Partially ordered set. A *partial ordering* is a relation $\sqsubseteq: L \times L \to \{\mathit{true}, \mathit{false}\}$ that is reflexive (i.e. $\forall l : l \sqsubseteq l$), transitive (i.e. $\forall l_1, l_2, l_3 : l_1 \sqsubseteq l_2 \wedge l_2 \sqsubseteq l_3 \Rightarrow l_1 \sqsubseteq l_3$), and anti-symmetric (i.e. $\forall l_1, l_2 : l_1 \sqsubseteq l_2 \wedge l_2 \sqsubseteq l_1 \Rightarrow l_1 = l_2$).

A *partially ordered set* $(L, \sqsubseteq)$ is a set L equipped with a partial ordering $\sqsubseteq$ (sometimes written $\sqsubseteq_L$). We shall write $l_2 \sqsupseteq l_1$ for $l_1 \sqsubseteq l_2$ and $l_1 \sqsubset l_2$ for $l_1 \sqsubseteq l_2 \wedge l_1 \neq l_2$.

Example: Integers. The intergers ordered in the usual way, i.e. for two integers i_1, i_2:

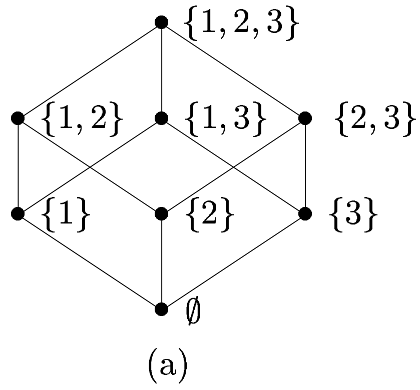
$$i_1 \sqsubseteq i_2 \text{ iff } i_1 \leq i_2$$

Example: Powerset. Take a (finite) set X and look at the set of all sub-sets of X, i.e. its *power set* $\mathcal{P}(X)$. A partial ordering on $\mathcal{P}(X)$ is given by *inclusion*, i.e. for two sub-sets $S_1, S_2 \in \mathcal{P}(X)$:

$$S_1 \sqsubseteq S_2 \text{ iff } S_1 \subseteq S_2$$

A subset Y of L has $l \in L$ as an *upper bound* if $\forall l' \in Y : l' \sqsubseteq l$ and as a *lower bound* if $\forall l' \in Y : l' \sqsupseteq l$. A *least upper bound* l of Y is an upper bound of Y that satisfies $l \sqsubseteq l_0$ whenever l_0 is another upper bound of Y; similarly, a *greatest lower bound* l of Y is a lower bound of Y that satisfies $l_0 \sqsubseteq l$ whenever l_0 is another lower bound of Y. Note that subsets Y of a partially ordered set L need not have least upper bounds nor greatest lower bounds but when they exist they are unique (since $\sqsubseteq$ is anti-symmetric) and they are denoted $\bigsqcup Y$ and $\bigsqcap Y$, respectively. Sometimes $\bigsqcup$ is called the *join operator* and $\bigsqcap$ the *meet operator* and we shall write $l_1 \sqcup l_2$ for $\bigsqcup\{l_1, l_2\}$ and similarly $l_1 \sqcap l_2$ for $\bigsqcap\{l_1, l_2\}$.

Complete lattice. A *complete lattice* $L = (L, \sqsubseteq) = (L, \sqsubseteq, \bigsqcup, \bigsqcap, \bot, \top)$ is a partially ordered set $(L, \sqsubseteq)$ such that all subsets have least upper bounds as well as greatest lower bounds. Furthermore, $\bot = \bigsqcup \emptyset = \bigsqcap L$ is the *least element* and $\top = \bigsqcap \emptyset = \bigsqcup L$ is the *greatest element.*



Example: Powerset. Take a (finite) set X and look again at its *power set* $\mathcal{P}(X)$. A partial ordering $\sqsubseteq$ on $\mathcal{P}(X)$ is given as above by *inclusion.* The *meet* and *join* operators are given by (set) *intersection*

$$S_1 \sqcap S_2 = S_1 \cap S_2$$

and (set) *union*

$$S_1 \sqcup S_2 = S_1 \cup S_2.$$

The least and greatest elements in $\mathcal{P}(X)$ are given by $\bot = \emptyset$ and $\top = X$.

