Tutorial 5: Spline Curves and Surfaces

1. A four knot, two dimensional Bezier curve is defined by the following table

<table>
<thead>
<tr>
<th></th>
<th>(x, y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>P₀</td>
<td>(0, 0)</td>
</tr>
<tr>
<td>P₁</td>
<td>(2, 3)</td>
</tr>
<tr>
<td>P₂</td>
<td>(3, -1)</td>
</tr>
<tr>
<td>P₃</td>
<td>(0, 0)</td>
</tr>
</tbody>
</table>

a. Use de Casteljau's construction to sketch the curve.

b. Calculate the coefficients \( a₀, a₁, a₂ \) and \( a₃ \) of the corresponding cubic spline patch:

\[
P(\mu) = a₃\mu³ + a₂\mu² + a₁\mu + a₀
\]

c. Differentiate the spline patch equation to find \( P'(\mu) \) and hence show that the gradient at \( P₃ \) is the same as the gradient of the line joining \( P₃ \) to \( P₂ \).

2. A Coons surface patch is to be drawn using the following array of points:

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-1)</td>
<td>(0, 0, 0)</td>
<td>(0, 10, 5)</td>
<td>(0, 20, 10)</td>
<td>(0, 30, 20)</td>
</tr>
<tr>
<td>(0)</td>
<td>(10, 0, 5)</td>
<td>(10, 10, 20)</td>
<td>(10, 25, 30)</td>
<td>(15, 35, 40)</td>
</tr>
<tr>
<td>(1)</td>
<td>(20, 0, 10)</td>
<td>(20, 12, 40)</td>
<td>(20, 30, 50)</td>
<td>(25, 40, 30)</td>
</tr>
<tr>
<td>(2)</td>
<td>(30, 0, 5)</td>
<td>(35, 15, 30)</td>
<td>(40, 35, 40)</td>
<td>(50, 50, 20)</td>
</tr>
</tbody>
</table>

We are interested in the patch constructed on the centre knots, \( P(0, 0), P(0, 1), P(1, 0) \) and \( P(1, 1) \).

a. Find the equations of the four cubic spline patches that bound the Coons Patch:

\[
P(\mu, 0), P(\mu, 1), P(0, \nu), P(1, \nu)
\]

These are each parametric cubic splines of the form:

\[
P = a₃\mu³ + a₂\mu² + a₁\mu + a₀ \quad \text{or} \quad P = a₃\nu³ + a₂\nu² + a₁\nu + a₀
\]

The parameters for either form can be found using:

\[
\begin{pmatrix}
  a₀ \\
  a₁ \\
  a₂ \\
  a₃
\end{pmatrix}
= \begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  -3 & -2 & 3 & -1 \\
  2 & 1 & -2 & 1
\end{pmatrix}
\begin{pmatrix}
  Pₙ \\
  Pₙ⁺₁ \\
  Pₙ⁺₂ \\
  Pₙ⁺₃
\end{pmatrix}
\]

b. Find the point at the centre of the patch using the equation:

\[
P(\mu, \nu) = P(\mu, 0)(1 - \nu) + P(\mu, 1)\nu + P(0, \nu)(1 - \mu) + P(1, \nu)\mu
\]

\[
- P(0, 0)(1 - \mu)(1 - \nu) - P(0, 1)(1 - \mu)\nu - P(1, 0)\mu(1 - \nu) - P(1, 1)\mu\nu
\]

NB: This numerical calculation is rather tedious unless you use a programmable calculator, spreadsheet or software such as MatLab (which is available on the lab machines).