Tensor-based Image Diffusions Derived from Generalizations of the Total Variation and Beltrami Functionals

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Motivation (1/2)

- Nonlinear diffusion models for Computer Vision
  - Class A: Directly-designed PDEs
    - Perona-Malik method [ieeeT-PAMI'90]
    - CLMC regularized PDE [Catte et al, siamJNA'92]
    - Coherence-enhancing diffusion [Weickert, IJCV'99]
    - Method of [Tschumperlé & Deriche, ieeeT-PAMI'05]

      ⋮
  - Class B: Variational Methods
    - Total Variation [Rudin, Osher & Fatemi, PhysicaD'92]
    - Vectorial Total Variation [Sapiro, CVIU'97]
    - Color Total Variation [Blomgren & Chan, ieeeT-IP'98]
    - Beltrami Flow [Sochen, Kimmel & Maladi, ieeeT-IP'98]

      ⋮
- For some methods of Class A: known connection with Class B, e.g. :
  - Perona-Malik model $\dfrac{\partial u(x,y,t)}{\partial t} = \operatorname{div}\left(g(\|\nabla u\|^2)\nabla u\right)$
  - $\min_u \int_\Omega \varphi(\|\nabla u\|^2) \mathrm{d}\boldsymbol{x} \rightsquigarrow \dfrac{\partial u}{\partial t} = \operatorname{div}\left(2\varphi'(\|\nabla u\|^2)\nabla u\right)$

  $g(s^2) = 2\varphi'(s^2)$
- But, for several types of PDE-based diffusion methods no variational interpretation existed

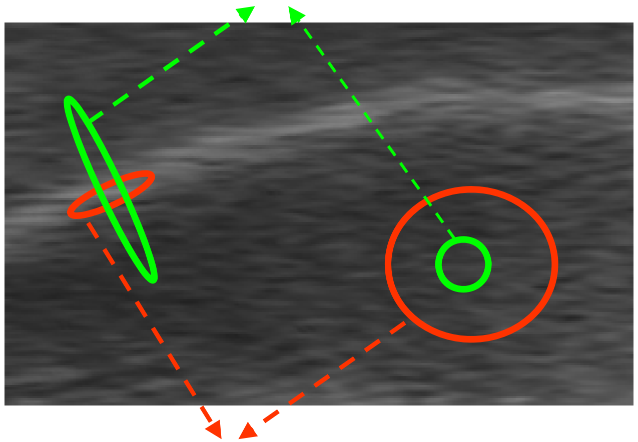
Motivation (2/2)

- Advantages of variational interpretation of diffusion methods
  - conceptually clear formalism
  - helps with the reduction of model parameters
  - easier application to problems that can be formulated as constrained energy minimization, e.g.:
    - image restoration, inpainting, interpolation
  - can lead to efficient implementations based on optimization techniques

- Advantages of using tensors in image diffusion
  - Structure tensor

    measure of the image variation & geometry in the neighborhood of each point
  - Diffusion tensor

    flexible adaptation to the image structures



Contributions

- We propose a novel generic functional that:
  - is designed for vector-valued images
  - generalizes several existing variational methods
  - is based on the structure tensor
  - leads to tensor-based nonlinear diffusions that contain regularizing convolutions

- As special cases, we propose 2 novel diffusion methods:
  - *Generalized Beltrami Flow*
  - *Tensor Total Variation*

