A complete and decidable axiomatisation for deontic interpreted systems*
Report RN/06/17

Alessio Lomuscio and Bożena Woźna

Department of Computer Science, University College London
Gower Street, London WC1E 6BT, United Kingdom
email: {A.Lomuscio,B.Wozna}@cs.ucl.ac.uk

Abstract. We solve the problem left open in [6] by providing a complete axiomatisation of deontic interpreted systems on a language that includes full CTL as well as the K_i, O_i and $\widehat{\mathrm{K}}_i^j$ modalities. Additionally we show that the logic employed enjoys the finite model property, hence decidability is guaranteed. To achieve these results we follow the technique used by Halpern and Emerson in [2].

1 Introduction

Concepts based on deontic notions are increasingly being used in specification and verification of large multi-agent systems. Because of their open and self-interested nature it is unrealistic to assume that a team of engineers in a single organisation may maintain control of a whole multi-agent system. This makes it difficult, even a priori, to verify either off-line or at runtime that each individual agent complies with a set of specifications. It seems more feasible, instead, to permit the agents to perform incorrect/unwanted/undesirable actions, only to flag all unwanted behaviours and reason about the properties that these may bring about in the system.

In other words, by adding a suitable set of deontic notions we can aim to verify not only what properties the system enjoys when each individual agent is performing following the intended specifications (as it is traditionally done in Software Engineering), but also what consequences result from the violation of some of these specifications by some agents. This shift to a more liberal, finer grained approach requires the introduction of suitable formal machinery both in terms of specification languages and verification tools.

Deontic interpreted systems [6] have recently been introduced for this objective. In their basic form they provide a computationally grounded semantics [12] to interpret a logic capturing epistemic, temporal and correctness notions. By using this formalism it is possible to give a semantical description of key scenarios [7] and use the logic to check whether or not particular properties hold on these

* The authors acknowledge support from the EPSRC (grant GR/S49353) and the Nuffield Foundation (grant NAL/690/G).

specifications. Specifically, deontic interpreted systems can be used to interpret a language that includes CTL modalities AU, EU, EX [2], epistemic modalities K_i [3], modalities representing correct functioning behaviour O_i, and modalities $\widehat{\mathrm{K}}_i^j$ representing knowledge under the assumption of correct behaviour. Automatic model checking tools for deontic interpreted systems have been developed [9, 8] supporting the automatic verification of state spaces of the region of 10^{40} and beyond [10, 11, 13, 8].

While the above results concern specification patters, verification tools, and concrete scenarios, important theoretical issues have so far been left open. In particular, the axiomatisation of deontic interpreted systems originally provided in [6]was limited to a language that did not include temporal operators. Furthermore, the bi-indexed modality $\widehat{\mathrm{K}}_i^j$, whose importance in practical verification is now well recognised, was not included in the language.

The difficulty of the problem is linked to two issues. First, the modality $\widehat{\mathrm{K}}_i^j$ is defined in terms of the intersection between two relations with different properties: modalities like these are known to be hard to treat. Second, any axiomatisation for deontic interpreted systems would have to include a logic for branching-time, but the standard procedure for axiomatising CTL involves a non-standard filtration procedure [2].

The contribution of the present work is to solve the problem left open in [6], i.e., to provide a complete axiomatisation of deontic interpreted systems on a language that includes full CTL as well as the K_i, O_i and $\widehat{\mathrm{K}}_i^j$ modalities. Additionally we show that the logic employed enjoys the finite model property, hence it is decidable. To show these results we extend the technique originally presented by Halpern and Emerson in [2] to the richer language above.

The rest of the paper is organised as follows. In Section 2 we present syntax and semantics of the logic. Section 3 is devoted to the construction of the underlying machinery to prove the main results of the paper. Sections 4 and 5 present a decidability theorem and a completeness proof for the logic.

2 Deontic interpreted systems

Deontic interpreted systems [6] constitute a semantics to interpret epistemic, correctness and temporal operators in a computational setting. They extend the framework of interpreted systems [3], popularised by Halpern and colleagues in the 90s to reason about knowledge, to modalities expressing correctness and knowledge under assumptions of correctness. Technically, deontic interpreted systems provide an interpretation to the operators O_i ($\mathrm{O}_i\phi$ representing "whenever agent i is working correctly ϕ is the case") and $\widehat{\mathrm{K}}_i^j$ ($\widehat{\mathrm{K}}_i^j\phi$ representing "agent i knows that ϕ under the assumption that agent j is working correctly") as well as the standard epistemic operators K_i and branching time operators of CTL already supported by interpreted systems. Semantically this is achieved simply by assuming that the local states of the agents are composed by two disjoint sets of allowed (or "green") and disallowed (or "red") local states. Loosely speaking an agent "is working correctly" whenever it is following its protocol (defined in interpreted systems as a function from local states to sets of actions) in its choice of actions. Given that the focus of this paper is to axiomatise the result trace based semantics resulting from this we refer to [6] and related papers for more details.

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{N}_+ = \{1, 2, \ldots\}$, $\mathcal{PV}$ be a set of propositional variables, and $\mathcal{AG} = \{1, \ldots, n\}$ a set of agents, for $n \in \mathbb{N}_+$.

Definition 1 (Syntax). *Let $p \in \mathcal{PV}$ and $i \in \mathcal{AG}$. The language $\mathcal{L}$ is defined by the following grammar:*

$$\varphi := p \mid \neg\varphi \mid \varphi \vee \varphi \mid \mathrm{EX}\varphi \mid \mathrm{E}(\varphi \mathrm{U} \varphi) \mid \mathrm{A}(\varphi \mathrm{U} \varphi) \mid \mathrm{K}_i\varphi \mid \mathrm{O}_i\varphi \mid \widehat{\mathrm{K}}_i^j\varphi$$

The language above extends CTL [1] with a standard epistemic operator K_i [3], and two further modalities: O_i and $\widehat{\mathrm{K}}_i^j$ [6]. The formula $\mathrm{EX}\alpha$ is read as "there exists a computation path such that at the next step of the path α holds", $\mathrm{E}(\alpha \mathrm{U} \beta)$ is read as "there exists a computation path such that β eventually occurs and α continuously holds until then", $\mathrm{K}_i\alpha$ is read as "agent i knows that α", $\mathrm{O}_i\alpha$ is read as "whenever agent i is functioning correctly α holds", and $\widehat{\mathrm{K}}_i^j\alpha$ is read as "agent i knows that α under the assumption that the agent j is functioning correctly".

The remaining operators can be introduced via abbreviations as usual, i.e., $\alpha \wedge \beta \stackrel{def}{=} \neg(\neg\alpha \vee \neg\beta)$, $\alpha \Rightarrow \beta \stackrel{def}{=} \neg\alpha \vee \beta$, $\alpha \Leftrightarrow \beta \stackrel{def}{=} (\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$, $\mathrm{AX}\alpha \stackrel{def}{=} \neg\mathrm{EX}\neg\alpha$, $\mathrm{EF}\alpha \stackrel{def}{=} \mathrm{E}(\top \mathrm{U} \alpha)$, $\mathrm{AF}\alpha \stackrel{def}{=} \mathrm{A}(\top \mathrm{U} \alpha)$, $\mathrm{EG}\alpha \stackrel{def}{=} \neg\mathrm{AF}\neg\alpha$, $\mathrm{AG}\alpha \stackrel{def}{=} \neg\mathrm{EF}\neg\alpha$, $\mathrm{A}(\alpha \mathrm{W} \beta) \stackrel{def}{=} \neg\mathrm{E}(\neg\alpha \mathrm{U} \neg\beta)$, $\mathrm{E}(\alpha \mathrm{W} \beta) \stackrel{def}{=} \neg\mathrm{A}(\neg\alpha \mathrm{U} \neg\beta)$, $\overline{\mathrm{K}}_i\alpha \stackrel{def}{=} \neg\mathrm{K}_i(\neg\alpha)$, $\overline{\mathrm{O}}_i\alpha \stackrel{def}{=} \neg\mathrm{O}_i(\neg\alpha)$.

Since most of the proofs of the paper are by induction on the length of the formula, below we give a definition of length that will be used throughout the paper.

Definition 2 (Length). *Let $\varphi \in \mathcal{L}$. The* length *of φ (denoted by $|\varphi|$) is defined inductively as follows:*

- *If $\varphi \in \mathcal{PV}$, then $|\varphi| = 1$,*
- *If φ is of the form $\neg\alpha$, $\mathrm{K}_i\alpha$, $\mathrm{O}_i\alpha$, or $\widehat{\mathrm{K}}_i^j\alpha$, then $|\varphi| = |\alpha| + 1$,*
- *If φ is of the form $\mathrm{EX}\alpha$, then $|\varphi| = |\alpha| + 2$,*
- *If φ is of the form $\alpha \vee \beta$ then $|\varphi| = |\alpha| + |\beta| + 1$,*
- *If φ is of the form $\mathrm{A}(\alpha \mathrm{U} \beta)$ or $\mathrm{E}(\alpha \mathrm{U} \beta)$, then $|\varphi| = |\alpha| + |\beta| + 2$.*

Let φ and ψ be $\mathcal{L}$ formulas. We say that ψ is a *subformula* of φ and denote by $\psi \in Sub(\varphi)$ if either (a) $\psi = \varphi$; or (b) φ is of the form $\neg\alpha$, $\mathrm{EX}\alpha$, $\mathrm{K}_i\alpha$, $\mathrm{O}_i\alpha$, or $\widehat{\mathrm{K}}_i^j\alpha$, and ψ is a subformula of α; or (c) φ is of the form $\alpha \vee \beta$, $\mathrm{E}(\alpha \mathrm{U} \beta)$, or $\mathrm{A}(\alpha \mathrm{U} \beta)$ and ψ is a subformula of either α or β.

Following [6] we interpret $\mathcal{L}$ on *deontic interpreted systems*. Whenever reasoning about models and other semantic structures (such as Hintikka's structures below) we assume that each agent $i \in \mathcal{AG}$ (respectively the environment e) is associated with a set of local states L_i (respectively L_e). These are partitioned into allowed (or green) $\mathcal{G}_i$ (respectively $\mathcal{G}_e$) and disallowed (red) $\mathcal{R}_i$ (respectively $\mathcal{R}_e$) states. The selection and execution of actions on global states generates by means of a transition function runs, or computational paths, that are represented below by means of the temporal relation T. Given our current interest is presently concerned with axiomatisations we will focus at the level of models as defined below. For more details on what below we refer to [3, 6].

Definition 3 (Deontic Interpreted Systems). *A* deontic interpreted system *(or a* model*) is a tuple $M = (S, T, (\mathrm{R}_i^K)_{i\in\mathcal{AG}}, (\mathrm{R}_i^O)_{i\in\mathcal{AG}}, (\mathrm{R}_i^j)_{i,j\in\mathcal{AG}}, \mathcal{V})$ where S is a set of states; $T \subseteq S \times S$ is a serial relation on S; $\mathrm{R}_i^K \subseteq S \times S$ is an equivalence relation for each agent $i \in \mathcal{AG}$; $\mathrm{R}_i^O \subseteq S \times S$ is a serial, i-jEuclidean and transitive relation for each agent $i \in \mathcal{AG}$; $\mathrm{R}_i^j \subseteq S \times S$ is a relation for each agent $i \in \mathcal{AG}$ defined by: $(s, s') \in R_i^j$ iff $(s, s') \in R_i^K \cap R_j^O$; $\mathcal{V} : S \longrightarrow 2^{\mathcal{PV}}$ is a valuation function, which assigns to each state a set of proposition variables that are assumed to be true at that state.*

We call $F = (S, T, (\mathrm{R}_i^K)_{i\in\mathcal{AG}}, (\mathrm{R}_i^O)_{i\in\mathcal{AG}}, (\mathrm{R}_i^j)_{i,j\in\mathcal{AG}})$ a frame.

Note that in the above definition of the deontic interpreted system we do not impose any conditions on the set of states, and do not specify how the epistemic and deontic relations are defined. We do this, because we can always construct local and environmental states to put any set S, with any equivalence relation on it, in the form traditionaly used in the multi-agent systems; namly to define S to be a subset of the product of local sets of states, one per each agent, and a set of states for environemnt [5, 6].

A *path* in M is an infinite sequence $\pi = (s_0, s_1, \ldots)$ of states such that $(s_i, s_{i+1}) \in T$ for each $i \in \mathbb{N}$. For a path $\pi = (s_0, s_1, \ldots)$, we take $\pi(k) = s_k$. By $\Pi(s)$ we denote the set of all the paths starting at $s \in S$.

