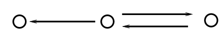
Complex Systems- Exercises 7

1. (i) When $K = 1$, explain whether or not every module is a loop linkage in the Kauffman boolean network.
(ii) Suppose for $K = 1$ the module diagram in a Kauffman boolean network consists of a single loop. If in the quenched model one of the coupling functions is constant, what can we say about the number and size of the cycles and the size of their basins of attraction?

Solution: (i) A module need not be a loop, e.g.,



(ii) The constant value will be passed on to all nodes after at most N updates to give a fixed point which will be the only cycle with a basin of attraction of size N.

2. Assume that $K = N$ and that we can compute probabilities for going from one configuration to another as if the system goes through a random walk, as in the notes. The probability of having an open trajectory at time t is q_t with

$$q_{t+1} = q_t(1 - \frac{t+1}{\Omega})$$

as in the notes.

- Show that the probability p_{t+1} of terminating the excursion at time $t+1$ is

  $$p_{t+1} = \frac{t+1}{\Omega} q_t.$$

  Solution: The excursion will terminate at $t+1$ if it has been still open at time t (with probability q_t) and if its next state is one of $\Sigma_0, \ldots, \Sigma_t$ (with probability $\frac{t+1}{\Omega}$).

- Show that starting with a given initial state the closure of the excursion at time t provides with equal probability all cycle lengths up to t.

  Solution: Since closure of the excursion at time t means that Σ_{t+1} is equal to Σ_i for some i with $0 \leq i \leq t$, which has probability $\frac{1}{\Omega}$ for any such i and thus produces a cycle of length $t - i$ with the same probability.

- Show that the probability $P(L)$ that a given initial state is in the basin of attraction of a cycle of length L is given by:

$$P(L) = \sum_{t=L}^{\Omega} \frac{p_t}{t}.$$

Solution: To get a cycle of length L the excursion must remain open at time $t = L - 1$. Thereafter, for each t with $L \leq t \leq \Omega$, the probability of getting a cycle of length L is the probability that the excursion has been open at time t, i.e., p_t, times the probability that the next state is Σ_{t-L} which has probability $1/t$. Since for distinct values of t these events are disjoint, the probability of their union is the sum

$$P(L) = \sum_{t=L}^{\Omega} \frac{p_t}{t}$$

as required.

- Using the asymptotic approximation $1-x \sim e^{-x}$ as $x \to 0$ and thus $1-\frac{t}{\Omega} \approx e^{-\frac{t}{\Omega}}$ for large Ω, show that for large Ω we have:

$$\frac{p_t}{t} \approx \frac{1}{\Omega} \exp\left(-\sum_{s=1}^{t-1} \frac{s}{\Omega}\right).$$

Solution: We have:

$$\frac{p_t}{t} = \frac{1}{\Omega} q_{t-1} = \frac{1}{\Omega} \prod_{s=t-1}^{1} (1 - \frac{s}{\Omega}) = \frac{1}{\Omega} \prod_{s=1}^{t-1} (1 - \frac{s}{\Omega}) =$$

$$\approx \frac{1}{\Omega} \exp -(\sum_{s=1}^{t-1} \frac{s}{\Omega}).$$

- Hence, show that if Ω is large we have the approximation:

$$P(L) \approx \int_L^{\infty} \frac{1}{\Omega} \exp(-x(x-1)/2\Omega)\, dx.$$

Solution: We have

$$\sum_{s=1}^{t-1} \frac{s}{\Omega} = \frac{t(t-1)}{2\Omega}.$$

For large Ω we pass to the continuum and replace the sum with the integral with its upper limit taken to be ∞.