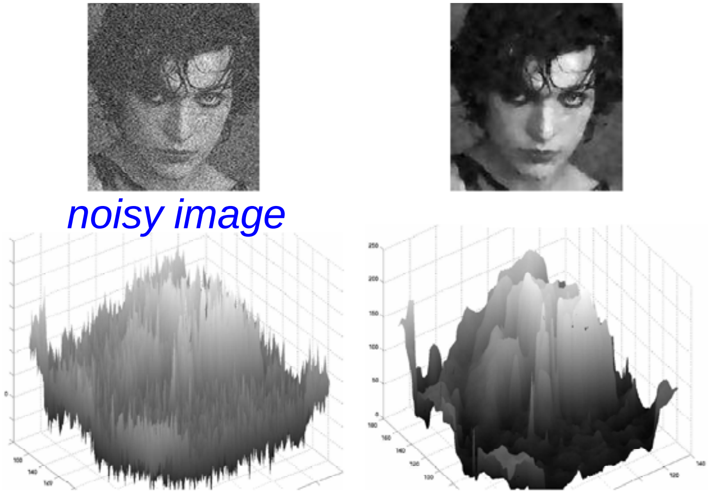
- These methods:
  - combine the advantages of variational and tensor-based diffusion approaches
  - yielded promising performance measures in denoising experiments

Generalization of the Beltrami Functional (1/2)

- Original Beltrami Flow
  [Sochen, Kimmel & Maladi, IEEE T-IP 98]
  - Interpretation of a vector-valued image *u* with *n* channels as a 2D surface embedded in R^{n+2}:

$$(x,y) \longrightarrow (x\,,\,y\,,\,u_1(x,y)\,,\,u_2(x,y)\,,\,\dots\,u_n(x,y))$$

  - Flow towards the minimization of the surface area: tensor-based diffusion



noisy image

embedded surface   instant from the flow

example for the simplest case n=1

Figure from [Tschumperle, Thesis'02]

  - It offers an elegant way to:
    - couple the image channels and
    - extend in the vector-valued case the properties of Total Variation
  - But, the diffusion tensor is not regularized (no neighborhood info)
    → limitations on the robustness to noise & edge enhancement

- To overcome these limitations, we generalize the Beltrami Functional …

Generalization of the Beltrami Functional (2/2)

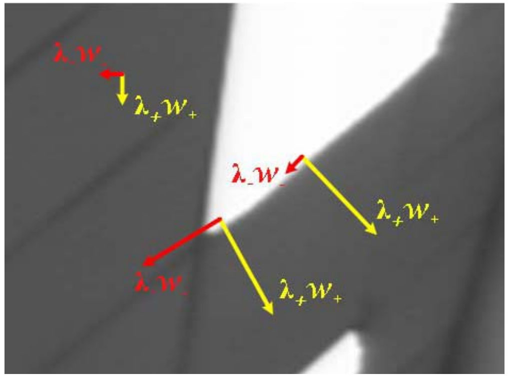
- Proposed generalization of the Beltrami functional:
  - We use higher dimensional mappings of the form:

$$\boldsymbol{x} \longrightarrow \left(\boldsymbol{x}, \underbrace{\mathcal{P}^{\boldsymbol{u}}(\boldsymbol{x})}\right)$$

image patch [Tschumperle & Brun, ICIP'09], *that contains weighted image values*
*not only at point **x***
but also at points in a window around it

  - In this way, each $\boldsymbol{x}$ contributes to the area of the embedded surface by considering the image variation in its neighborhood
  - If the patch sampling step → 0, the area of the embedded surface tends to:

$$A[\boldsymbol{u}] = \int_\Omega \sqrt{(\alpha^2 + \lambda_1)(\alpha^2 + \lambda_2)}\mathrm{d}\boldsymbol{x}$$



    - $\lambda_i = \lambda_i(J_K(\nabla \boldsymbol{u}))$: eigenvalues of the structure tensor $J_K(\nabla \boldsymbol{u}) = K * \sum \nabla u_i \otimes \nabla u_i$