Definition 4 (Satisfaction). *Let M be a model, s a state, and α, $\beta \in \mathcal{L}$. The satisfaction relation $\models$, indicating truth of a formula in model M at state s, is defined inductively as follows:*

$(M, s) \models p$ iff $p \in \mathcal{V}(s)$, $\quad (M, s) \models \alpha \wedge \beta$ iff $(M, s) \models \alpha$ and $(M, s) \models \beta$,
$(M, s) \models \neg\alpha$ iff $(M, s) \not\models \alpha$, $\quad (M, s) \models \mathrm{EX}\alpha$ iff $(\exists \pi \in \Pi(s))(M, \pi(1)) \models \alpha$,
$(M, s) \models \mathrm{E}(\alpha \mathrm{U} \beta)$ iff $(\exists \pi \in \Pi(s))(\exists m \geq 0)[(M, \pi(m)) \models \beta$ and $(\forall j < m)(M, \pi(j)) \models \alpha]$,
$(M, s) \models \mathrm{A}(\alpha \mathrm{U} \beta)$ iff $(\forall \pi \in \Pi(s))(\exists m \geq 0)[(M, \pi(m)) \models \beta$ and $(\forall j < m)(M, \pi(j)) \models \alpha]$,
$(M, s) \models \mathrm{K}_i\alpha$ iff $(\forall s' \in S)$ $(s\mathrm{R}_i^K s'$ implies $(M, s') \models \alpha)$,
$(M, s) \models \mathrm{O}_i\alpha$ iff $(\forall s' \in S)$ $(sR_i^O s'$ implies $(M, s') \models \alpha)$,
$(M, s) \models \widehat{\mathrm{K}}_i^j\alpha$ iff $(\forall s' \in S)$ $(s\mathrm{R}_i^j s'$ implies $(M, s') \models \alpha)$.

We conclude this section with a definition of validity/satisfiability problems.

Definition 5 (Validity and Satisfiability). *Let M be a model and $\varphi \in \mathcal{L}$.* **(a)** *φ is* valid *in M (written $M \models \varphi$), if $M, s \models \varphi$ for all states $s \in S$.* **(b)** *φ is* satisfiable *in M, if $M, s \models \varphi$ for some state $s \in S$.* **(c)** *φ is* valid *(written $\models \varphi$), if φ is valid in all the models M.* **(d)** *φ is* satisfiable *if it is satisfiable in some model M. In this case M is said to be a model for φ.*

In the next section we prove that $\mathcal{L}$ has the *finite model property* (FMP), that is, we show that any satisfiable $\mathcal{L}$ formula is also satisfiable on a finite model. This result allows us to provide a decidability algorithm for $\mathcal{L}$ (see Section 4), which we use later on to prove that the language has a complete axiomatic system.

3 Finite Model Property (FMP)

The standard procedure for showing the FMP in modal logic is to construct a filtration of an arbitrary model of a satisfiable formula and show that this filtrated model is itself a model for the formula. As it is well-known, while this procedure produces the intended result for a number of logics, it fails in others, for instance in the case of CTL. More refined techniques for showing the FMP exist; notably the construction given in [2] via *Hintikka structures* guarantees the result. Indeed, given that the logic in study here is an extension of CTL, here we follow the procedure given in [2] and show it can be extended to extensions of CTL.

We start by defining two auxiliary structures: a *Hintikka structure* for a given $\mathcal{L}$ formula, and the *quotient structure* for a given model. As in the previous section and in rest of the paper we assume to be dealing with a set of agents defined on local states, and protocols.

Definition 6 (Hintikka structure). *A* Hintikka structure *for* φ *is a tuple* $H = (S, T, (\mathrm{R}_i^K)_{i\in\mathcal{AG}}, (\mathrm{R}_i^O)_{i\in\mathcal{AG}}, (\mathrm{R}_i^j)_{i,j\in\mathcal{AG}}, \mathbb{L})$ *where* S *is a set of states,* T, R_i^K, R_i^O *and* R_i^j *are binary relations on* S, *and* $\mathbb{L} : S \to 2^{\mathcal{L}}$ *is a labelling function assigning a set of formulas to each state such that* $\varphi \in \mathbb{L}(s)$ *for some* $s \in S$ *and the following conditions are satisfied:*

H.1. *if* $\neg\alpha \in \mathbb{L}(s)$, *then* $\alpha \notin \mathbb{L}(s)$
H.2. *if* $\neg\neg\alpha \in \mathbb{L}(s)$, *then* $\alpha \in \mathbb{L}(s)$
H.3. *if* $(\alpha \wedge \beta) \in \mathbb{L}(s)$, *then* $\alpha \in \mathbb{L}(s)$ *and* $\beta \in \mathbb{L}(s)$
H.4. *if* $\neg(\alpha \wedge \beta) \in \mathbb{L}(s)$, *then* $\neg\alpha \in \mathbb{L}(s)$ *or* $\neg\beta \in \mathbb{L}(s)$
H.5. *if* $\mathrm{E}(\alpha \mathrm{U} \beta) \in \mathbb{L}(s)$, *then* $\beta \in \mathbb{L}(s)$ *or* $\alpha \wedge \mathrm{EXE}(\alpha\mathrm{U}\beta) \in \mathbb{L}(s)$
H.6. *if* $\neg\mathrm{E}(\alpha\mathrm{U}\beta) \in \mathbb{L}(s)$, *then* $\neg\beta \wedge \neg\alpha \in \mathbb{L}(s)$ *or* $\neg\beta \wedge \neg\mathrm{EXE}(\alpha\mathrm{U}\beta) \in \mathbb{L}(s)$
H.7. *if* $\mathrm{A}(\alpha\mathrm{U}\beta) \in \mathbb{L}(s)$, *then* $\beta \in \mathbb{L}(s)$ *or* $\alpha \wedge \neg\mathrm{EX}(\neg\mathrm{A}(\alpha\mathrm{U}\beta)) \in \mathbb{L}(s)$
H.8. *if* $\neg\mathrm{A}(\alpha\mathrm{U}\beta) \in \mathbb{L}(s)$, *then* $\neg\beta \wedge \neg\alpha \in \mathbb{L}(s)$ *or* $\neg\beta \wedge \mathrm{EX}(\neg\mathrm{A}(\alpha\mathrm{U}\beta)) \in \mathbb{L}(s)$
H.9. *if* $\mathrm{EX}\alpha \in \mathbb{L}(s)$, *then* $(\exists t \in S)((s,t) \in T$ *and* $\alpha \in \mathbb{L}(t))$
H.10. *if* $\neg\mathrm{EX}\alpha \in \mathbb{L}(s)$, *then* $(\forall t \in S)((s,t) \in T$ *implies* $\neg\alpha \in \mathbb{L}(t))$
H.11. *if* $\mathrm{E}(\alpha\mathrm{U}\beta) \in \mathbb{L}(s)$, *then* $(\exists \pi \in \Pi(s))(\exists n \geq 0)(\beta \in \mathbb{L}(\pi(n))$ *and* $(\forall j < n)\alpha \in \mathbb{L}(\pi(j)))$
H.12. *if* $\mathrm{A}(\alpha\mathrm{U}\beta) \in \mathbb{L}(s)$, *then* $(\forall \pi \in \Pi(s))(\exists n \geq 0)(\beta \in \mathbb{L}(\pi(n))$ *and* $(\forall j < n)\alpha \in \mathbb{L}(\pi(j)))$
H.13. *if* $\mathrm{K}_i\alpha \in \mathbb{L}(s)$ *and* $s\mathrm{R}_i^K t$, *then* $\alpha \in \mathbb{L}(t)$
H.14. *if* $\neg\mathrm{K}_i\alpha \in \mathbb{L}(s)$, *then there exists* $t \in S$ *such that* $(s\mathrm{R}_i^K t$ *and* $\neg\alpha \in \mathbb{L}(t))$
H.15. *if* $\mathrm{K}_i\alpha \in \mathbb{L}(s)$, *then* $\alpha \in \mathbb{L}(s)$
H.16. *if* $\mathrm{K}_i\alpha \in \mathbb{L}(s)$ *and* $s\mathrm{R}_i^K t$, *then* $\mathrm{K}_i\alpha \in \mathbb{L}(t)$
H.17. *if* $s\mathrm{R}_i^K t$ *and* $s\mathrm{R}_i^K u$ *and* $\mathrm{K}_i\alpha \in \mathbb{L}(t)$, *then both* $\alpha \in \mathbb{L}(u)$ *and* $\mathrm{K}_i\alpha \in \mathbb{L}(u)$
H.18. *if* $\mathrm{O}_i\alpha \in \mathbb{L}(s)$ *and* $s\mathrm{R}_i^O t$, *then* $\alpha \in \mathbb{L}(t)$
H.19. *if* $\neg\mathrm{O}_i\alpha \in \mathbb{L}(s)$, *then there exists* $t \in S$ *such that* $(s\mathrm{R}_i^O t$ *and* $\neg\alpha \in \mathbb{L}(t))$
H.20. *if* $\mathrm{O}_i\alpha \in \mathbb{L}(s)$ *and* $(sR_i^O t)$, *then* $\mathrm{O}_i\alpha \in \mathbb{L}(t)$
H.21. *if* $s\mathrm{R}_i^O t$ *and* $s\mathrm{R}_j^O u$ *and* $\mathrm{O}_i\alpha \in \mathbb{L}(u)$, *then both* $\alpha \in \mathbb{L}(t)$ *and* $\mathrm{O}_i\alpha \in \mathbb{L}(u)$
H.22. *if* $\widehat{\mathrm{K}}_i^j\alpha \in \mathbb{L}(s)$ *and* $s\mathrm{R}_i^j t$, *then* $\alpha \in \mathbb{L}(t)$
H.23. *if* $\widehat{\mathrm{K}}_i^j\alpha \in \mathbb{L}(s)$ *and* $sR_i^j t$, *then* $\widehat{\mathrm{K}}_i^j\alpha \in \mathbb{L}(t)$

H.24. *if* $s\mathrm{R}_i^j t$ *and* $s\mathrm{R}_i^j u$ *and* $\widehat{\mathrm{K}}_i^j\alpha \in \mathbb{L}(t)$*, then both* $\alpha \in \mathbb{L}(u)$ *and* $\widehat{\mathrm{K}}_i^j\alpha \in \mathbb{L}(u)$
H.25. *if* $\mathrm{K}_i\alpha \in \mathbb{L}(s)$*, then* $\widehat{\mathrm{K}}_i^j\alpha \in \mathbb{L}(s)$
H.26. *if* $\mathrm{O}_j\alpha \in \mathbb{L}(s)$*, then* $\widehat{\mathrm{K}}_i^j\alpha \in \mathbb{L}(s)$

Note that the set of formulas $\mathbb{L}(s)$ is a propositional tableau [4] for each state s. Note also that, intuitively, the rules H14, H15, H16 and H17 correspond to the seriality, reflexivity, transitivity and Euclidean property for the epistemic case, respectively; the rules H19, H20 and H21 correspond to the seriality, transitivity and i-jEuclidean property for the deontic case, respectively; and the rules H23 and H24 correspond to the transitivity and Euclidean property for the intersection of epistemic and deontic concepts, respectively. Note further that the Hintikka structure differs from a the deontic interpreted system in that the assignment $\mathbb{L}$ is not restricted to propositional variables, nor it is required to contain p or $\neg p$ for any $p \in \mathcal{PV}$.

We have the following result:

Lemma 1 (Hintikka's Lemma for $\mathcal{L}$). *A formula $\varphi \in \mathcal{L}$ is satisfiable (i.e., φ has a model) if and only if there is a Hintikka structure for φ.*

Proof. It is easy to check that any model $M = (S, T, (\mathrm{R}_i^K)_{i\in\mathcal{AG}}, (\mathrm{R}_i^O)_{i\in\mathcal{AG}}, (\mathrm{R}_i^j)_{i,j\in\mathcal{AG}}, \mathcal{V})$ for φ is a Hintikka structure for φ, when we extend $\mathcal{V}$ to cover all formulae which are true in a state, i.e., in M we replace $\mathcal{V}$ by $\mathbb{L}$ that is defined as: $\alpha \in \mathbb{L}(s)$ if $(M, s) \models \alpha$, for all $s \in S$.

