# Complex Systems- Exercises 3 (solutions)

1. 1. Suppose $P \in \mathbb{R}^{n\times n}$ is a stochastic matrix.

   (i) Show that the 2-step transition matrix $P^{(2)} = P \circ P = P^2$ is a stochastic matrix.

   (ii) By using induction, show that $P^n$ is a stochastic matrix for any positive integer $n$.

**Solution:** (i) We need to check that all entries of $P^2$ are non-negative and the sum of all entries in every row is $1$. Since all entries of $P$ are non-negative we have $(P^2)_{ij} = \sum_{k=1}^{n} P_{ik}P_{kj} \geq 0$. Also for every row with index $i$:

$$\sum_{j=1}^{n}(P^2)_{ij} = \sum_{j=1}^{n}\sum_{k=1}^{n} P_{ik}P_{kj} = \sum_{k=1}^{n}(\sum_{j=1}^{n} P_{ik}P_{kj})$$

$$\sum_{k=1}^{n}(P_{ik}\sum_{j=1}^{n} P_{kj}) = \sum_{k=1}^{n}(P_{ik}\cdot 1) = \sum_{k=1}^{n} P_{ik} = 1.$$

(ii) We need to check that if $P^k$ for a positive integer $k$ is stochastic, so is $P^{k+1}$. Let $Q = P^k$. Then $P^{k+1} = Q \circ P$ and the proof follows exactly as in (i) with the first instance of $P$ in $P^2 = P \circ P$ replaced with $Q$.

2. Find the communicating classes of the stochastic matrix

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/3 & 1/6 & 1/6 & 1/3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1}$$

on the set of states $\{1, 2, 3, 4\}$ and decide if $P$ is irreducible or not.

**Solution:** The communicating classes are the following sets:

- $\{1, 2\}$, since $1$ and $2$ communicate with each other but no other state can be accessed from them,

- $\{3\}$, since $3$ cannot be accessed from any other state, and,
- $\{4\}$, since no other state can be accessed from $4$.

Thus, $P$ is not irreducible since it has more than one communicating class.

3. Suppose $0 < p, q < 1$ and consider

$$P = \begin{pmatrix} p & 1-p \\ 1-q & q \end{pmatrix} \tag{2}$$

- Check that $P$ has an eigenvalue $1$ and an eigenvalue $\lambda$ with $|\lambda| < 1$. Determine the stationary distribution $\pi$ of $P$.
- By taking the two left eigenvectors of $P$ as the new basis of $\mathbb{R}^2$, show that given any initial probability vector $p$ we have $\lim_{n\to\infty} pP^n = \pi$.

**Solution:** (i) The solution of $\det(P - \lambda I) = 0$ leads to $\lambda^2 - \lambda(p+q) + p + q - 1 = 0$. Clearly $1$ is an eigenvalue corresponding to left eigenvector $\pi$ with $\pi P = \pi$, i.e., $P^T \pi^T = \pi^T$. A simple calculation shows that

$$\pi = \left( \frac{1-q}{2-p-q}, \frac{1-p}{2-p-q} \right)$$

which is the stationary distribution. The other eigenvalue is $\lambda = p + q - 1$ since the product of the two eigenvalues is the constant term in the discriminant. By summing the two inequalities $0 < p < 1$and $0 < q < 1$ and subtracting $1$ from all sides we get $-1 < \lambda < 1$.
(ii) Let $x = (1, -1)$ be the eigenvector corresponding to $\lambda = p + q - 1$. Then we can write $p = a\pi + bx$ where $a, b$ are real numbers. Then we have

$$pP^n = (a\pi + bx)P^n = a\pi P^n + bxP^n = a\pi + b\lambda^n x$$

as $n \to \infty$, we have $\lambda^n \to 0$. Thus we get

$$\lim_{n\to\infty} pP^n = a\pi$$

Since $pP^n$ is a probability vector for all $n \in \mathbb{N}$, it follows that $a = 1$ as required. Alternatively, we can argue as follows. Since $\pi$ and $p$ are both probability vectors, the sum of the two components of $p - \pi$ is zero and thus $p - \pi = c(1, -1) = cx$ for some real number $c$. Therefore,

$$pP^n - \pi = (p - \pi)P^n = cxP^n = c\lambda^n x \to 0,$$

as $n \to \infty$.

