Complex Systems- Exercises 8

1. Suppose we only have two neurons in the Hopfield network. Assume we have (i) $w_{12} = w_{21} = 1$ or (ii) $w_{12} = w_{21} = -1$.

In the case of asynchronous updating, show that for (i) there are two attracting fixed points namely $[1, 1]$ and $[-1, -1]$ and that all orbits converge to one of these, whereas for (ii), the attracting fixed points are $[-1, 1]$ and $[1, -1]$ and all orbits converge to one of these.

In the case of synchronous updating, show that for both (i) and (ii), the fixed points do not attract nearby points and there are orbits which oscillate forever.

2. Consider the energy function

$$E = -\frac{1}{2} \sum_{i,j=1}^{N} w_{ij} x_i x_j$$

in the Hopfield network. We update x_m to x'_m and denote the new energy by E'. Show that $E' - E = \sum_{i \neq m} w_{mi} x_i (x_m - x'_m)$.

3. Suppose we have a Hopfield network with N nodes that has stored p patterns $\vec{x}^k$, for $k = 1, \ldots, p$. Consider for any three values $1 \leq k_1, k_2, k_3 \leq p$, the mixture state

$$\vec{x}^{\text{ mix}} = \text{sgn}(\vec{x}^{k_1} + \vec{x}^{k_2} + \vec{x}^{k_3})$$

(i) Show that on average x_i^{mix} has the same sign as $x_i^{k_1}$ three times out of four.
(ii) Deduce that the Hamming distance of $\vec{x}^{\text{ mix}}$ from $\vec{x}^{k_1}$ is $N/4$.
(iii) Show also that $\sum_{i=1}^{N} x_i^{k_1} x_i^{\text{ mix}} = N/2$ on average.
(iv) Compute h_i^{mix} as on page 13 of the notes to derive:

$$h_i^{\text{ mix}} = \frac{1}{N} \sum_{j,\ell} x_i^{\ell} x_j^{\ell} x_j^{\text{ mix}} = \frac{1}{2} x_i^{k_1} + \frac{1}{2} x_i^{k_2} + \frac{1}{2} x_i^{k_3} + \text{ cross terms}$$

(v) Conclude that $\vec{x}^{\text{ mix}}$ is indeed an attractor of the network in most cases.