Complex Systems- Exercises 3(solutions)
Attractors and Chaos

1. Show that the tail map sends open balls to open balls, i.e. show that for any $x \in \Sigma^{\mathbb{N}}$ and any integer $n \geq 1$, we have:

$$\sigma[O(x, 1/2^n)] = O(\sigma x, 1/2^{n-1}).$$

Solution: $y \in \sigma[O(x, 1/2^n)]$ iff $y_i = x_{i+1}$ for $i < n-1$ iff $y \in O(\sigma x, 1/2^{n-1})$.
2. Check that the tail map is continuous. (Hint: Show that the pre-image of any open ball is an open set.)

Solution: $\sigma^{-1}(O(x, 1/2^n)) = O(0x, 1/2^{n+1}) \cup O(1x, 1/2^{n+1})$.
3. Check that the tail map satisfies the following:

- Sensitive to initial conditions (with $\delta = 1/2$).**Solution:** We claim this is true for $\delta = 1/2$. Take any open ball, say of radius $1/2^m$, around $x \in \Sigma^{\mathbb{N}}$. Then there exists $y \in O(x, 1/2^n)$ with $x_m \neq y_m$ (simply replace x_m with $1 - x_m$). We have $d(\sigma^m x, \sigma^m y) = 1 > 1/2$.

- Topologically transitive (show that $\forall$ open $U \neq \emptyset$. $\exists n$. $\sigma^n(U) = \Sigma^{\mathbb{N}}$). **Solution:** Take any open ball say $O(x, 1/2^n) \subset U$. Then $\sigma^n[O(x, 1/2^n)] = \Sigma^{\mathbb{N}}$. But $\sigma^n[O(x, 1/2^n)] \subset \sigma^n[U]$, so $\sigma^n[U] = \Sigma^{\mathbb{N}}$. Thus, $\sigma^n[U] \cap V = \Sigma^{\mathbb{N}} \cap V = V$.

- Its periodic orbits are dense in $\Sigma^{\mathbb{N}}$. **Solution:** For any finite string $k \in \Sigma^n$, we have the periodic point $k^\omega \in \Sigma^{\mathbb{N}}$ of period n since clearly $\sigma^n(k^\omega) = k^\omega$. So, if $O(x, 1/2^m)$ is any open ball, then it contains the periodic point $(x_0 x_1, \ldots x_{m-1})^\omega$.

- It has a dense orbit.

  Solution: Here is an element of $\Sigma^{\mathbb{N}}$ with a dense orbit with respect to σ:

  $$0\,1\;00\,01\,10\,11\;000\,001\,010\,011\,100\,101\,110\,111\;\;\ldots\ldots,$$

  namely concatenate all blocks of length one followed by those with length 2 followed by those with length 3 ad infinitum.

4. Suppose $g : Y \to Y$ is a dynamical system with a semi-conjugacy:

$$\begin{array}{ccc} \Sigma^{\mathbb{N}} & \xrightarrow{\sigma} & \Sigma^{\mathbb{N}} \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow{g} & Y \end{array}$$

From such a semi-conjugacy, show that we can deduce the following results about g from the corresponding properties of σ:

(i) g is topologically transitive.

(ii) Periodic orbits of g are dense.

(iii) g has a dense orbit.

Solution: (i) Suppose $U, V \subset Y$ are non-empty open subsets. Then, $h^{-1}(U), h^{-1}(V) \subset \Sigma^{\mathbb{N}}$ are non-empty subsets of $\Sigma^{\mathbb{N}}$. Hence, since σ is topologically transitive, there exists $n > 0$ such that $\sigma^n(h^{-1}(U)) \cap h^{-1}(V) \neq \emptyset$. Let $x \in \sigma^n[h^{-1}(U)] \cap h^{-1}(V)$. Then $h(x) \in V$ and $h(x) \in h[\sigma^n[h^{-1}(U)]]$ But $h \circ \sigma^n = g^n \circ h$ by the conjugacy relation. Hence, $h(x) \in g^n[h[h^{-1}(U)]] = g^n[U]$. Thus, $g^n[U] \cap V \neq \emptyset$.

Note. The map h is not necessarily invertible so h^{-1} may not exist as a function. However, for a subset $U \subset Y$ the pre-image $h^{-1}(U) := \{x \in X : h(x) \in U\}$ always exists.

(ii) First note that h sends any periodic point of $\Sigma^{\mathbb{N}}$ to a periodic point of g. In fact, if $\sigma^n(x) = x$ then from $h \circ \sigma^n = g^n \circ h$ we get $h \circ \sigma^n(x) = g^n \circ h(x)$, i.e., $h(x) = g^n(h(x))$ as claimed. Now, let $O \subset Y$ be any non-empty open subset. Then $h^{-1}(O) \subset \Sigma^{\mathbb{N}}$ is a non-empty open subset and thus contains a periodic point, say $y \in \Sigma^{\mathbb{N}}$, of σ. Thus, $h(y)$ is a periodic point of g and we have:

$h(y) \in h[h^{-1}(O)] = O$.

(iii) If $x \in \Sigma^{\mathbb{N}}$ has a dense orbit then $h(x) \in Y$ has a dense orbit wrt g.

(*) 5. Consider $Q_d : \mathbb{R} \to \mathbb{R} : x \mapsto x^2 + d$ Show that for $d < 1/4$, the map Q_d is conjugate via a linear map of type $L : x \mapsto \alpha x + \beta$ to $F_c : \mathbb{R} \to \mathbb{R} : x \mapsto cx(1-x)$ for a unique $c > 1$.

Hint: Find α, β, c in terms of d such that:

$$F_c \circ L = L \circ Q_d$$

Solution: Let $L : x \mapsto \alpha x + \beta$. Then $L^{-1} : y \mapsto (y - \beta)/\alpha$. We have:

$$F_c L(x) = F_c(\alpha x + \beta) = c(\alpha x + \beta)(1 - \alpha x - \beta).$$

Thus, we have $F_c \circ L = L \circ Q_d$ when

$$c(\alpha x + \beta)(1 - \alpha x - \beta) = \alpha(x^2 + d) + \beta.$$

Or:

$$-c\alpha^2 x^2 + (c\alpha - 2c\alpha\beta)x + c\beta(1 - \beta) = \alpha x^2 + (\alpha d + \beta).$$

To have the above equality for **all** values of x we equate the coefficients of (i) x^2, (ii) x and (iii) the constant terms on both sides to get respectively:

(i) $-c\alpha^2 = \alpha$.

(ii) $c\alpha - 2c\alpha\beta = 0$.

(iii) $c\beta - c\beta^2 = \alpha d + \beta$.

Since $\alpha \neq 0$, we obtain $c\alpha = -1$ from (i). Thus, (ii) now gives $\beta = 1/2$. Then substitution in (iii) gives:

$$4\alpha^2 d + 2\alpha + 1 = 0$$

which has real roots iff $d < 1/4$. We get $\alpha = (-1 \pm \sqrt{1 - 4d})/4d$. We choose the negative square root so that α remains non-zero (and hence c defined) to finally get $\alpha = (-1 - \sqrt{1 - 4d})/4d$, $c = 4d/(\sqrt{1 - 4d} + 1)$ and $\beta = 1/2$.