For the converse, suppose that $H = (S, T, (\mathrm{R}_i^K)_{i\in\mathcal{AG}}, (\mathrm{R}_i^O)_{i\in\mathcal{AG}}, (\mathrm{R}_i^j)_{i,j\in\mathcal{AG}}, \mathbb{L})$ is an Hintikka structure for φ. Let $M = (S, T', (\mathrm{R}_i'^K)_{i\in\mathcal{AG}}, (\mathrm{R}_i'^O)_{i\in\mathcal{AG}}, (\mathrm{R}_i'^j)_{i,j\in\mathcal{AG}}, \mathcal{V})$, where T' is serial closure of T; $\mathrm{R}_i'^K$ is reflexive and Euclidean closure of R_i^K; $\mathrm{R}_i'^O$ is serial, transitive and i-jEuclidean closure of R_i^O; and $\mathrm{R}_i'^j$ is transitive and Euclidean closure of R_i^j; $\mathcal{V} : S \to 2^{\mathcal{PV}}$ is defined by $\mathcal{V}(s) = \{p \mid p \in \mathbb{L}(s)\}$. We now show by the induction on the structure of formulas that if $\psi \in Sub(\varphi)$, then $\psi \in \mathbb{L}(s)$ implies $M, s \models \psi$ and $\neg\psi \in \mathbb{L}(s)$ implies $M, s \models \neg\psi$.

1. ψ is a primitive proposition p. The result follows directly from the definition of $\mathcal{V}$ and the fact that $\mathbb{L}(s)$ is a propositional tableau, so we cannot have both p and $\neg p$ in $\mathbb{L}(s)$.
2. ψ is of the form $\neg\alpha$ or $\alpha\wedge\beta$. Then, the result follows easily using the induction hypothesis and the fact that $\mathbb{L}(s)$ is a propositional tableau.
3. ψ is of the form $\mathrm{EX}\beta$. If $\mathrm{EX}\beta \in \mathbb{L}(s)$ then by the rule $H9$ of the definition of the Hintikka structure H, there is some state t such that $(s, t) \in T$ and $\beta \in \mathbb{L}(t)$. So, by the induction hypothesis we have that $M, t \models \beta$, and thereby we have that $M, s \models \mathrm{EX}\beta$. If $\neg\mathrm{EX}\beta \in \mathbb{L}(s)$, then by the rule $H10$ of the definition of the Hintikka structure H, for all state t such that $(s, t) \in T$ we have that $\neg\beta \in \mathbb{L}(t)$. So, by the induction hypothesis we have that $M, t \models \neg\beta$ for all state t such that $(s, t) \in T$, and thereby we have that $M, s \models \neg\mathrm{EX}\beta$.
4. ψ is of the form $\mathrm{E}(\alpha\mathrm{U}\beta)$. If $\mathrm{E}(\alpha\mathrm{U}\beta) \in \mathbb{L}(s)$, then by the rule $H11$ of the definition of the Hintikka structure H, there exist a path π that starts at state s and a state $\pi(n)$ with $n \geq 0$ such that $\beta \in \mathbb{L}(\pi(n))$ and $\alpha \in \mathbb{L}(\pi(j))$

for all $j < n$. Since by the induction hypothesis we have that $M, \pi(n) \models \beta$ and $M, \pi(j) \models \alpha$ for all $j < n$, we must have that $M, s \models \mathrm{E}(\alpha U\beta)$. If $\neg\mathrm{E}(\alpha \mathrm{U}\beta) \in \mathbb{L}(s)$, then by the rule $H6$ we have that $\neg\alpha \wedge \neg\beta \in \mathbb{L}(s)$ or $\neg\beta \wedge \neg\mathrm{EXE}(\alpha\mathrm{U}\beta) \in \mathbb{L}(s)$. Let suppose that $\neg\alpha \wedge \neg\beta \in \mathbb{L}(s)$. Then by rules $H1 - H4$ and by the induction hypothesis we have that $M, s \models \neg\alpha\wedge\neg\beta$, which implies that $M, s \models \neg\mathrm{E}(\alpha\mathrm{U}\beta)$. Let suppose now that $\neg\beta \wedge \neg\mathrm{EXE}(\alpha\mathrm{U}\beta) \in \mathbb{L}(s)$. By the rule $H3$ and by the induction hypothesis we have that $M, s \models \neg\beta$ and $M, s \models \neg\mathrm{EXE}(\alpha\mathrm{U}\beta)$, which implies that $M, s \models \neg\mathrm{E}(\alpha\mathrm{U}\beta)$.

5. α is of the form $\mathrm{A}(\alpha\mathrm{U}\alpha_2)$. If $\mathrm{A}(\alpha\mathrm{U}\alpha_2) \in \mathbb{L}(s)$, then by the rule $H12$ of the definition of the Hintikka structure H, for all paths π that start at state s there exists a state $\pi(n)$ with $n \geq 0$ such that $\beta \in \mathbb{L}(\pi(n))$ and $\alpha \in \mathbb{L}(\pi(j))$ for all $j < n$. Since by the induction hypothesis we have that $M, \pi(n) \models \beta$ and $M, \pi(j) \models \alpha$ for all $j < n$ and path π that start at s, we must have that $M, s \models \mathrm{A}(\alpha U\beta)$. If $\neg\mathrm{A}(\alpha\mathrm{U}\beta) \in \mathbb{L}(s)$, then by the rule $H8$ we have that $\neg\alpha \wedge \neg\beta \in \mathbb{L}(s)$ or $\neg\beta \wedge \mathrm{EX}(\neg\mathrm{A}(\alpha\mathrm{U}\beta)) \in \mathbb{L}(s)$. Let suppose that $\neg\alpha \wedge \neg\beta \in \mathbb{L}(s)$. Then by rules $H1 - H4$ and by the induction hypothesis we have that $M, s \models \neg\alpha \wedge \neg\beta$, which implies that $M, s \models \neg\mathrm{A}(\alpha\mathrm{U}\beta)$. Let suppose now that $\neg\beta \wedge \mathrm{EX}(\neg\mathrm{A}(\alpha\mathrm{U}\beta)) \in \mathbb{L}(s)$. By the rule $H3$ and by the induction hypothesis we have that $M, s \models \neg\beta$ and $M, s \models \mathrm{EX}(\neg\mathrm{A}(\alpha\mathrm{U}\beta))$, which implies that $M, s \models \neg\mathrm{A}(\alpha\mathrm{U}\beta)$.
6. ψ is of the form $\mathrm{K}_i\beta$. Let supose that $\mathrm{K}_i\beta \in \mathbb{L}(s)$. We want to show that $M, s \models \mathrm{K}_i\beta$. It suffices to show that $M, t \models \beta$ for all state t such that $s\mathrm{R}_i^{\prime K}t$. But since $\mathrm{R}_i^{\prime K}$ is a reflexive and Euclidean closure of R_i^K, if $s\mathrm{R}_i^{\prime K}t$ then either (a) $s\mathrm{R}_i^K t$, or (b) there exists a state v such that $v\mathrm{R}_i^K s$ and $v\mathrm{R}_i^K t$. Let first assume that (a) holds. Since $\mathrm{K}_i\beta \in \mathbb{L}(s)$, by rule $H13$ we have that $\beta \in \mathbb{L}(t)$. Let assume now that (b) holds. Since $\mathrm{K}_i\beta \in \mathbb{L}(s)$, by rule $H17$, we have that $\beta \in \mathbb{L}(t)$. So, by the induction hypothesis we have that $M, t \models \beta$. Since this holds for an arbitary t such that $s\mathrm{R}_i^{\prime K}t$ we can conclude that $M, s \models \mathrm{K}_i\beta$.
   Now, let supose that $\neg K_i\beta \in \mathbb{L}(s)$. By the rule $H14$ of the definition of the Hintikka structure H, there exists state t such that $s\mathrm{R}_i^K t$ and $\neg\beta \in \mathbb{L}(t)$. Since $\mathrm{R}_i^K \subseteq \mathrm{R}_i^{\prime K}$, and since by the induction hypothesis we have that $M, t \models \neg\beta$, we must have $M, s \models \neg\mathrm{K}_i\beta$.
7. ψ is of the form $\mathrm{O}_i\beta$. Let supose that $\mathrm{O}_i\beta \in \mathbb{L}(s)$. We want to show that $M, s \models \mathrm{O}_i\beta$. It suffices to show that $M, t \models \beta$ for all state t such that $s\mathrm{R}_i^{\prime O}t$. But since $\mathrm{R}_i^{\prime O}$ is a serial, transitive and i-jEuclidean closure of R_i^O, if $s\mathrm{R}_i^{\prime O}t$ then there exists $k > 0$ and there exists a sequence of states $x_0, \ldots, x_k$ such that $s = x_0$, $t = x_k$, and for all $n \in \{0, \ldots, k-1\}$ either $x_n\mathrm{R}_i^0 x_{n+1}$, or there exists a state v such that $v\mathrm{R}_i^O x_n$ and $v\mathrm{R}_j^O x_{n+1}$. An intuction on n, using rules $H18$, $H20$ and $H21$ shows that we must have $\mathrm{O}_i\beta \in \mathbb{L}(x_n)$ for all $n \in \{0, \ldots, k\}$, and $\beta \in \mathbb{L}(x_n)$ for all $n \in \{1, \ldots, k\}$. In particular, we have that $\mathrm{O}_i\beta \in \mathbb{L}(t)$ and $\beta \in \mathbb{L}(t)$. So, by the induction hypothesis we have that $M, t \models \beta$. Since this holds for an arbitary t such that $s\mathrm{R}_i^{\prime O}t$ we can conclude that $M, s \models \mathrm{O}_i\beta$.
   Now, let supose that $\neg\mathrm{O}_i\beta \in \mathbb{L}(s)$. By the rule $H19$, we have that there exists state t such that $s\mathrm{R}_i^O t$ and $\neg\beta \in \mathbb{L}(t)$. Since $\mathrm{R}_i^O \subseteq \mathrm{R}_i^{\prime O}$, and since by the induction hypothesis we have that $M, t \models \neg\beta$, we must have $M, s \models \neg\mathrm{O}_i\beta$.

8. ψ is of the form $\widehat{\mathrm{K}}_i^j\beta$. Let supose that $\widehat{\mathrm{K}}_i^j\beta \in \mathbb{L}(s)$. We want to show that $M, s \models \widehat{\mathrm{K}}_i^j\beta$. It suffices to show that $M, t \models \beta$ for all state t such that $s\mathrm{R}_i'^j t$. But since $\mathrm{R}_i'^j$ is a transitive and Euclidean closure of R_i^j, if $s\mathrm{R}_i'^O t$ then there exists $k > 0$ and there exists a sequence of states $x_0, \ldots, x_k$ such that $s = x_0$, $t = x_k$, and for all $n \in \{0, \ldots, k-1\}$ either $x_n\mathrm{R}_i^j x_{n+1}$, or there exists a state v such that $v\mathrm{R}_i^j x_n$ and $v\mathrm{R}_j^j x_{n+1}$. An intuction on n, using rules $H23$ and $H24$ shows that we must have $\widehat{\mathrm{K}}_i^j\beta \in \mathbb{L}(x_n)$ for all $n \in \{0, \ldots, k\}$, and $\beta \in \mathbb{L}(x_n)$ for all $n \in \{1, \ldots, k\}$. In particular, we have that $\widehat{\mathrm{K}}_i^j\beta \in \mathbb{L}(t)$ and $\beta \in \mathbb{L}(t)$. So, by the induction hypothesis we have that $M, t \models \beta$. Since this holds for an arbitary t such that $s\mathrm{R}_i'^j t$ we can conclude that $M, s \models \widehat{\mathrm{K}}_i^j\beta$.
   Now, let supose that $\neg\widehat{\mathrm{K}}_i^j\beta \in \mathbb{L}(s)$. By the rule $H1$, we have that $\widehat{\mathrm{K}}_i^j\beta \notin \mathbb{L}(s)$. Since any model for a given formula is a Hintikka structure for the formula, by the Contrapostion Law, we have that $M, s \models \neg\widehat{\mathrm{K}}_i^j\beta$.

□

We now proceed to define a quotient structure for a given model. The quotient construction depends on an equivalence relation of states on a given model. To define this we use the *Fischer-Ladner closure* of a formula $\varphi \in \mathcal{L}$ (denoted by $FL(\varphi)$) as $FL(\varphi) = CL(\varphi) \cup \{\neg\alpha \mid \alpha \in CL(\varphi)\}$, where $CL(\varphi)$ is the smallest set of formulas that contains φ and satisfy the following conditions:

(a). if $\neg\alpha \in CL(\varphi)$, then $\alpha \in CL(\varphi)$,
(b). if $\alpha \wedge \beta \in CL(\varphi)$, then $\alpha, \beta \in CL(\varphi)$,
(c). if $\mathrm{E}(\alpha \mathrm{U} \beta) \in CL(\varphi)$, then $\alpha, \beta, \mathrm{EXE}(\alpha\mathrm{U}\beta) \in CL(\varphi)$,
(d). if $\mathrm{A}(\alpha \mathrm{U} \beta) \in CL(\varphi)$, then $\alpha, \beta, \mathrm{AXA}(\alpha\mathrm{U}\beta) \in CL(\varphi)$,
(e). if $\mathrm{EX}\alpha \in CL(\varphi)$, then $\alpha \in CL(\varphi)$,
(g). if $\mathrm{O}_i\alpha \in CL(\varphi)$, then $\alpha \in CL(\varphi)$,
(f). if $\mathrm{K}_i\alpha \in CL(\varphi)$, then $\alpha \in CL(\varphi)$,
(h). if $\widehat{\mathrm{K}}_i^j\alpha \in CL(\varphi)$, then $\alpha \in CL(\varphi)$.

