Rank and Nullity theorem

Rank and Nullity of a matrix

- Given a linear map $f : \mathbb{R}^n \to \mathbb{R}^m$, its **image space** or **range space** is defined as $\text{im}(f) = \{f(x) : x \in \mathbb{R}^n\}$ and its **kernel** or **null space** is defined by $\text{null}(f) = \{x : f(x) = 0\}$.
- Similarly, the same notions are defined for a matrix $A \in \mathbb{R}^{m \times n}$ which represents f.
- Dually, we have similar definitions $\text{Im}(A^T)$ and $\text{null}(A^T)$ for the transpose $A^T \in \mathbb{R}^{n \times m}$
- $\text{im}(f)$ and $\text{null}(f)$ are both subspaces (i.e., closed under vector addition and scalar multiplication).
- The celebrated Rank-Nullity theorem says that for any linear map $f : \mathbb{R}^n \to \mathbb{R}^m$ we have: $\dim(\text{im}(f)) + \dim(\text{null}(f)) = n$.
- This theorem is an immediate consequence of the following results on matrices.

Rank and nullity of matrices

- Let $A \in \mathbb{R}^{m \times n}$, we show that
- column rank$(A) =$ row rank$(A) :=$ rank(A).
- column rank$(A) +$ nullity$(A) = n$.
- column rank$(A) +$ nullity$(A^T) = m$.
- These properties are consequence of the following facts:

(i) An elementary row or column operation does not change the column rank or the row rank of A.

(ii) Using these elementary operations A can be reduced to the following block matrix of an identity matrix $I_{r \times r}$ of dimension r and three zero matrices $0_{r \times (n-r)} \in \mathbb{R}^{r \times (n-r)}$, $0_{(m-r) \times r} \in \mathbb{R}^{(m-r) \times r}$ and $0_{(m-r) \times (n-r)} \in \mathbb{R}^{(m-r) \times (n-r)}$:

$$\begin{bmatrix} I_{r \times r} & 0_{r \times (n-r)} \\ 0_{(m-r) \times r} & 0_{(m-r) \times (n-r)} \end{bmatrix}$$

Elementary operations preserve rank and nullity I

- **Claim 1.** An elementary row operation on $A \in \mathbb{R}^{m \times n}$ does not change its column rank.
- **Proof.** Since the column rank of A is the maximum number of columns of A that are linearly independent, it is sufficient (by an exercise in Tutorial Sheet 4) to show that an elementary row operation does not change the linear independence of the columns of A.
- Recall that columns of A are linearly independent iff $Ax = 0$ implies $x = 0$.
- Let A' be the result of an elementary row operation on A.
- We show that $Ax = 0$ iff $A'x = 0$ which proves that columns of A and those of A' are either both linearly independent or both linearly dependent (why?).
- We show this for the elementary row operation of subtracting λ times the second row a^2 from the first row a^1:
- $Ax = 0$ iff $a^1 \cdot x = a^2 \cdot x = \cdots = a^m \cdot x = 0$ iff $(a^1 - \lambda a^2) \cdot x = a^2 \cdot x = \cdots = a^m \cdot x = 0$ iff $A'x = 0$.
- The proof for the other elementary operations is similar.

Elementary operations preserve rank and nullity II

- **Claim 2.** An elementary row operation does not change the linear independence of the rows of a matrix.
- **Proof.** By taking the transpose of the matrix, we can equivalently show that an elementary column operation does not change the linear independence of the columns of a matrix $A \in \mathbb{R}^{m \times n}$.
- Again the case of swapping two rows or multiplying a row by a non-zero number are trivial.
- We show it for $A'' = [a_1 - \lambda a_2, a_2, \cdots, a_n]$ obtained by the elementary column operation of subtracting λ times the second column a_2 from the first column a_1 of A.
- Recall that $Ax = \sum_{i=1}^{n} x_i a_i$.
- If $a_1, a_2, \cdots, a_n$ are linearly independent and if $A''x = 0$, then $x_1(a_1 - \lambda a_2) + x_2 a_2 + \cdots x_n a_n = 0$, i.e., $x_1 a_1 + (x_2 - \lambda x_1) a_2 + \cdots x_n a_n = 0$, which by the linear independence of a_i's implies $x_1 = x_2 - \lambda x_1 = \ldots = x_n = 0$, i.e., $x_1 = x_2 = \ldots = x_n = 0$. Thus, the columns of A'' are linearly independent.

Elementary operations preserve rank and nullity III

- Now for the converse, assume that the columns of $A'' = [a_1 - \lambda a_2, a_2, \cdots, a_n]$ are linearly independent and that $Ax = 0$.
- Then $x_1 a_1 + x_2 a_2 + x_3 a_1 + \cdots + x_n a_n = 0$ and thus $x_1(a_1 - \lambda a_2) + (x_2 + \lambda x_1)a_2 + \cdots + x_n a_n = 0$ which gives
- $x_1 = x_2 + \lambda x_1 = x_3 = \cdots = x_n = 0$,
- which implies $x_i = 0$ for all i.
- Thus, the columns of A are linearly independent, which completes the proof of that elementary row operations do not change the column or row rank of a matrix. By considering A^T, the same is true for elementary column operations.
- Finally, we know that using Gauss-Jordan technique we can reduce any $m \times n$ matrix by elementary row and column operations to a diagonal matrix where all its non-zero entries are on the diagonal.
- This completes the proof of the rank-nullity theorem.