Generalized Functional based on the Structure Tensor

- $$E[\boldsymbol{u}] = \int_\Omega \psi\left(\,\lambda_1(J_K(\nabla \boldsymbol{u})),\, \lambda_2(J_K(\nabla \boldsymbol{u}))\,\right) \mathrm{d}\boldsymbol{x}$$
  - $\psi(\lambda_1, \lambda_2)$: cost function (increasing)
  - $J_K(\nabla \boldsymbol{u}) = K * \sum_{i=1}^{N} \nabla u_i \otimes \nabla u_i$: 2x2 structure tensor with:
    - eigenvalues λ_1, λ_2 , eigenvectors $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2$ (depend on *K*)
- Difficulty in the theoretical analysis:

  In contrast to most variational methods, Euler-Lagrange equations not applicable here
- Theorem: we have shown that the functional minimization leads to:

$$\begin{cases} \partial u_i / \partial t = \mathrm{div}\left(D_K \nabla u_i\right), \quad i = 1, .., N, \\ D_K = K * \left(2\dfrac{\partial \psi}{\partial \lambda_1} \boldsymbol{\theta}_1 \otimes \boldsymbol{\theta}_1 + 2\dfrac{\partial \psi}{\partial \lambda_2} \boldsymbol{\theta}_2 \otimes \boldsymbol{\theta}_2 \right) \end{cases}$$

novel general type of anisotropic diffusion

Tensor Total Variation

- 1^{st} special case of the novel generic functional:

$$E[\boldsymbol{u}] = \int_{\Omega} \psi\,(\,\lambda_1(J_K(\nabla \boldsymbol{u})), \lambda_2(J_K(\nabla \boldsymbol{u}))\,)\,\mathrm{d}\boldsymbol{x}$$

with $\psi(\lambda_1, \lambda_2) = \sqrt{\lambda_1} + \sqrt{\lambda_2}$

  - Steepest descent (applying the proved theorem):

$$\frac{\partial u_i}{\partial t} = \mathrm{div}\left(\left[K * \left(\frac{1}{\sqrt{\lambda_1}}\boldsymbol{\theta}_1 \otimes \boldsymbol{\theta}_1 + \frac{1}{\sqrt{\lambda_2}}\boldsymbol{\theta}_2 \otimes \boldsymbol{\theta}_2\right)\right]\nabla u_i\right)\ ,\ i = 1, .., N$$

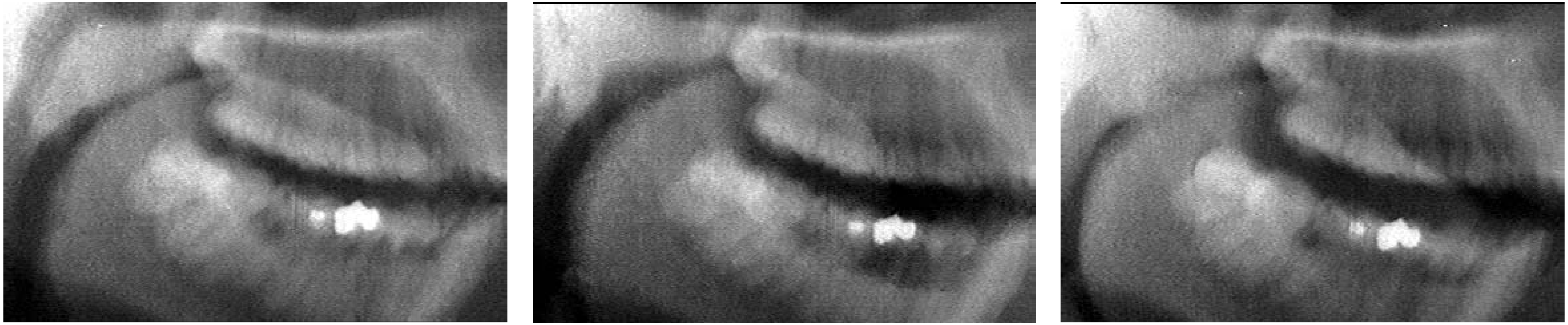
  - Classic TV: special sub-case with: N=1(graylevel images) and $K = \delta(\boldsymbol{x})$

(a) Noisy Input
(PSNR=20 dB)