Observe that for a given formula $\varphi \in \mathcal{L}$, $FL(\varphi)$ forms a finite set of formulae, as the following lemma shows (the size of a finite set A — denoted by $Card(A)$ — is defined as the number of elements of A).

Lemma 2. *Given a formula $\varphi \in \mathcal{L}$, $Card(FL(\varphi)) \leq 2(|\varphi|)$.*

Proof. Straightforward by induction on the length of φ. □

Definition 7 (Fischer-Ladner's equivalence relation). *Let $\varphi \in \mathcal{L}$ and $M = (S, T, (\mathrm{R}_i^K)_{i\in\mathcal{AG}}, (\mathrm{R}_i^O)_{i\in\mathcal{AG}}, (\mathrm{R}_i^j)_{i,j\in\mathcal{AG}}, \mathcal{V})$ be a model for φ. The relation $\leftrightarrow_{FL(\varphi)}$ on a set of states S is defined as follows:*

$$s \leftrightarrow_{FL(\varphi)} s' \text{ if } (\forall\alpha \in FL(\varphi))((M, s) \models \alpha \text{ iff } (M, s') \models \alpha)$$

By $[s]$ we denote the set $\{w \in S \mid w \leftrightarrow_{FL(\varphi)} s\}$.

Observe that $\leftrightarrow_{FL(\varphi)}$ is indeed an equivalence relation, so using it we can define the quotient structure for a given model for $\mathcal{L}$.

Definition 8 (Quotient structure). *Let* $\varphi \in \mathcal{L}$, $M = (S, T, (\mathrm{R}_i^K)_{i\in\mathcal{AG}}, (\mathrm{R}_i^O)_{i\in\mathcal{AG}}, (\mathrm{R}_i^j)_{i,j\in\mathcal{AG}}, \mathcal{V})$ *be a model for* φ, *and* $\leftrightarrow_{FL(\varphi)}$ *a Fischer-Ladner's equivalence relation. The* quotient structure of M by $\leftrightarrow_{FL(\varphi)}$ *is the tuple* $M_{\leftrightarrow_{FL(\varphi)}} = (S', T', (\mathrm{R}_i'^K)_{i\in\mathcal{AG}}, (\mathrm{R}_i'^O)_{i\in\mathcal{AG}}, (\mathrm{R}_i'^j)_{i,j\in\mathcal{AG}}, \mathbb{L}')$, *where*

- $S' = \{[s] \mid s \in S\}$,
- $T' = \{([s],[s']) \in S' \times S' \mid (\exists w \in [s])(\exists w' \in [s'])$ *such that* $(w, w') \in T\}$,
- $\mathrm{R}_i'^K$ *be a transitive closure of* $\{([s],[s']) \in S' \times S' \mid (\exists w \in [s])(\exists w' \in [s'])$ *such that* $(w, w') \in \mathrm{R}_i^K\}$,
- $\mathrm{R}_i'^O = \{([s],[s']) \in S' \times S' \mid (\exists w \in [s])(\exists w' \in [s'])$ *such that* $(w, w') \in \mathrm{R}_i^O\}$,
- $\mathrm{R}_i'^j = \{([s],[s']) \in S' \times S' \mid (\forall w \in [s])(\forall w' \in [s'])$ *such that* $(w, w') \in \mathrm{R}_i^j\}$,
- $\mathbb{L}' : S' \to 2^{FL(\varphi)}$ *is defined by:* $\mathbb{L}'([s]) = \{\alpha \in FL(\varphi) \mid (M, s) \models \alpha\}$.

Note that the set S' is finite as it is the result of collapsing states satisfying formulas that belong to the finite set $FL(\varphi)$. In fact we have $Card(S') \leq 2^{Card(FL(\varphi))}$. Note also that the relation T' is serial, $\mathrm{R}_i'^K$ is reflexive, symmetric and transitive (i.e., it is an equivalence relation), $\mathrm{R}_i'^O$ is serial, transitive and $i-j$ Euclidean, and $\mathrm{R}_i'^j$ is transitive and Euclidean. Further, since $\mathcal{L}$ is an extension of CTL, the resulting quotient structure may not be a model. In particular, the following lemma holds.

Lemma 3. *The quotient construction does not preserve satisfiability of formulas of the form* $\mathrm{A}(\alpha \mathrm{U} \beta)$, *where* $\alpha, \beta \in \mathcal{L}$. *In particular, there is a model* M *for* $\mathrm{A}(\top \mathrm{U} p)$ *with* $p \in \mathcal{PV}$ *such that* $M_{\leftrightarrow_{FL(\varphi)}}$ *is not a model for* $\mathrm{A}(\top \mathrm{U} p)$.

Proof. [Sketch] Consider the following model $M = (S, T, R_1^K, R_1^O, R_1^1, \mathcal{V})$ for $\mathrm{A}(\top \mathrm{U} p)$, where $S = \{s_0, s_1, \ldots, \}$, $T = \{(s_0, s_0)\} \cup \{(s_i, s_{i-1}) \mid i > 0\}$, $R_1^K = R_1^O = R_1^1 = \{(s_i, s_i) \mid i \geq 0\}$, $p \in \mathcal{V}(s_0)$ and $p \notin \mathcal{V}(s_i)$ for all $i > 0$. It is easy to observe that in the quotient structure of M, i.e., in $M_{\leftrightarrow_{FL(\mathrm{A}(\top \mathrm{U} p))}}$, two distinct states s_i and s_j, for all $i, j > 0$, will be identified. As a result of that, a cycle along which p is always false will appear in $M_{\leftrightarrow_{FL(\mathrm{A}(\top \mathrm{U} p))}}$. This implies that $\mathrm{A}(\top \mathrm{U} p)$ does not hold along the cycle. □

Although the quotient structure of a given model M by $\leftrightarrow_{FL(\varphi)}$ may not be a model, it satisfies another important property, which allows us to view it as a *pseudo-model*; it can be unwound into a proper model. This observation can be used to show that the language $\mathcal{L}$ has the FMP property. To make this idea precise, we introduce the following auxiliary definitions.

An *interior* (respectively *frontier*) node of a *directed acyclic graph* (DAG)[1] is one which has (respectively does not have) a T-successor. The *root* of a DAG is the node (if it exists) from which all other nodes are reachable via the T relation. A *fragment* $M' = (S', T', (\mathrm{R}_i'^K)_{i\in\mathcal{AG}}, (\mathrm{R}_i'^O)_{i\in\mathcal{AG}}, (\mathrm{R}_i'^j)_{i,j\in\mathcal{AG}}, \mathbb{L}')$ of a Hintikka structure is a structure such that (S', T') generates a finite DAG whose interior nodes satisfy $H1$-$H10$ and $H13$-$H26$, and the frontier nodes satisfy $H1$-$H8$ and $H13$-$H26$. Given $M = (S, T, (\mathrm{R}_i^K)_{i\in\mathcal{AG}}, (\mathrm{R}_i^O)_{i\in\mathcal{AG}}, (\mathrm{R}_i^j)_{i,j\in\mathcal{AG}}, \mathbb{L})$ and $M' = (S', T', (\mathrm{R}_i'^K)_{i\in\mathcal{AG}}, (\mathrm{R}_i'^O)_{i\in\mathcal{AG}}, (\mathrm{R}_i'^j)_{i,j\in\mathcal{AG}}, \mathbb{L}')$, we say that M is *contained*

[1] Recall that a directed acyclic graph is a directed graph such that for any node v, there is no nonempty directed path starting and ending on v.

in M', and write $M \subseteq M'$, if $S \subseteq S'$, $T = T' \cap (S \times S)$, $\mathrm{R}_i^K = \mathrm{R}_i'^K \cap (S \times S)$, $\mathrm{R}_i^O = \mathrm{R}_i'^O \cap (S \times S)$, $\mathrm{R}_i^j = \mathrm{R}_i'^j \cap (S \times S)$, $\mathbb{L} = \mathbb{L}'|S$.

Definition 9 (Pseudo-model). *Let $\varphi \in \mathcal{L}$. A* pseudo-model $M = (S, T, (\mathrm{R}_i^K)_{i\in\mathcal{AG}}, (\mathrm{R}_i^O)_{i\in\mathcal{AG}}, (\mathrm{R}_i^j)_{i,j\in\mathcal{AG}}, \mathbb{L})$ *for φ is defined in the same manner as a Hintikka structure for φ in Definition 6, except that condition $H12$ is replaced by the following condition $H'12$: $(\forall s \in S)$ if $\mathrm{A}(\alpha \mathrm{U} \beta) \in \mathbb{L}(s)$, then there is a fragment $(S', T', (\mathrm{R}_i'^K)_{i\in\mathcal{AG}}, (\mathrm{R}_i'^O)_{i\in\mathcal{AG}}, (\mathrm{R}_i'^j)_{i,j\in\mathcal{AG}}, \mathbb{L}') \subseteq M$ such that: (a) (S', T') generates a finite DAG with root s; (b) for all frontier nodes $t \in S'$, $\beta \in \mathbb{L}'(t)$; (c) for all interior nodes $u \in S'$, $\alpha \in \mathbb{L}'(u)$.*

We have the following.

Lemma 4. *Let $\varphi \in \mathcal{L}$, $FL(\varphi)$ be the Fischer-Ladner closure of φ, $M = (S, T, (\mathrm{R}_i^K)_{i\in\mathcal{AG}}, (\mathrm{R}_i^O)_{i\in\mathcal{AG}}, (\mathrm{R}_i^j)_{i,j\in\mathcal{AG}}, \mathcal{V})$ a model for φ, and $M_{\leftrightarrow_{FL(\varphi)}} = (S', T', (\mathrm{R}_i'^K)_{i\in\mathcal{AG}}, (\mathrm{R}_i'^O)_{i\in\mathcal{AG}}, (\mathrm{R}_i'^j)_{i,j\in\mathcal{AG}}, \mathbb{L})$ the quotient structure of M by $\leftrightarrow_{FL(\varphi)}$. Then, $M_{\leftrightarrow_{FL(\varphi)}}$ is a pseudo-model for φ.*

Proof. This can be shown by induction on the structure of φ. The proof for the CTL part of $\mathcal{L}$ follows immediately from Lemma 3.8 in [2]. Consider now φ to be of the following forms:

H.13 . $\varphi = \mathrm{K}_i\alpha$. Let $\mathrm{K}_i\alpha \in \mathbb{L}([s])$ and $[s]\mathrm{R}_i'^K[t]$ for an arbitrary $[t] \in S'$. Since $[s]\mathrm{R}_i'^K[t]$, by the definition of $\mathrm{R}_i'^K$ there exists $k > 0$ and there exists a sequence $[x_0] \dots [x_k]$ of states such that $[s] = [x_0]$, $[x_k] = [t]$ and $[x_j]R_i[x_{j+1}]$ with $R_i = \{([s], [s']) \in S' \times S' \mid (\exists w \in [s])(\exists w' \in [s'])$ such that $(w, w') \in \mathrm{R}_i^K\}$ for all $j \in \{0, \dots, k-1\}$. We will first show that if $\mathrm{K}_i\alpha \in \mathbb{L}([x_0])$ and $[x_0]R_i[x_1]$, then $\mathrm{K}_i\alpha \in \mathbb{L}([x_1])$ and $\alpha \in \mathbb{L}([x_1])$.
Since $[x_0]R_i[x_1]$, by the definition of R_i we have that there exist $x_0' \in [x_0]$ and $x_1' \in [x_1]$ such that $x_0'\mathrm{R}_i^K x_1'$. Without losing of generality we can take $x_0' = x_0$ and $x_1' = x_1$, and thereby we have that

$$x_0 \mathrm{R}_i^K x_1 \tag{1}$$

Since $\mathrm{K}_i\alpha \in \mathbb{L}([x_0])$, by the definition of $\leftrightarrow_{FL(\varphi)}$ and $\mathbb{L}$ we have that $M, x_0 \models \mathrm{K}_i\alpha$ (in fact we have $M, x \models \mathrm{K}_i\alpha$ for all $x \in [x_0]$). Thus, by the definition of $\models$ we have that

$$M, t \models \alpha \text{ for all state } t \text{ such that } x_0 \mathrm{R}_i^K t \tag{2}$$

So, in paricular, since (1) holds, we have that $M, x_1 \models \alpha$. Thus by the definitions of $\leftrightarrow_{FL(\varphi)}$ and $\mathbb{L}$, we have that $\alpha \in \mathbb{L}([x_1])$. Now, consider any state y such that $x_1\mathrm{R}_i^K y$. Since (1) holds and the relation R_i^K is transitive, we have that $x_0\mathrm{R}_i^K y$. Thus, since (2) holds we have that $M, y \models \alpha$. Since this holds for any y such that $x_1\mathrm{R}_i^K y$, we have that $M, x_1 \models \mathrm{K}_i\alpha$. Thus, by the definitions of $\leftrightarrow_{FL(\varphi)}$ and $\mathbb{L}$, we have that $K_i\alpha \in \mathbb{L}([x_1])$. Now, by induction on $0 \leq j < k$, we conclude that if $\mathrm{K}_i\alpha \in \mathbb{L}([x_j])$ and $[x_j]R_i[x_{j+1}]$, then $\mathrm{K}_i\alpha \in \mathbb{L}([x_{j+1}])$ and $\alpha \in \mathbb{L}([x_{j+1}])$. This implies that $\mathrm{K}_i\alpha \in \mathbb{L}([t])$ and $\alpha \in \mathbb{L}([t])$. So, conditions $H13$ and $H16$ are fulfilled.

