average.

Solution: On average $3N/4$ of the components of $\vec{x}^{\text{ mix}}$ and $\vec{x}^{k_1}$ are the same, which means that on average $3N/4$ of the products $x_i^{k_1} x_i^{\text{mix}}$ are $+1$. Also on average $N/4$ of the components of $\vec{x}^{\text{ mix}}$ and $\vec{x}^{k_1}$ are different, which means that on average $N/4$ of the products $x_i^{k_1} x_i^{\text{mix}}$ are -1. Therefore on average $\sum_{i=1}^{N} x_i^{k_1} x_i^{\text{mix}} = (3N/4) \times (1) + (N/4) \times (-1) = N/2$.

(iv) Compute h_i^{mix} as on page 13 of the notes to derive:

$$h_i^{\text{mix}} = \frac{1}{N}\sum_{j,\ell} x_i^\ell x_j^\ell x_j^{\text{mix}} = \frac{1}{2}x_i^{k_1} + \frac{1}{2}x_i^{k_2} + \frac{1}{2}x_i^{k_3} + \text{ cross terms}$$

Solution: We have:

$$h_i^{\text{mix}} = \frac{1}{N}\sum_{1\le j\le N, 1\le \ell\le p} x_i^\ell x_j^\ell x_j^{\text{mix}}$$

in which we separate the contributions of the three terms $\ell = k_1$, $\ell = k_2$ and $\ell = k_3$ from the rest to obtain:

$$h_i^{\text{mix}} = \frac{1}{N}\left(\sum_{j=1}^{N} x_i^{k_1} x_j^{k_1} x_j^{\text{mix}} + x_i^{k_2} x_j^{k_2} x_j^{\text{mix}} + x_i^{k_3} x_j^{k_3} x_j^{\text{mix}}\right) + \text{cross terms}$$

$$= \frac{1}{N}\left(x_i^{k_1}\sum_{j=1}^{N} x_j^{k_1} x_j^{\text{mix}} + x_i^{k_2}\sum_{j=1}^{N} x_j^{k_2} x_j^{\text{mix}} + x_i^{k_3}\sum_{j=1}^{N} x_j^{k_3} x_j^{\text{mix}}\right) + \text{cross terms}$$

By part (iii) each of the three sums above is $N/2$ and we obtain:

$$h_i^{\text{mix}} = \frac{1}{2}x_i^{k_1} + \frac{1}{2}x_i^{k_2} + \frac{1}{2}x_i^{k_3} + \text{ cross terms}$$

(v) Conclude that $\vec{x}^{\text{ mix}}$ is indeed an attractor of the network in most cases.

Solution: Similar to the case of a single pattern in the notes, here , assuming that the cross talks will be less than $1/2$ in absolute value, which will be for the majority of cases for sufficiently small p, we get:

$$\text{sgn}(h_i^{\text{mix}}) = \text{sgn}\frac{1}{2}(x_i^{k_1} + x_i^{k_2} + x_i^{k_3}) = \text{sgn}(x_i^{\text{mix}})$$

and it follows that x_i^{mix} does not change sign and hence $\vec{x}^{\text{ mix}}$ is in most cases an attractor of the network.