4. Show that $\pi P = \pi \iff \pi(aI + (1-a)P) = \pi$, for $0 < a < 1$, where $I \in \mathbb{R}^{n\times n}$ is the identity matrix.

**Solution:** Let $\pi P = \pi$. Then,

$$\pi(aI + (1-a)P) = a\pi + (1-a)\pi P = a\pi + (1-a)\pi = \pi.$$

On the other hand, if $\pi(aI+(1-a)P) = \pi$, then rewriting the equation we obtain:

$$a\pi + (1-a)\pi P = \pi \qquad \Rightarrow \qquad (1-a)\pi = (1-a)\pi P,$$

from which the result follows after diving by $1 - a > 0$.

5. Show that if $\pi$ satisfies the detailed balanced condition for a stochastic matrix $P$, then it is a stationary distribution.

**Solution:** Suppose

$$\pi_i P_{ij} = \pi_j P_{ji}, \text{ for } 1 \le i, j \le N$$

Then

$$(\pi P)_i = \sum_{j=1}^{n} \pi_j P_{ji} = \sum_{j=1}^{n} \pi_i P_{ij} = \pi_i \sum_{j=1}^{n} P_{ij} = \pi_i$$

6. Rewrite the stochastic updating rule for the stochastic Hopfield network to obtain the probability of flipping:

$$\Pr(x_i \to -x_i) = \frac{1}{1 + \exp(\Delta E/T)}, \tag{3}$$

where $\Delta E = E' - E$ is the change in energy.

**Solution:** We have:

$$\Pr(x_i) = \frac{1}{1 + \exp(-2h_i x_i/T)}$$

Note also from Exercise 2(ii) in sheet 2 that when $x_i \to -x_i$, we have:

$$\Delta E = E' - E = 2h_i x_i$$

Thus, when we have $x_i \to -x_i$

$$\text{Pr}(x_i \to -x_i) = \text{Pr}(-x_i|x_i) = \text{Pr}(-x_i|\Delta E = 2h_i x_i)$$
$$= \frac{1}{1+\exp(2h_i x_i/T)}{\restriction_{\Delta E = 2h_i x_i}} = \frac{1}{1+\exp(\Delta E/T)}$$

7. Show that, with respect to the transition matrix for flipping nodes in a stochastic Hopfield network, the distribution

$$\pi(x) = \text{Pr}(x) = \frac{\exp(-E(x)/T)}{Z}, \tag{4}$$

satisfies the detailed balanced condition.

**Solution:** Assume $E(x_i)$ and $E(-x_i)$ denote the energies of the network when node $i$ has values $x_i$ and $-x_i$ respectively while all other nodes keep their values unchanged. Then, using the result of Exercise 5, we have:

$$\text{Pr}(x_i)\text{Pr}(x_i \to -x_i) = \frac{e^{-E(x_i)/T}}{Z} \frac{1}{1+e^{(E(-x_i)-E(x_i))/T}}$$
$$= \frac{1}{Z} \cdot \frac{1}{e^{E(x_i)/t} + e^{E(-x_i)/T}} = \frac{e^{-E(-x_i)/T}}{Z} \frac{1}{1+e^{(E(x_i)-E(-x_i))/T}} = \text{Pr}(-x_i)\text{Pr}(-x_i \to x_i).$$

8. Suppose we have a stochastic Hopfield network with $N$ nodes and $q$ is the uniform distribution on the nodes, i.e., $q(i) = 1/N$ for $1 \leq i \leq N$. Check that the following probabilistic transition rule is an example of Gibbs sampling:

- At each point in time, select a node $i$ with probability $q(i)$;
- flip the value $x_i$ of $i$ with probability:

$$\text{Pr}(x_i \to -x_i) = \frac{1}{1+\exp(\Delta E/T)},$$

  where $\Delta E = E' - E$ is the change in energy.

**Solution:** By Exercise 7, we know that $\pi(x) = \frac{\exp(-E(x)/T)}{Z}$ is the stationary distribution of the stochastic network and the conditional probability distribution for flipping a node is as given in the present problem. Therefore, by the definition of Gibbs sampling we indeed have an example of it here.