H.14 . $\varphi = \neg \mathrm{K}_i\alpha$. Let $\neg \mathrm{K}_i\alpha \in \mathbb{L}([s])$. Then, by the definitions of $\leftrightarrow_{FL(\varphi)}$ and $\mathbb{L}$, we have that $M, s \models \neg \mathrm{K}_i\alpha$ (in fact we have $M, s' \models \neg \mathrm{K}_i\alpha$ for all the state $s' \in [s]$). So, by the definition of $\models$, we have that there exists $t \in S$ such that $s\mathrm{R}_i^K t$ and $(M, t) \models \neg\alpha$. Consider an equivalence class of $\leftrightarrow_{FL(\varphi)}$ that is generated by t, i.e., the state $[t]$ of S'. Since $s\mathrm{R}_i^K t$, by the definition of $\mathrm{R}_i'^K$ we have that $[s]\mathrm{R}_i'^K[t]$. Since $(M, t) \models \neg\alpha$, by the definitions of $\leftrightarrow_{FL(\varphi)}$ and $\mathbb{L}$, we have that $\neg\alpha \in \mathbb{L}([t])$. Therefore, we can conclude that there exists state $[t] \in S'$ such that $[s]\mathrm{R}_i'^K[t]$ and $\neg\alpha \in \mathbb{L}([t])$. So, condition $H14$ is fulfilled.

H.15 . $\varphi = \mathrm{K}_i\alpha$. Let $\mathrm{K}_i\alpha \in \mathbb{L}([s])$. Then, by the definitions of $\leftrightarrow_{FL(\varphi)}$ and $\mathbb{L}$, we have that $M, s \models \mathrm{K}_i\alpha$ (in fact we have $M, s' \models \mathrm{K}_i\alpha$ for all the state $s' \in [s]$). Thus, by the definition of $\models$ we have that

$$M, t \models \alpha \text{ for all state } t \text{ such that } s\mathrm{R}_i^K t \tag{3}$$

So, since R_i^K is reflexive, we have that $M, s \models \alpha$. Then, by the definitions of $\leftrightarrow_{FL(\varphi)}$ and $\mathbb{L}$, we have that $\alpha \in \mathbb{L}([s])$, which implies that condition $H15$ is fulfilled.

H.17 . Let $[s]\mathrm{R}_i'^K[t]$, $[s]\mathrm{R}_i'^K[u]$ and $\mathrm{K}_i\alpha \in \mathbb{L}([u])$. Since $[s]\mathrm{R}_i'^K[u]$ and $\mathrm{R}_i'^K$ is symmetric, we have that $[u]\mathrm{R}_i'^K[s]$. Further, since $\mathrm{R}_i'^K$ is transitive and $[u]\mathrm{R}_i'^K[s]$ and $[s]\mathrm{R}_i'^K[t]$, we have that $[u]\mathrm{R}_i'^K[t]$. Thus, since $\mathrm{K}_i\alpha \in \mathbb{L}([u])$, by case H.13 of the proof, we have that $\alpha \in \mathbb{L}([t])$ and $\mathrm{K}_i\alpha \in \mathbb{L}([t])$. So, condition $H17$ is fulfilled.

H.18 . $\varphi = \mathrm{O}_i\alpha$. Let $\mathrm{O}_i\alpha \in \mathbb{L}([s])$ and $[s]\mathrm{R}_i'^O[t]$ for an arbitrary $[t] \in S'$. Since $[s]\mathrm{R}_i'^O[t]$, by the definition of $\mathrm{R}_i'^O$, there exist states $s' \in [s]$ and $t' \in [t]$ such that $s'\mathrm{R}_i^O t'$. Since $\mathrm{O}_i\alpha \in \mathbb{L}([s])$, by the definitions of $\leftrightarrow_{FL(\varphi)}$ and $\mathbb{L}$ we have that $M, s'' \models \mathrm{O}_i\alpha$ for all $s'' \in [s]$. So, in particular we have that $(M, s') \models \mathrm{O}_i\alpha$. By the definition of $\models$, we have that $(M, t'') \models \alpha$ for all $t'' \in S$ such that $s'\mathrm{R}_i^O t''$. In particular, since $s'\mathrm{R}_i^O t'$, we have that $(M, t') \models \alpha$. Thus, since $[t'] = [t]$, by the definitions of $\leftrightarrow_{FL(\varphi)}$ and $\mathbb{L}$ we have that $\alpha \in \mathbb{L}([t])$. So, the condition $H18$ is fulfilled.

H.19 . $\varphi = \neg \mathrm{O}_i\alpha$. Let $\neg \mathrm{O}_i\alpha \in \mathbb{L}([s])$. Then, by the definitions of $\leftrightarrow_{FL(\varphi)}$ and $\mathbb{L}$, we have that $M, s \models \neg \mathrm{O}_i\alpha$ (in fact we have $M, s' \models \neg \mathrm{O}_i\alpha$ for all the state $s' \in [s]$). So, by the definition of $\models$, we have that there exists a state $t \in S$ such that $s\mathrm{R}_i^O t$ and $(M, t) \models \neg\alpha$. Consider an equivalence class of $\leftrightarrow_{FL(\varphi)}$ that is generated by t, i.e., the state $[t]$ of S'. Since $s\mathrm{R}_i^O t$, by the definition of $\mathrm{R}_i'^O$ we have that $[s]\mathrm{R}_i'^O[t]$. Since $(M, t) \models \neg\alpha$, by the definitions of $\leftrightarrow_{FL(\varphi)}$ and $\mathbb{L}$, we have that $\neg\alpha \in \mathbb{L}([t])$. Therefore, we can conclude that there exists state $[t] \in S'$ such that $[s]\mathrm{R}_i'^O[t]$ and $\neg\alpha \in \mathbb{L}([t])$. So, condition $H19$ is fulfilled.

H.20 . Let $[s]\mathrm{R}_i'^O[t]$ and $\mathrm{O}_i\alpha \in \mathbb{L}([s])$. By case H.18 of the proof, we have that $\alpha \in \mathbb{L}([t])$. Now, consider any $[t'] \in S'$ such that $[t]\mathrm{R}_i'^O[t']$. Since $\mathrm{R}_i'^O$ is transitive, we have that $[s]\mathrm{R}_i'^O[t']$. So, again by case H.18 , we have that $\alpha \in \mathbb{L}([t'])$ for each $[t']$ such that $[t]\mathrm{R}_i'^O[t']$. Thus, by the definition of $\leftrightarrow_{FL(\varphi)}$, $\mathbb{L}$ and $\models$, we can conclude that $\mathrm{O}_i\alpha \in \mathbb{L}([t])$. So, condition $H20$ is fulfilled.

H.21 . Let $[s]\mathrm{R}_i'^O[t]$ and $[s]\mathrm{R}_j'^O[u]$, and $\mathrm{O}_i\alpha \in \mathbb{L}([u])$. Since $\mathrm{R}_i'^O$ is $i - j$Euclidean, we have that $[u]\mathrm{R}_i'^O[t]$. Thus, since $\mathrm{O}_i\alpha \in \mathbb{L}([u])$ holds, by case H.18 we have $\alpha \in \mathbb{L}([t])$, and by case H.20 we have $\mathrm{O}_i\alpha \in \mathbb{L}([t])$. So, condition $H21$ is fulfilled.

H.22 . $\varphi = \widehat{\mathrm{K}}_i^j\alpha$. Let $\widehat{\mathrm{K}}_i^j\alpha \in \mathbb{L}([s])$ and $[s]\mathrm{R}_i'^j[t]$ for an arbitrary $[t] \in S'$. Since $[s]\mathrm{R}_i'^j[t]$, by the definition of $\mathrm{R}_i'^j$ we have that $s'\mathrm{R}_i^j t'$ for all states $s' \in [s]$ and $t' \in [t]$. Since $\widehat{\mathrm{K}}_i^j\alpha \in \mathbb{L}([s])$, by the definition of $\leftrightarrow_{FL(\varphi)}$ and $\mathbb{L}$ we have that $M, s' \models \widehat{\mathrm{K}}_i^j\alpha$ for all $s' \in [s]$. Thus, by the definition of $\models$ we have that $M, x \models \alpha$ for all states x such that $s'\mathrm{R}_i^j x$. So, since $s'\mathrm{R}_i^j t'$ for all $t' \in [t]$, we can conclude that $M, t' \models \alpha$. Thus, since $t' \in [t]$, by the definition of $\leftrightarrow_{FL(\varphi)}$ and $\mathbb{L}$ we have that $\alpha \in \mathbb{L}([t])$. Therefore, we can conclude that condition $H22$ is fulfilled.

H.23 . Let $[s]\mathrm{R}_i'^j[t]$ and $\widehat{\mathrm{K}}_i^j\alpha \in \mathbb{L}([s])$. By case H.22 of the proof, we have that $\alpha \in \mathbb{L}([t])$. Now, consider any $[t'] \in S'$ such that $[t]\mathrm{R}_i'^j[t']$. Since $\mathrm{R}_i'^j$ is transitive, we have that $[s]\mathrm{R}_i'^j[t']$. So again by case H.22 , we have that $\alpha \in \mathbb{L}([t'])$ for each $[t']$ such that $[t]\mathrm{R}_i'^j[t']$. Thus, by the definition of $\leftrightarrow_{FL(\varphi)}$, $\mathbb{L}$ and $\models$, we can conclude that $\widehat{\mathrm{K}}_i^j\alpha \in \mathbb{L}([t])$. So, condition $H23$ is fulfilled.

H.24 . Let $[s]\mathrm{R}_i'^j[t]$ and $[s]\mathrm{R}_i'^j[u]$, and $\widehat{\mathrm{K}}_i^j\alpha \in \mathbb{L}([u])$. Since $\mathrm{R}_i'^j$ is euclidean, we have that $[u]\mathrm{R}_i'^j[t]$. Since $\widehat{\mathrm{K}}_i^j\alpha \in \mathbb{L}([u])$ holds, by case H.22 of the proof, we have $\alpha \in \mathbb{L}([t])$, and by case H.23 we have $\widehat{\mathrm{K}}_i^j\alpha \in \mathbb{L}([t])$. So, condition $H24$ is fulfilled.

H.25 . $\varphi = \mathrm{K}_i\alpha$. Let $(M, s) \models \mathrm{K}_i\alpha$, and $\mathrm{K}_i\alpha \in \mathbb{L}([s])$. By the definition of $\models$, we have that $(M, t) \models \alpha$ for all $t \in S$ such that $s\mathrm{R}_i^K t$. Consider the following two sets $K(s, i) = \{t \mid (s\mathrm{R}_i^K t)$ and $(M, t) \models \alpha\}$ and $O(s, i, j) = \{t \in K(s, i) \mid (s\mathrm{R}_j^O t)\}$, where $i, j \in \{1 \ldots, n\}$. By the definition of $K(s, i)$ and $O(s, j)$, we have that $O(s, i, j) = \{t \mid (s\mathrm{R}_i^j t)$ and $M, t \models \alpha\}$. Therefore, by the definition of $\models$ we have that $(M, s) \models \widehat{\mathrm{K}}_i^j\alpha$. Thus, by the definitions of $\leftrightarrow_{FL(\varphi)}$ and $\mathbb{L}$, we have that $\widehat{\mathrm{K}}_i^j\alpha \in \mathbb{L}([s])$. So, condition $H25$ is fulfilled.