Properties of functions. A function $f : L_1 \to L_2$ between partially ordered sets $L_1 = (L_1, \sqsubseteq_1)$ and $L_2 = (L_2, \sqsubseteq_2)$ is *surjective* (or *onto* or *epic*) if

$$\forall l_2 \in L_2 : \exists l_1 \in L_1 : f(l_1) = l_2$$

and it is *injective* (or 1–1 or *monic*) if

$$\forall l, l' \in L_1 : f(l) = f(l') \Rightarrow l = l'$$

The function f is *monotone* (or *isotone* or *order-preserving*) if

$$\forall l, l' \in L_1 : l \sqsubseteq_1 l' \quad \Rightarrow \quad f(l) \sqsubseteq_2 f(l')$$

It is an *additive* function (or a *join morphism*, sometimes called a *distributive* function) if

$$\forall l_1, l_2 \in L_1 : f(l_1 \sqcup l_2) = f(l_1) \sqcup f(l_2)$$

and it is called a *multiplicative* function (or a *meet morphism*) if

$$\forall l_1, l_2 \in L_1 : f(l_1 \sqcap l_2) = f(l_1) \sqcap f(l_2)$$

The function f is a *completely additive* function (or a *complete join morphism*) if for all $Y \subseteq L_1$:

$$f(\bigsqcup_1 Y) = \bigsqcup_2 \{f(l') \mid l' \in Y\} \text{ whenever } \bigsqcup_1 Y \text{ exists}$$

and it is *completely multiplicative* (or a *complete meet morphism*) if for all $Y \subseteq L_1$:

$$f(\bigsqcap_1 Y) = \bigsqcap_2 \{f(l') \mid l' \in Y\} \text{ whenever } \bigsqcap_1 Y \text{ exists}$$

The function f is *affine* if for all *non-empty* $Y \subseteq L_1$

$$f(\bigsqcup_1 Y) = \bigsqcup_2 \{f(l') \mid l' \in Y\} \text{ whenever } \bigsqcup_1 Y \text{ exists (and } Y \neq \emptyset)$$

and it is *strict* if $f(\bot_1) = \bot_2$; note that a function is completely additive if and only if it is both affine and strict.

Cartesian product. Let $L_1 = (L_1, \sqsubseteq_1)$ and $L_2 = (L_2, \sqsubseteq_2)$ be partially ordered sets. Define $L = (L, \sqsubseteq)$ by

$$L = \{(l_1, l_2) \mid l_1 \in L_1 \wedge l_2 \in L_2\}$$

and

$$(l_{11}, l_{21}) \sqsubseteq (l_{12}, l_{22}) \text{ iff } l_{11} \sqsubseteq_1 l_{12} \wedge l_{21} \sqsubseteq_2 l_{22}$$

If additionally each $L_i = (L_i, \sqsubseteq_i, \bigsqcup_i, \bigsqcap_i, \bot_i, \top_i)$ is a complete lattice then so is $L = (L, \sqsubseteq, \bigsqcup, \bigsqcap, \bot, \top)$ and furthermore

$$\bigsqcup Y = (\bigsqcup_1 \{l_1 \mid \exists l_2 : (l_1, l_2) \in Y\} , \bigsqcup_2 \{l_2 \mid \exists l_1 : (l_1, l_2) \in Y\})$$

and $\bot = (\bot_1, \bot_2)$ and similarly for $\bigsqcap Y$ and $\top$. We often write $L_1 \times L_2$ for L and call it the *cartesian product* of L_1 and L_2.

Total function space. Let $L_1 = (L_1, \sqsubseteq_1)$ be a partially ordered set and let S be a set. Define $L = (L, \sqsubseteq)$ by

$$L = \{f : S \to L_1 \mid f \text{ is a total function}\}$$

and

$$f \sqsubseteq f' \text{ iff } \forall s \in S : f(s) \sqsubseteq_1 f'(s)$$

If additionally $L_1 = (L_1, \sqsubseteq_1, \bigsqcup_1, \bigsqcap_1, \bot_1, \top_1)$ is a complete lattice then so is $L = (L, \sqsubseteq, \bigsqcup, \bigsqcap, \bot, \top)$ and furthermore

$$\bigsqcup Y = \lambda s. \bigsqcup_1 \{f(s) \mid f \in Y\}$$

and $\bot = \lambda s.\bot_1$ and similarly for $\bigsqcap Y$ and $\top$. We often write $S \to L_1$ for L and call it the *total function space* from S to L_1.

Monotone function space. Again let $L_1 = (L_1, \sqsubseteq_1)$ and $L_2 = (L_2, \sqsubseteq_2)$ be partially ordered sets. Now define $L = (L, \sqsubseteq)$ by

$$L = \{f : L_1 \to L_2 \mid f \text{ is a monotone function}\}$$

and

$$f \sqsubseteq f' \text{ iff } \forall l_1 \in L_1 : f(l_1) \sqsubseteq_2 f'(l_1)$$

If additionally each $L_i = (L_i, \sqsubseteq_i, \bigsqcup_i, \bigsqcap_i, \bot_i, \top_i)$ is a complete lattice then so is $L = (L, \sqsubseteq, \bigsqcup, \bigsqcap, \bot, \top)$ and furthermore

$$\bigsqcup Y = \lambda l_1. \bigsqcup_2 \{f(l_1) \mid f \in Y\}$$

and $\bot = \lambda l_1.\bot_2$ and similarly for $\bigsqcap Y$ and $\top$. We often write $L_1 \to L_2$ for L and call it the *monotone function space* from L_1 to L_2.