(b) TV PDE
(PSNR=26.5 dB, t=16.4)

(c) Tensor TV PDE
(PSNR=27.1 dB, t=9.6)

Tensor Total Variation: Example



Input sequence

Output sequence

Application of Tensor Total Variation in a sequence of X-ray images

Generalized Beltrami Flow

- 2nd special case of the novel generic functional:

$$E[\boldsymbol{u}] = \int_\Omega \psi\left(\, \lambda_1(J_K(\nabla \boldsymbol{u})), \lambda_2(J_K(\nabla \boldsymbol{u}))\,\right) \mathrm{d}\boldsymbol{x}$$

with $\psi(\lambda_1, \lambda_2) = \sqrt{(\alpha^2 + \lambda_1)\,(\alpha^2 + \lambda_2)}$

  - Steepest descent (applying the proved theorem):

$$\frac{\partial u_i}{\partial t} = \mathrm{div}\left(\left[K * \left(\sqrt{\frac{\alpha^2+\lambda_2}{\alpha^2+\lambda_1}}\boldsymbol{\theta}_1 \otimes \boldsymbol{\theta}_1 + \sqrt{\frac{\alpha^2+\lambda_1}{\alpha^2+\lambda_2}}\boldsymbol{\theta}_2 \otimes \boldsymbol{\theta}_2\right)\right] \nabla u_i\right)$$

  - Classic Beltrami flow [Sochen et. al, IEEE T-IP 98]: special sub-case with $K = \delta(\boldsymbol{x})$ and minimization in the space of embeddings

(a) Noisy Input (PSNR=20 dB) | (b) Beltrami Flow (PSNR=23.4 dB) | (c) Gener. Beltrami Flow (PSNR=24.0 dB)

Other Interesting Special Cases

- Other special cases of the novel generic functional:

$$E[\boldsymbol{u}] = \int_\Omega \psi\left(\lambda_1(J_K(\nabla \boldsymbol{u})), \lambda_2(J_K(\nabla \boldsymbol{u}))\right) \mathrm{d}\boldsymbol{x}$$ with:

  - $\psi(\lambda_1, \lambda_2) = \phi(\lambda_1 + \lambda_2)$: Steepest descent:

$$\partial u_i / \partial t = \operatorname{div}\left(2\left[K * \varphi'(K * \|\nabla \boldsymbol{u}\|^2)\right] \nabla u_i\right)$$

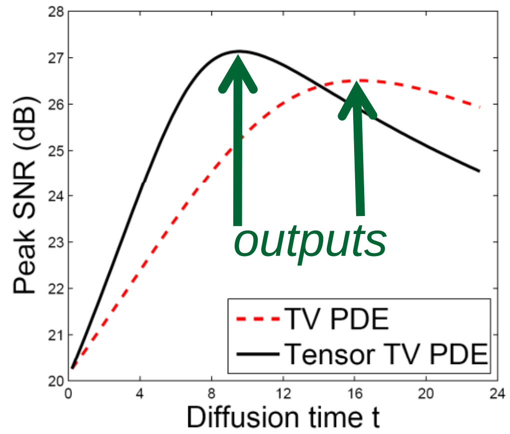
→novel regularization of the Perona-Malik model

→regularization of Sapiro's Vectorial TV: $\psi = \sqrt{\lambda_1 + \lambda_2}$

  - $K = \delta(\boldsymbol{x})$ (no regularizing convolution):
    - Studied in [Blomgren & Chan T-IP'98, Tschumperlé & Deriche, T-PAMI'05]
    - The corresponding diffusion is anisotropic only if the image channels are $N \geq 2$
    - No incorporation of neighborhood info

Denoising Experiments: Framework

- Experimental Framework
  - take a noise-free reference image
  - add gaussian noise
  - input in the compared diffusion methods
  - compute PSNR during each PDE flow and output the image with the maximum PSNR