H.26 . $\varphi = \mathrm{O}_j\alpha$. Let $(M, s) \models \mathrm{O}_j\alpha$, and $\mathrm{O}_j\alpha \in \mathbb{L}([s])$. By the definition of $\models$, we have that $(M, t) \models \alpha$ for all $t \in S$ such that $s\mathrm{R}_j^O t$. Consider the following two sets $O(s, j) = \{t \mid (s\mathrm{R}_j^O t)$ and $(M, t) \models \alpha\}$ and $K(s, i, j) = \{t \in O(s, j) \mid (s\mathrm{R}_i^K t)\}$, where $i, j \in \{1 \ldots, n\}$. By the definition of $K(s, i, j)$ and $O(s, i)$, we have that $K(s, i, j) = \{t \mid (s\mathrm{R}_i^j t)$ and $M, t \models \alpha\}$. Therefore, by the definition of $\models$ we have that $(M, s) \models \widehat{\mathrm{K}}_i^j\alpha$. Thus, by the definitions of $\leftrightarrow_{FL(\varphi)}$ and $\mathbb{L}$, we have that $\widehat{\mathrm{K}}_i^j\alpha \in \mathbb{L}([s])$. So, condition $H26$ is fulfilled.

□

We can now prove the main claim of the section, i.e., the fact that $\mathcal{L}$ has the finite model property.

Theorem 1 (FMP for $\mathcal{L}$). *Let $\varphi \in \mathcal{L}$. Then the following are equivalent: (1) φ is satisfiable; (2) There is a finite pseudo-model for φ; (3) There is a Hintikka structure for φ.*

Proof. [sketch] (3) ⇒ (1) follows from Lemma 1. (1) ⇒ (2) follows from Lemma 4. To prove (2) ⇒ (3) it is enough to construct a Hintikka structure for φ by "unwinding" the pseudo-model for φ. This can be done in the same way as is described in [2] for the proof of Theorem 4.1. □

4 Decidability for $\mathcal{L}$

Let φ be a $\mathcal{L}$ formula, and $FL(\varphi)$ the Fischer-Ladner closure of φ. We define $\Delta \subseteq FL(\varphi)$ to be *maximal* if for every formula $\alpha \in FL(\varphi)$, either $\alpha \in \Delta$ or $\neg\alpha \in \Delta$.

Theorem 2. *There is an algorithm for deciding whether any $\mathcal{L}$ formula is satisfiable.*

Proof. Given a formula $\varphi \in \mathcal{L}$, we will construct a finite pseudo-model for φ of size less or equal $2^{2\cdot|\varphi|}$. We proceed as follows.

1. Build a structure $M' = (S', T', (\mathrm{R}_i'^K)_{i\in\mathcal{AG}}, (\mathrm{R}_i'^O)_{i\in\mathcal{AG}}, (\mathrm{R}_i'^j)_{i,j\in\mathcal{AG}}, \mathbb{L}')$ in the following way:
   - $S' = \{\Delta \mid \Delta \subseteq FL(\varphi)$ and Δ is maximal and satisfies rules $H1$-$H8$, $H13$, $H24$, $H25\}$;
   - $T' \subseteq S' \times S'$ is a relation such that $(\Delta_1, \Delta_2) \in T'$ iff $\neg\mathrm{EX}\alpha \in \Delta_1$ implies that $\neg\alpha \in \Delta_2$;
   - for each agent $i \in \mathcal{AG}$, $\mathrm{R}_i'^K \subseteq S' \times S'$ is a relation such that $(\Delta_1, \Delta_2) \in \mathrm{R}_i'^K$ iff $\{\alpha \mid \mathrm{K}_i\alpha \in \Delta_1\} \subseteq \Delta_2$;
   - for each agent $i \in \mathcal{AG}$, $\mathrm{R}_i'^O \subseteq S' \times S'$ is a relation such that $(\Delta_1, \Delta_2) \in \mathrm{R}_i'^O$ iff $\{\alpha \mid \mathrm{O}_i\alpha \in \Delta_1\} \subseteq \Delta_2$;
   - for each agent $i, j \in \mathcal{AG}$, $\mathrm{R}_i'^j \subseteq S' \times S'$ is a relation such that $(\Delta_1, \Delta_2) \in \mathrm{R}_i'^j$ iff $\{\alpha \mid \widehat{\mathrm{K}}_i^j\alpha \in \Delta_1\} \subseteq \Delta_2$;
   - $\mathbb{L}' : S \to 2^{FL(\varphi)}$ is a function defined by $\mathbb{L}'(\Delta) = \Delta$.

   It is easy to observe that M', as constructed above, satisfies properties $H1$-$H8$, $H15$, $H25$, $H26$; properties $H10$, $H13$, $H18$, and $H22$ (because of the definition of T', $\mathrm{R}_i'^K$, $\mathrm{R}_i'^O$, and $\mathrm{R}_i'^j$ respectively). Note also that since $Card(FL(\varphi)) \leq 2 \cdot |\varphi|$ (see Lemma 2), S' has at most $2^{2\cdot|\varphi|}$ elements.
2. Test the above structure M' for fulfilment of the properties $H9$, $H11$, $H'12$, $H14$, $H16$, $H17$, $H19$-$H21$, $H23$ and $H24$ by repeatedly applying the following deletion rules until no more states in M' can be deleted.

   H9 Delete any state which has no T-successors.

   H11-H12' Delete any state $\Delta_1 \in S'$ such that $\mathrm{E}(\alpha\mathrm{U}\beta) \in \Delta_1$ (respectively $\mathrm{A}(\alpha\mathrm{U}\beta) \in \Delta_1$) and there does not exist a fragment $M'' \subseteq M'$ such that: (i) (S'', T'') generates a finite DAG with root Δ_1; (ii) for all frontier nodes $\Delta_2 \in S''$, $\beta \in \Delta_2$; (iii) for all interior nodes $\Delta_3 \in S''$, $\alpha \in \Delta_3$.

   H14 Delete any state $\Delta_1 \in S'$ such that $\neg\mathrm{K}_i\alpha \in \Delta_1$, and Δ_1 does not have any $\mathrm{R}_i'^K$ successor $\Delta_2 \in S'$ with $\neg\alpha \in \Delta_2$.

   H16 Delete any state $\Delta_1 \in S'$ such that $\Delta_1\mathrm{R}_i'^K\Delta_2$ and $\mathrm{K}_i\alpha \in \Delta_1$ and $\neg K_i\alpha \in \Delta_2$.

   H17 Delete any state $\Delta_1 \in S'$ such that $\Delta_1\mathrm{R}_i'^K\Delta_2$ and $\Delta_1\mathrm{R}_i'^K\Delta_3$ and $\alpha \in \Delta_2$ and $\mathrm{K}_i\neg\alpha \in \Delta_3$

- H19 Delete any state $\Delta_1 \in S'$ such that $\neg \mathrm{O}_i\alpha \in \Delta_1$, and Δ_1 does not have any $\mathrm{R}_i'^O$ successor $\Delta_2 \in S'$ with $\neg\alpha \in \Delta_2$.
- H20 Delete any state $\Delta_1 \in S'$ such that $\Delta_1 \mathrm{R}_i'^O \Delta_2$ and $\mathrm{O}_i\alpha \in \Delta_1$ and $\neg O_i\alpha \in \Delta_2$.
- H21 Delete any state $\Delta_1 \in S'$ such that $\Delta_1 R_i^O \Delta_2$ and $\Delta_1 R_j^O \Delta_3$ and $\mathrm{O}_i\neg\alpha \in \Delta_3$ and $\alpha \in \Delta_2$.
- H23 Delete any state $\Delta_1 \in S'$ such that $\Delta_1 \mathrm{R}_i'^j \Delta_2$ and $\widehat{\mathrm{K}}_i^j\alpha \in \Delta_1$ and $\neg\widehat{\mathrm{K}}_i^j\alpha \in \Delta_2$.
- H24 Delete any state $\Delta_1 \in S'$ such that $\Delta_1 R_i'^j \Delta_2$ and $\Delta_1 R_i'^j \Delta_3$ and $\alpha \in \Delta_2$ and $\widehat{\mathrm{K}}_i^j\neg\alpha \in \Delta_3$.

We call the above two points a *decidability algorithm for $\mathcal{L}$*.

Claim (1). The decidability algorithm for $\mathcal{L}$ terminates.
Proof. The termination is obvious given that the initial set S' is finite.

Claim (2). Let $M = (S, T, (\mathrm{R}_i^K)_{i\in\mathcal{AG}}, (\mathrm{R}_i^O)_{i\in\mathcal{AG}}, (\mathrm{R}_i^j)_{i,j\in\mathcal{AG}}, \mathbb{L})$ be the resulting structure of the algorithm. The formula $\varphi \in \mathcal{L}$ is satisfiable iff $\varphi \in s$, for some $s \in S$.
Proof. In order to show the part right-to-left of the above property, note that either the resulting structure is a pseudo-model for φ, or $S = \emptyset$ (this can be shown inductively on the structure of the algorithm). So, if $\varphi \in s$ for some $s \in S$, φ is satisfiable by Theorem 1.

Conversely, if φ is satisfiable, then there exists a model M^* such that $M^* \models \varphi$. Let $M^*_{\leftrightarrow_{FL(\varphi)}} = M' = (S', T', (\mathrm{R}_i'^K)_{i\in\mathcal{AG}}, (\mathrm{R}_i'^O)_{i\in\mathcal{AG}}, (\mathrm{R}_i'^j)_{i,j\in\mathcal{AG}}, \mathbb{L}')$ be the quotient structure of M^* by $\leftrightarrow_{FL(\varphi)}$. By Theorem 1 we have that M' is a pseudo-model for φ. Moreover, by the definition of $\mathbb{L}'$ in the quotient structure, $\mathbb{L}'(s)$ is maximal with respect to $FL(\varphi)$ for all $s \in S'$. Now, let $M'' = (S'', T'', (\mathrm{R}_i''^K)_{i\in\mathcal{AG}}, (\mathrm{R}_i''^O)_{i\in\mathcal{AG}}, (\mathrm{R}_i''^j)_{i,j\in\mathcal{AG}}, \mathbb{L}'')$ be a structure defined by step 1 of the decidability algorithm, and $f : S' \to S''$ a function defined by $f(s) = \mathbb{L}'(s)$. The following conditions hold:

1. If $(s,t) \in T'$, then $(f(s), f(t)) \in T''$;
   Proof (via contradiction): Let $(s,t) \in T'$ and $(f(s), f(t)) \notin T''$. By the definition of T'' we have that $\neg\mathrm{EX}\alpha \in f(s)$ and $\alpha \in f(t)$. Then, by the definition of f, we have that $\neg\mathrm{EX}\alpha \in \mathbb{L}'(s)$ and $\alpha \in \mathbb{L}'(t)$. So, by the definition of $\mathbb{L}'$ in the quotient structure we have that $M^*, s \models \neg\mathrm{EX}\alpha$ and $M^*, t \models \alpha$, which contradict the fact that $(s,t) \in T'$.
2. If $(s,t) \in \mathrm{R}_i'^K$, then $(f(s), f(t)) \in \mathrm{R}_i''^K$;
   Proof (via contradiction): Let $(s,t) \in \mathrm{R}_i'^K$ and $(f(s), f(t)) \notin \mathrm{R}_i''^K$. By the definition of $\mathrm{R}_i''^K$ we have that $\mathrm{K}_i\alpha \in f(s)$ and $\alpha \notin f(t)$. Then, by the definition of f, we have that $\mathrm{K}_i\alpha \in \mathbb{L}'(s)$ and $\alpha \notin \mathbb{L}'(t)$. So, by the definition of $\mathbb{L}'$ in the quotient structure we have that $M^*, s \models \mathrm{K}_i\alpha$ and $M^*, t \models \neg\alpha$, which contradict the fact that $(s,t) \in \mathrm{R}_i'^K$.
3. If $(s,t) \in \mathrm{R}_i'^O$, then $(f(s), f(t)) \in \mathrm{R}_i''^O$;
   Proof (via contradiction): Let $(s,t) \in \mathrm{R}_i'^O$ and $(f(s), f(t)) \notin \mathrm{R}_i''^O$. By the definition of $\mathrm{R}_i''^O$ we have that $\mathrm{O}_i\alpha \in f(s)$ and $\alpha \notin f(t)$. Then, by the

definition of f, we have that $\mathrm{O}_i\alpha \in \mathbb{L}'(s)$ and $\alpha \notin \mathbb{L}'(t)$. So, by the definition of $\mathbb{L}'$ in the quotient structure we have that $M^*, s \models \mathrm{O}_i\alpha$ and $M^*, t \models \neg\alpha$, which contradict the fact that $(s,t) \in \mathrm{R}_i'^{O}$.

