Vector norms

The ℓ_p norms of a vector in $\mathbb{R}^n$

- For $p > 0$, the ℓ_p norm of any vector $v \in \mathbb{R}^n$ is defined as

$$\|v\|_p = \left(\sum_{i=1}^{n} |v_i|^p \right)^{1/p}$$

- $p = 1$, ℓ_1 norm: $\|x\|_1 = \sum_{i=1}^{n} |x_i|$
- $p = 2$, ℓ_2 norm: $\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$
- $p = \infty$, ℓ_∞ norm: $\|x\|_\infty = \max_{1 \le i \le m} |x_i|$
- Note that $\|x\|_\infty \le \|x\|_2 \le \|x\|_1$.
- This follows from: $\max_{1 \le i \le n} |x_i|^2 \le \sum_{i=1}^{n} x_i^2$,
- and $\sum_{i=1}^{n} x_i^2 \le (\sum_{i=1}^{n} |x_i|)^2$.

The ℓ_∞ norm

- As $p \to \infty$, we have $\|v\|_p \to \|v\|_\infty := \max_{1 \le i \le n} |v_i|$.
- If $v = \mathbf{0}$ then this is trivial, so assume $v \neq \mathbf{0}$.
- Let $m \in \{1, 2, 3, \ldots, n\}$ be such that $|v_m| = \max_{1 \le i \le n} |v_i|$.
- $$\|v\|_p = |v_m| \left(\sum_{i=1}^{n} \frac{|v_i|^p}{|v_m|^p} \right)^{1/p}.$$
- We have $\frac{|v_i|^p}{|v_m|^p} \le 1$ for $1 \le i \le n$ and at least one of them is one, since $\frac{|v_m|^p}{|v_m|^p} = 1$, i.e., the sum is between 1 and n.
- So, $|v_m| \le |v_m| \left(\sum_{i=1}^{n} \frac{|v_i|^p}{|v_m|^p} \right)^{\frac{1}{p}} \le |v_m|(n)^{\frac{1}{p}} \to |v_m|$ as $p \to \infty$, (since $n^{\frac{1}{p}} \to 1$ as $p \to \infty$).
- Thus, $\|v\|_p \to |v_m| = \|v\|_\infty$ as $p \to \infty$.

Cauchy-Schwartz inequality

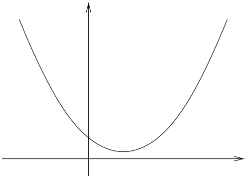
- For all $u, v \in \mathbb{R}^n$ we have

$$|u \cdot v|^2 \leq (u \cdot u)(v \cdot v), \text{ i.e., } |u \cdot v| \leq \|u\|_2 \|v\|_2.$$

- For a proof, consider the vector $\lambda u + v$ for any $\lambda \in \mathbb{R}$.
- Since the length of any vector is nonnegative, we have:

$$\forall \lambda.\, 0 \leq (\lambda u + v) \cdot (\lambda u + v) = (u \cdot u)\lambda^2 + 2(u \cdot v)\lambda + (v \cdot v)$$

- Thus, the above quadratic $a\lambda^2 + b\lambda + c$ in λ with the three coefficients $a = u \cdot u$, $b = 2(u \cdot v)$ and $c = v \cdot v$ is always non-negative, i.e., it cannot have two distinct real roots.
- So we must have: $b^2 - 4ac \leq 0$.



- Thus, $|u \cdot v|^2 \leq (u \cdot u)(v \cdot v)$ as required.