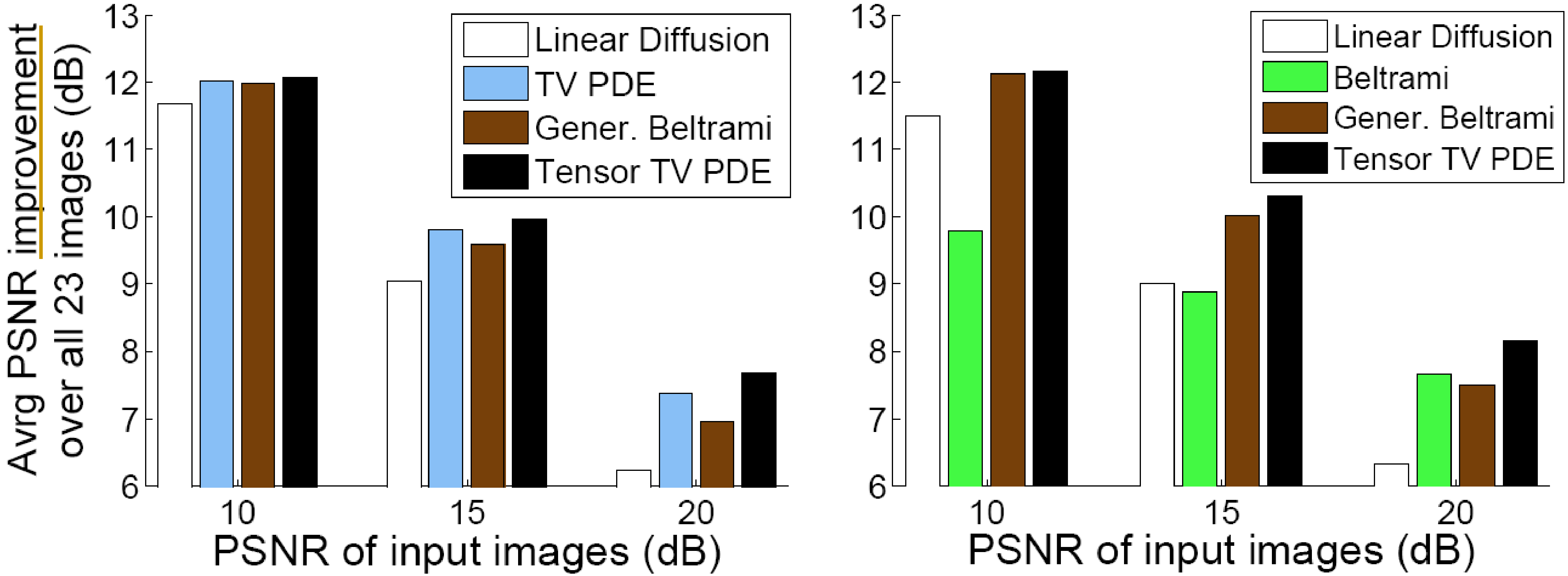
- This framework has been repeated for reference images from a dataset of *CIPR*: www.cipr.rpi.edu/resource/stills/kodak.html

23 natural images of size 768 x 512 pixels

…

- Both graylevel & color versions of images have been used

Denoising Experiments: Performance Measures



(a) Graylevel images (b) Color images

Summary & Conclusions

- We introduced a generic functional that
  - is based on the image structure tensor
  - generalizes Total Variation & Beltrami Functionals
- We proved that its minimization leads to a novel general type of anisotropic diffusion
- We proposed two novel anisotropic diffusion methods
- Several denoising experiments showed the potential of the novel approach

- The proposed framework opens various new directions for future research
  - Many other special cases of the generic functional might be promising
  - Thanks to the variational interpretation, such regularized tensor-based diffusions can be applied to other problems, e.g.:
    - image restoration, inpainting and interpolation

Thank You for Your Attention!

Questions?

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