4. If $(s,t) \in \mathrm{R}_i'^{j}$, then $(f(s), f(t)) \in \mathrm{R}_i''^{j}$;
   Proof (via contradiction): Let $(s,t) \in \mathrm{R}_i'^{j}$ and $(f(s), f(t)) \notin \mathrm{R}_i''^{j}$. By the definition of $\mathrm{R}_i''^{j}$ we have that $\widehat{\mathrm{K}}_i^j\alpha \in f(s)$ and $\alpha \notin f(t)$. Then, by the definition of f, we have that $\widehat{\mathrm{K}}_i^j\alpha \in \mathbb{L}'(s)$ and $\alpha \notin \mathbb{L}'(t)$. So, by the definition of $\mathbb{L}'$ in the quotient structure we have that $M^*, s \models \widehat{\mathrm{K}}_i^j\alpha$ and $M^*, t \models \neg\alpha$, which contradict the fact that $(s,t) \in \mathrm{R}_i'^{j}$.

Thus, the image of M' under f is contained in M'', i.e., $M' \subseteq M''$. It remains to show that if $s \in S'$, then $f(s) \in S''$ will not be eliminated in step 2 of the decidability algorithm. This can be checked by induction on the order in which states of S'' are eliminated. For instance, assume that $s \in S'$, and $\mathrm{A}(\alpha \mathrm{U} \beta) \in f(s)$. By the definition of f, we have that $\mathrm{A}(\alpha \mathrm{U} \beta) \in \mathbb{L}'(s)$. Now, since M' is a pseudo-model, by Definition 9 we have that there exists a fragment rooted at s that is contained in M' and it satisfies property $H'12$. Thus, since f preserves the above condition (a), we have that there exists a fragment rooted at $f(s)$ that is contained in M'' and it satisfies property $H'12$. This implies that $f(s) \in S''$ will not be eliminated in step 2b of the decidability algorithm. Other cases can be proven similarly. Therefore, it follows that for some $s \in S$ we have $\varphi \in \mathbb{L}(s)$.

□

5 A Complete Axiomatic System for $\mathcal{L}$

An *axiomatic system* consists of a collection of *axioms* and *inference rules*. An axiom is a formula, and an inference rule has the form "from formulas $\varphi_1, \ldots, \varphi_m$ infer formula φ". We say that φ is *provable* (written $\vdash \varphi$) if there is a sequence of formulas ending with φ, such that each formula is either an instance of an axiom, or follows from other provable formulas by applying an inference rule. We say that a formula φ is *consistent* if $\neg\varphi$ is not provable. A finite set $\{\varphi_1, \ldots, \varphi_m\}$ of formulas is *consistent* if and only if the conjunction $\varphi_1 \wedge \ldots \wedge \varphi_m$ of its members is consistent, and an infinite set of formulas is *consistent* if all of its finite subsets are consistent. A set F of formulas is a *maximal consistent set* if it is consistent and for all $\varphi \notin F$, the set $F \cup \{\varphi\}$ is inconsistent. An axiom system is *sound* (resp. *complete*) with respect to the class of models, if $\vdash \varphi$ implies $\models \varphi$ (resp. if $\models \varphi$ implies $\vdash \varphi$).

Definition 10 (Axiomatisation of deontic interpreted systems). *Let $i \in \{1, \ldots, n\}$. Consider the following axiomatic system for $\mathcal{L}$:*

PC. *All substitution instances of classical tautologies.*
$\mathbf{X_1}$. $\mathrm{EX}\top$
$\mathbf{X_2}$. $\mathrm{EX}(\alpha \vee \beta) \Leftrightarrow \mathrm{EX}\alpha \vee \mathrm{EX}\beta$
$\mathbf{U_1}$. $\mathrm{E}(\alpha \mathrm{U} \beta) \Leftrightarrow \beta \vee (\alpha \wedge \mathrm{EXE}(\alpha \mathrm{U} \beta))$

$\mathbf{U_2}$. $\mathrm{A}(\alpha \mathrm{U} \beta) \Leftrightarrow \beta \vee (\alpha \wedge \mathrm{AXA}(\alpha \mathrm{U} \beta))$
$\mathbf{K}_{\mathrm{K_i}}$. $(\mathrm{K}_i \alpha \wedge \mathrm{K}_i(\alpha \Rightarrow \beta)) \Rightarrow \mathrm{K}_i \beta$
$\mathbf{T}_{\mathrm{K_i}}$. $\mathrm{K}_i \alpha \Rightarrow \alpha$
$\mathbf{4}_{\mathrm{K_i}}$. $\mathrm{K}_i \alpha \Rightarrow \mathrm{K}_i \mathrm{K}_i \alpha$
$\mathbf{5}_{\mathrm{K_i}}$. $\neg \mathrm{K}_i \alpha \Rightarrow \mathrm{K}_i \neg \mathrm{K}_i \alpha$
$\mathbf{K}_{\mathrm{O_i}}$. $(\mathrm{O}_i \alpha \wedge \mathrm{O}_i(\alpha \Rightarrow \beta)) \Rightarrow \mathrm{O}_i \beta$
$\mathbf{D}_{\mathrm{O_i}}$. $\mathrm{O}_i \alpha \Rightarrow \neg \mathrm{O}_i \neg \alpha$
$\mathbf{4}_{\mathrm{O_i}}$. $\mathrm{O}_i \alpha \Rightarrow \mathrm{O}_i \mathrm{O}_i \alpha$
$\mathbf{5}_{\mathrm{O_i}}^{\mathbf{i-j}}$. $\neg \mathrm{O}_i \alpha \Rightarrow \mathrm{O}_j \neg \mathrm{O}_i \alpha$
$\mathbf{K}_{\widehat{\mathrm{K}}_\mathrm{i}^\mathrm{j}}$. $(\widehat{\mathrm{K}}_i^j \alpha \wedge \widehat{\mathrm{K}}_i^j(\alpha \Rightarrow \beta)) \Rightarrow \widehat{\mathrm{K}}_i^j \beta$
$\mathbf{4}_{\widehat{\mathrm{K}}_\mathrm{i}^\mathrm{j}}$. $\widehat{\mathrm{K}}_i^j \alpha \Rightarrow \widehat{\mathrm{K}}_i^j \widehat{\mathrm{K}}_i^j \alpha$
$\mathbf{5}_{\widehat{\mathrm{K}}_\mathrm{i}^\mathrm{j}}$. $\neg \widehat{\mathrm{K}}_i^j \alpha \Rightarrow \widehat{\mathrm{K}}_i^j \neg \widehat{\mathrm{K}}_i^j \alpha$
$\mathrm{O} - \widehat{\mathrm{K}}_\mathbf{i}^\mathbf{j}$. $\mathrm{O}_j \alpha \Rightarrow \widehat{\mathrm{K}}_i^j \alpha$
$\mathrm{K} - \widehat{\mathrm{K}}_\mathbf{i}^\mathbf{j}$. $\mathrm{K}_i \alpha \Rightarrow \widehat{\mathrm{K}}_i^j \alpha$
MP. *From* α *and* $\alpha \Rightarrow \beta$ *infer* β*,*
$\mathbf{Nec}_{\mathrm{K}_i}$. *From* α *infer* $\mathrm{K}_i \alpha$*,*
$\mathbf{Nec}_{\mathrm{O}_i}$. *From* α *infer* $\mathrm{O}_i \alpha$*,*
$\mathbf{R1}_X$. *From* $\alpha \Rightarrow \beta$ *infer* $\mathrm{EX}\alpha \Rightarrow \mathrm{EX}\beta$
$\mathbf{R2}_X$. *From* $\gamma \Rightarrow (\neg \beta \wedge \mathrm{EX}\gamma)$ *infer* $\gamma \Rightarrow \neg \mathrm{A}(\alpha \mathrm{U} \beta)$
$\mathbf{R3}_X$. *From* $\gamma \Rightarrow (\neg \beta \wedge \mathrm{AX}(\gamma \vee \neg \mathrm{E}(\alpha \mathrm{U} \beta)))$ *infer* $\gamma \Rightarrow \neg \mathrm{E}(\alpha \mathrm{U} \beta)$

We note that the system above includes the axiomatisation for CTL [2], S5 [3] for K_i and $\mathrm{KD45}^{i-j}$ [6] for O_i. The fragment for the operators $\widehat{\mathrm{K}}_i^j$, previously not explored, is K45. In line with the traditional interpretation of these axioms in an epistemic setting these are to be interpreted from the point of view of an external observer ascribing properties to the system. They both seem in line with the interpretation of the modality of knowledge under the assumption of correct behaviour. Further note that axioms $4_{\widehat{\mathrm{K}}_i^j}$, and $5_{\widehat{\mathrm{K}}_i^j}$ are to be expected given that both the underlying relations are transitive and Euclidean.

The interaction axioms $\mathrm{O}_i - \widehat{\mathrm{K}}_i^j$ and $\mathrm{K}_i - \widehat{\mathrm{K}}_i^j$ regulate the relationship between $\mathrm{O}_i, \mathrm{K}_i$ and $\widehat{\mathrm{K}}_i^j$. They were both discussed in [6] and correspond to our intuition regarding the meaning of the modalities. Note also that they closely match the interaction axioms for distributed versus standard knowledge, which again confirms our intuition given that distributed knowledge is defined on the intersection of the relations for standard knowledge.

The inference rules for all the components are also entirely expected — note that while Necessitation for $\widehat{\mathrm{K}}_i^j$ is not explicitly listed, it may easily be deduced from Nec_{K_i} or Nec_{O_i}.

Theorem 3. *The axiomatic system for $\mathcal{L}$ is sound and complete with respect to the deontic interpreted systems, i.e.* $\models \varphi$ *iff* $\vdash \varphi$*, for any formula* $\varphi \in \mathcal{L}$*.*

Proof. Soundness can be checked inductively as standard. For completeness, we show that any consistent formula φ is satisfiable. To do this, we first consider the structure $M = (S, T, (\mathrm{R}_i^K)_{i \in \mathcal{AG}}, (\mathrm{R}_i^O)_{i \in \mathcal{AG}}, (\mathrm{R}_i^j)_{i,j \in \mathcal{AG}}, \mathbb{L})$ for φ as defined in step

1 of the decidability algorithm. We then execute step 2 of the algorithm, obtaining a pseudo-model for φ. Crucially we show below that if a state $s \in S$ is eliminated at step 2 of the algorithm, then the formula $\psi_s = \bigwedge_{\alpha \in s} \alpha$ is inconsistent. Observe now that that for any $\alpha \in FL(\varphi)$ we have $\vdash \alpha \Leftrightarrow \bigvee_{\{s \mid \alpha \in s \text{ and } \psi_s \text{ is consistent}\}} \psi_s$. In particular, $\vdash \varphi \Leftrightarrow \bigvee_{\{s \mid \varphi \in s \text{ and } \psi_s \text{ is consistent}\}} \psi_s$. Thus, if φ is consistent, then ψ_s is consistent as well for some $s \in S$. It follows by Claim 2 of Theorem 2 that this particular s is present in the pseudo-model resulting from the execution of the algorithm. So, by Theorem 1, φ is satisfiable. Note that pseudo-models share the structural properties of models, i.e., their underlying frames have the same properties.

It remains to show that if a state $s \in S$ is eliminated at step 2 of the algorithm then the formula ψ_s is inconsistent. Before we do it, we need some auxiliary claims.

Claim (3). Let $s \in S$ and $\alpha \in FL(\varphi)$. Then, $\alpha \in s$ iff $\vdash \psi_s \Rightarrow \alpha$.
Proof. ('if'). Let $\alpha \in s$. By the definition of S, we have that any s in S is maximal. Thus, $\neg\alpha \notin s$. So, $\vdash \psi_s \Rightarrow \alpha$.