Chains. A subset $Y \subseteq L$ of a partially ordered set $L = (L, \sqsubseteq)$ is a *chain* if

$$\forall l_1, l_2 \in Y : (l_1 \sqsubseteq l_2) \vee (l_2 \sqsubseteq l_1)$$

Thus a chain is a (possibly empty) subset of L that is totally ordered. We shall say that it is a *finite chain* if it is a finite subset of L.

A sequence $(l_n)_n = (l_n)_{n \in \mathbf{N}}$ of elements in L is an *ascending chain* if

$$n \leq m \Rightarrow l_n \sqsubseteq l_m$$

Writing $(l_n)_n$ also for $\{l_n \mid n \in \mathbf{N}\}$ it is clear that an ascending chain also is a chain. Similarly, a sequence $(l_n)_n$ is a *descending chain* if

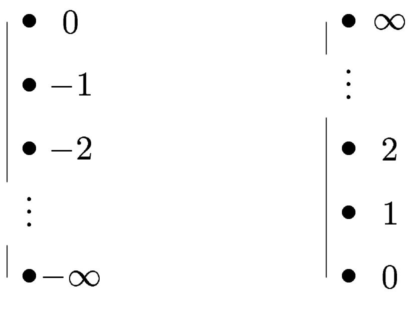
$$n \leq m \Rightarrow l_n \sqsupseteq l_m$$

We shall say that a sequence $(l_n)_n$ *eventually stabilises* if and only if

$$\exists n_0 \in \mathbf{N} : \forall n \in \mathbf{N} : n \geq n_0 \Rightarrow l_n = l_{n_0}$$

For the sequence $(l_n)_n$ we write $\bigsqcup_n l_n$ for $\bigsqcup\{l_n \mid n \in \mathbf{N}\}$ and similarly we write $\bigsqcap_n l_n$ for $\bigsqcap\{l_n \mid n \in \mathbf{N}\}$.

Ascending Chain and Descending Chain Conditions. We shall say that a partially ordered set $L = (L, \sqsubseteq)$ has finite *height* if and only if all chains are finite. It has finite height *at most* h if all chains contain at most $h + 1$ elements; it has finite height h if additionally there is a chain with $h + 1$ elements. The partially ordered set L satisfies the *Ascending Chain Condition* if and only if all ascending chains eventually stabilise.



(a) (b)

Reductive and extensive functions. Consider a monotone function $f : L \to L$ on a complete lattice $L = (L, \sqsubseteq, \bigsqcup, \bigsqcap, \bot, \top)$. A *fixed point* of f is an element $l \in L$ such that $f(l) = l$ and we write

$$\mathit{Fix}(f) = \{l \mid f(l) = l\}$$

for the set of fixed points. The function f is *reductive at* l if and only if $f(l) \sqsubseteq l$ and we write

$$\mathit{Red}(f) = \{l \mid f(l) \sqsubseteq l\}$$

for the set of elements upon which f is reductive; we shall say that f itself is *reductive* if $\mathit{Red}(f) = L$. Similarly, the function f is *extensive at* l if and only if $f(l) \sqsupseteq l$ and we write

$$\mathit{Ext}(f) = \{l \mid f(l) \sqsupseteq l\}$$

Since L is a complete lattice it is always the case that the set $\mathit{Fix}(f)$ will have a greatest lower bound in L and we denote it by $\mathit{lfp}(f)$:

$$\mathit{lfp}(f) = \bigsqcap \mathit{Fix}(f)$$

Similarly, the set $\mathit{Fix}(f)$ will have a least upper bound in L and we denote it by $\mathit{gfp}(f)$:

$$\mathit{gfp}(f) = \bigsqcup \mathit{Fix}(f)$$

If L satisfies the Ascending Chain Condition then there exists n such that $f^n(\bot) = f^{n+1}(\bot)$ and hence $\mathit{lfp}(f) = f^n(\bot)$. (Indeed any monotone function f over a partially ordered set satisfying the Ascending Chain Condition is also continuous.) Similarly, if L satisfies the Descending Chain Condition then there exists n such that $f^n(\top) = f^{n+1}(\top)$ and hence $\mathit{gfp}(f) = f^n(\top)$.

Fixed points and solutions. Given some equation(s) over some domain, e.g.

$$6x^3 - 3x^2 - x = 7$$

look at it as a "*recursive*" equation:

$$6x^3 - 3x^2 - 7 = x$$

or simply:

$$f(x) = x.$$

If x therefore is a *fixed point* of f it is also a *solution* to the original equation.

Knaster-Tarski Fixpoint Theorem. Let L be a complete lattice and $f : L \mapsto L$ an order-preserving map. Then

$$\bigsqcup\{x \in L \mid x \sqsubseteq f(x)\} \in \mathit{Fix}(f).$$

B.A. Davey and H.A. Priestley: *Introduction to Lattices and Order*, Cambridge 1990.