('only if'). Let $\vdash \psi_s \Rightarrow \alpha$. So, since s is maximal we have that $\alpha \in s$. □

Claim (4). Let $s, t \in S$, both of them be maximal and propositionally consistent, and $s\mathrm{R}_i^K t$ (respectively $s\mathrm{R}_i^O t$ and $s\mathrm{R}_i^j t$). If $\alpha \in t$, then $\neg K_i \neg\alpha \in s$ (respectively $\neg O_i \neg\alpha \in s$ and $\neg\widehat{\mathrm{K}}_i^j \neg\alpha \in s$).
Proof.[By contraposition] Let $\alpha \in t$ and $\neg K_i \neg\alpha \notin s$. Then, since s is maximal we have that $K_i \neg\alpha \in s$. Thus, since $s\mathrm{R}_i^K t$, we have that $\neg\alpha \in t$. This contradicts the fact that $\alpha \in t$, since t is propositionally consistent.

The same proof applies to O_i and $\widehat{\mathrm{K}}_i^j$. □

Claim (5). Let $s \in S$ be a maximal and consistent set of formulas and α such that $\vdash \alpha$. Then $\alpha \in s$.
Proof. Suppose $\alpha \notin s$ and $\vdash \alpha$. Since s is maximal then $\neg\alpha \in s$. So $\neg\alpha \wedge \psi_s$ is consistent where ψ_s where $\psi_s \in s$. So by definition of consistency we have that $\nvdash \neg(\neg\alpha \wedge \psi_s)$, so $\nvdash \alpha \vee \neg\psi_s$. But we have $\vdash \alpha \vee \psi_s$, so this is a contradiction.
□

We now show, by induction on the structure of the decidability algorithm for $\mathcal{L}$, that if a state $s \in S$ is eliminated at step 2 of the decidability algorithm, then $\vdash \neg\psi_s$.

Claim (6). If ψ_s is consistent, then s is not eliminated at step 2 of the decidability algorithm for $\mathcal{L}$.
Proof.

- H9 Let $\mathrm{EX}\alpha \in s$ and ψ_s be consistent. By the same reasoning as in the proof of Claim 4(a) in [2], we conclude that s satisfies $H9$. So s is not eliminated.
- H11-H'12 Let $\mathrm{E}(\alpha \mathrm{U} \beta) \in s$ (resp. $\mathrm{A}(\alpha \mathrm{U} \beta) \in s$) and suppose s is eliminated at step 2 because $H11$ (resp. $H'12$) is not satisfied. Then ψ_s is inconsistent. The proof showing that fact is the same as the proof of Claim 4(c) (resp. Claim 4(d)) in [2].

H14 Let $\neg \mathrm{K}_i\alpha \in s$ and ψ_s be consistent. Consider the set $S_{\neg\alpha} = \{\neg\alpha\} \cup \{\beta \mid \mathrm{K}_i\beta \in s\}$. We will show that $S_{\neg\alpha}$ is consistent. Suppose that $S_{\neg\alpha}$ is inconsistent. Then, $\vdash \beta_1 \wedge \ldots \wedge \beta_m \Rightarrow \alpha$, where $\beta_j \in \{\beta \mid \mathrm{K}_i\beta \in s\}$ for $j \in \{1, \ldots, m\}$. By rule Nec_{K_i} we have $\vdash \mathrm{K}_i((\beta_1 \wedge \ldots \wedge \beta_m) \Rightarrow \alpha)$. By axioms K_{K_i} and PC we have $\vdash (\mathrm{K}_i\beta_1 \wedge \ldots \wedge \mathrm{K}_i\beta_m) \Rightarrow \mathrm{K}_i\alpha$. Thus, since each $\mathrm{K}_i\beta_j \in s$ for $j \in \{1, \ldots, m\}$ and s is maximal and consistent, we have $\mathrm{K}_i\alpha \in s$. This contradicts the fact that ψ_s is consistent. So, $S_{\neg\alpha}$ is consistent. Now, since each set of formulas can be extended to a maximal one, we have that $S_{\neg\alpha}$ is contained in some maximal set t. Thus $\neg\alpha \in t$, and moreover, by the definition of R_i^K in M and the definition of $S_{\neg\alpha}$ we have that $s\mathrm{R}_i^K t$. Thus, s satisfies $H14$, and it is not eliminated by step $(H14)$ of the decidability algorithm.

H16 Suppose that ψ_s is consistent and s is eliminated at step (H16) of the decidability algorithm. Then, we have that $s\mathrm{R}_i^K t$, $\mathrm{K}_i\alpha \in s$ and $\neg\mathrm{K}_i\alpha \in t$. Thus, since s and t are maximal and propositionally consistent, by Claim 4 we have that $\neg\mathrm{K}_i\mathrm{K}_i\alpha \in s$. By axiom 4_{K_i} and Claim 5 we have that $\mathrm{K}_i\alpha \Rightarrow \mathrm{K}_i\mathrm{K}_i\alpha \in s$. So, since $\mathrm{K}_i\alpha \in s$ we have that $\mathrm{K}_i\mathrm{K}_i\alpha \in s$. So s is inconsistent. Therefore s cannot be eliminated at step (H16) of the decidability algorithm.

H17 Suppose that s is consistent and it is eliminated at step (f) of the decidability algorithm. Thus, we have that $s\mathrm{R}_i^K t$, $s\mathrm{R}_i^K u$, $\alpha \in t$, and $\mathrm{K}_i\neg\alpha \in u$. So, since $s\mathrm{R}_i^K t$, $\alpha \in t$, s and t are maximal and propositionally consistent, by Claim 4 we have that $\neg K_i \neg\alpha \in s$. Since s is maximal and consistent, by axiom 5_{K_i} and Claim 5, we have that $\neg\mathrm{K}_i\neg\alpha \Rightarrow \mathrm{K}_i\neg\mathrm{K}_i\neg\alpha \in s$. Therefore, we have that $\mathrm{K}_i\neg\mathrm{K}_i\neg\alpha \in s$. Thus, since $s\mathrm{R}_i^K u$, we have that $\neg\mathrm{K}_i\neg\alpha \in u$. But this is a contradictions given that $\mathrm{K}_i\neg\alpha \in u$ an u is propositionally consistent. So s is inconsistent. Therefore s cannot be eliminated at step (f) of the decidability algorithm.

H19 Let $\neg\mathrm{O}_i\alpha \in s$ and ψ_s be consistent. Consider the set $S_{\neg\alpha} = \{\neg\alpha\} \cup \{\beta \mid \mathrm{O}_i\beta \in s\}$. We will show that $S_{\neg\alpha}$ is consistent. Suppose that $S_{\neg\alpha}$ is inconsistent. Then, $\vdash \beta_1 \wedge \ldots \wedge \beta_m \Rightarrow \alpha$, where $\beta_j \in \{\beta \mid \mathrm{O}_i\beta \in s\}$ for $j \in \{1, \ldots, m\}$. By rule Nec_{O_i} we have $\vdash \mathrm{O}_i((\beta_1 \wedge \ldots \wedge \beta_m) \Rightarrow \alpha)$. By axioms K_{O_i} and PC we have $\vdash (\mathrm{O}_i\beta_1 \wedge \ldots \wedge \mathrm{O}_i\beta_m) \Rightarrow \mathrm{O}_i\alpha$. Since each $\mathrm{O}_i\beta_j \in s$ for $j \in \{1, \ldots, m\}$ and s is maximal and consistent, we have $\mathrm{O}_i\alpha \in s$. This contradicts the fact that ψ_s is consistent. So, $S_{\neg\alpha}$ is consistent. Now, since each set of formulas can be extended to a maximal one, we have that $S_{\neg\alpha}$ is contained in some maximal set t. Thus $\neg\alpha \in t$, and moreover, by the definition of R_i^O in M and the definition of $S_{\neg\alpha}$ we have that $s\mathrm{R}_i^O t$. Thus, s satisfies $H19$, and it is not eliminated by step $(H19)$ of the decidability algorithm.

H20 Suppose that ψ_s is consistent and s is eliminated at step (g) of the decidability algorithm. Then, we have that $s\mathrm{R}_i^O t$, $\mathrm{O}_i\alpha \in s$ and $\neg\mathrm{O}_i\alpha \in t$. Thus, since s and t are maximal and propositionally consistent, by Claim 4 we have that $\neg\mathrm{O}_i\mathrm{O}_i\alpha \in s$. By axiom 4_{O_i} and Claim 5 we have that $\mathrm{O}_i\alpha \Rightarrow \mathrm{O}_i\mathrm{O}_i\alpha \in s$. So, since $\mathrm{O}_i\alpha \in s$ we have that $\mathrm{O}_i\mathrm{O}_i\alpha \in s$. So s is inconsistent. Therefore s cannot be eliminated at step $(H20)$ of the decidability algorithm.

H21 If ψ_s is consistent, s cannot be eliminated at step $(H21)$ of the decidability algorithm. The proof can be done similarly to the one in $(H17)$ by using axiom $5^{i-j}_{O_i}$.

H23 If ψ_s is consistent, s cannot be eliminated at step $(H24)$ of the decidability algorithm. The proof can be done similarly to the one in $(H20)$ by using axiom $4_{\widehat{K}^j_i}$.

H24 If ψ_s is consistent, s cannot be eliminated at step $(H25)$ of the decidability algorithm. The proof can be done similarly to the one in $(H17)$ by using axiom $5_{\widehat{K}^j_i}$.

□

We have now shown that only states s with ψ_s inconsistent are eliminated. This ends the completeness proof. □

6 Conclusion

We have given a complete axiomatisation of deontic interpreted systems on a language that includes full CTL as well as the the K_i, O_i and $\widehat{K}^j_i$ modalities. Thereby, we have solved the problem left open in [6]. Further, we have shown that the language considered here has the finite model property, so it is decidable.

The $\widehat{K}^j_i$ modality can be straightforwardly extended to $\widehat{K}^X_i$ [6] representing knowledge of i under the assumption of correctness of all agents in X. We believe that the technique of this paper can be extended to $\widehat{K}^X_i$ without difficulty. For clarity this is not presented in this paper.

References

1. E. Clarke and E. Emerson. Design and synthesis of synchronization skeletons for branching-time temporal logic. In *Proceedings of Workshop on Logic of Programs*, volume 131 of *LNCS*, pages 52–71. Springer-Verlag, 1981.
2. E. A. Emerson and J. Y. Halpern. Decision procedures and expressiveness in the temporal logic of branching time. *Journal of Computer and System Sciences*, 30(1):1–24, 1985.
3. R. Fagin, J. Y. Halpern, Y. Moses, and M. Y. Vardi. *Reasoning about Knowledge.* MIT Press, Cambridge, 1995.
4. J. Halpern and Y. Moses. A guide to completeness and complexity for modal logics of knowledge and belief. *Artificial Intelligence*, 54:319–379, 1992.
5. A. Lomuscio and M. Ryan. On the relation between interpreted systems and Kripke models. In M. Pagnucco, W. R. Wobcke, and C. Zhang, editors, *Agent and Multi-Agent Systems - Proceedings of the AI97 Workshop on the theoretical and practical foundations of intelligent agents and agent-oriented systems*, volume 1441 of *LNAI*. Springer Verlag, May 1998.
6. A. Lomuscio and M. Sergot. Deontic interpreted systems. *Studia Logica*, 75(1):63–92, 2003.

7. A. Lomuscio and M. Sergot. A formalisation of violation, error recovery, and enforcement in the bit transmission problem. *Journal of Applied Logic*, 2(1):93–116, March 2004.
8. W. Nabialek, A. Niewiadomski, W. Penczek, A. Pólrola, and M. Szreter. VerICS 2004: A model checker for real time and multi-agent systems. In *Proceedings of the International Workshop on Concurrency, Specification and Programming (CS&P'04)*, volume 170 of *Informatik-Berichte*, pages 88–99. Humboldt University, 2004.
9. F. Raimondi and A. Lomuscio. MCMAS - A tool for verification of multi-agent systems. http://www.cs.ucl.ac.uk/staff/f.raimondi/MCMAS/.
10. F. Raimondi and A. Lomuscio. Symbolic model checking of deontic interpreted systems via OBDD's. In *Proceedings of DEON04, Seventh International Workshop on Deontic Logic in Computer Science*, volume 3065 of *Lecture Notes in Computer Science*. Springer Verlag, 2004.
11. F. Raimondi and A. Lomuscio. Automatic verification of multi-agent systems by model checking via OBDDs. *Journal of Applied Logic*, 2005. To appear in Special issue on Logic-based agent verification.
12. M. Wooldridge and A. Lomuscio. A computationally grounded logic of visibility, perception, and knowledge. *Logic Journal of the IGPL*, 9(2):273–288, 2001.
13. B. Woźna, A. Lomuscio, and W. Penczek. Bounded model checking for deontic interpreted systems. In *Proc. of the 2nd Workshop on Logic and Communication in Multi-Agent Systems (LCMAS'04)*, volume 126 of *ENTCS*, pages 93–114. Elsevier, 2004.