Tutorial 1: Analysis of three dimensional space.

Solutions

Q1. The three points $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$ can be used to generate two direction vectors in a number of different ways. For example, we could take

$\mathbf{P}_2 - \mathbf{P}_1 = (5, -10, 5)$ $\qquad$ $\mathbf{P}_3 - \mathbf{P}_1 = (15, 0, 5)$

to give the required direction vectors. Scaling a vector by a constant does not affect its direction so we can divide these direction vectors by 2 and 3 respectively to obtain the equivalent direction vectors:

(1, -2, 1) $\qquad$ (3, 0, 1)

We can find the cross product to obtain the normal vector:

$$(1,-2,1)\times(3,0,1) = (-2,2,6)$$

Scaling by a half gives the simpler normal vector (-1, 1, 3).

N.B. The rule for obtaining the cross product of two vectors can be represented in a number of different ways. For example

$$(a_1, a_2, a_3)\times(b_1, b_2, b_3) = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$$

The same rule expressed using **i**, **j**, **k** notation is:

$$(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\times(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) = a_2b_3 - a_3b_2\mathbf{i} + a_3b_1 - a_1b_3\mathbf{j} + a_1b_2 - a_2b_1\mathbf{k}$$

If you are familiar with matrices, the cross product can be represented as the determinant of a matrix:

$$(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\times(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

Q2. We have a general point in the plane, **P**, and a point $\mathbf{P}_1$ known to be in the plane from part 1:

$\mathbf{P} = (x, y, z)$ $\mathbf{P}_1 = (10, 20, 5)$

The difference between these vectors gives a vector parallel to the plane:

$$\mathbf{P} - \mathbf{P}_1 = (x - 10, y - 20, z - 5)$$

This vector is therefore perpendicular to the plane normal **n** = (-1, 1, 3), so we have:

$$\mathbf{n} \cdot (\mathbf{P} - \mathbf{P}_1) = 0$$

Which can be used to find the Cartesian plane equation as follows:

$$\begin{aligned} & \mathbf{n} \cdot (\mathbf{P} - \mathbf{P}_1) = 0 \\ \Rightarrow \quad & (-1,1,3) \cdot (x - 10, y - 20, z - 5) = 0 \\ \Rightarrow \quad & -x + 10 + y - 20 + 3(z - 5) = 0 \\ \Rightarrow \quad & -x + y + 3z - 25 = 0 \end{aligned}$$

The same equation can be obtained using $\mathbf{P}_2$ instead:

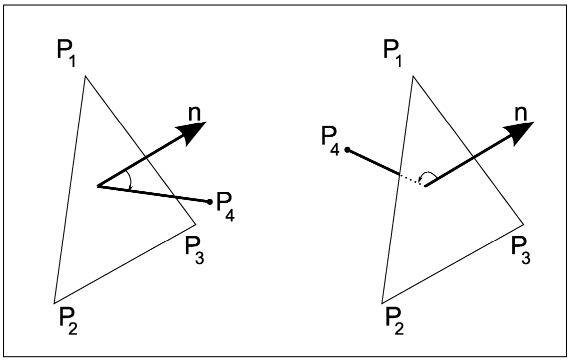
$$\mathbf{P} - \mathbf{P}_2 = (x - 15, y - 10, z - 10)$$

$$\begin{aligned} & \mathbf{n} \cdot (\mathbf{P} - \mathbf{P}_2) = 0 \\ \Rightarrow \quad & (-1,1,3) \cdot (x - 15, y - 10, z - 10) = 0 \\ \Rightarrow \quad & -x + 15 + y - 10 + 3(z - 10) = 0 \\ \Rightarrow \quad & -x + y + 3z - 25 = 0 \end{aligned}$$

Q3. The following pseudocode gives an example of how the coefficients of a Cartesian plane equation might be obtained using three given points in a plane:

```
TYPE Vector = Array [0..2] of REAL;

PROCEDURE PlaneEquation(P1,P2,P3: Vector;VAR a,b,c,d: REAL);

   VAR d1,d2: Vector;

   (* Find two vectors parallel to the plane *)
   FOR j:0 .. 2
      d1[j] = P2[j]-P1[j];
      d2[j] = P3[j]-P1[j];
   END FOR

   (* Find the normal vector to the plane n = [a,b,c] = d1 X d2 *)
   a := d1[1]*d2[2] - d1[2]*d2[1]
   b := d1[2]*d2[0] - d1[0]*d2[2]
   c := d1[0]*d2[1] - d1[1]*d2[0]
   (* take the dot product with P-P1 *)
   d := -(a*P1[0]+b*P1[1]+c*P1[2])

END PlaneEquation;
```

Q4. Let **n** be a surface normal to the tetrahedron. If it is an inner normal, it points in the same direction as a vector from the face defined by $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ to the fourth point $\mathbf{P}_4$. This is illustrated on the left in the following diagram:

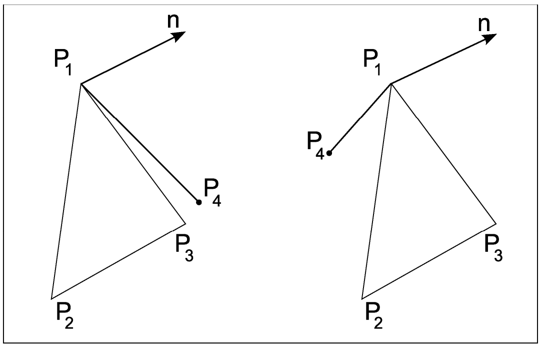


The right hand side illustrates the case where **n** is *not* an inner surface normal, i.e. an outward surface normal. From these diagrams, we can see that, wherever **n** is placed on the face, the angle it makes with the vector to $\mathbf{P}_4$ will be acute when **n** is an inner surface normal and obtuse otherwise. Using the fact that the dot product of two vectors **a**, **b** can be expressed in terms of the angle θ between them:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

if $\theta < 90$, we have $\cos\theta > 0$ and hence $\mathbf{a} \cdot \mathbf{b} > 0$, otherwise we have $\mathbf{a} \cdot \mathbf{b} < 0$ when θ is obtuse.

It does not matter where **n** is placed in the plane, so we can place it at $\mathbf{P}_1$, the two cases can therefore be illustrated as follows:



Take the dot product of the normal vector and the vector from $\mathbf{P}_1$ to $\mathbf{P}_4$, i.e.:

$$\mathbf{n} = (-1,1,3)$$
$$\mathbf{P}_4 - \mathbf{P}_1 = (20,0,5)$$
$$\mathbf{n} \cdot (\mathbf{P}_4 - \mathbf{P}_1) = -20 + 15 = -5$$

Since the result is negative the angle between these two vectors is bigger than 90, and so the normal vector must be the outward surface normal. An inner surface normal can be obtained by negating the outer normal, i.e. the inner surface normal can be given by (1, -1, -3).

Q5. In the parametric plane equation, we have a starting point, which we can choose as $\mathbf{P}_1$, two parameters μ, ν and two direction vectors $\mathbf{d}_1$, $\mathbf{d}_2$ that are parallel to the plane. The direction vectors might be chosen as follows for example:

$$\mathbf{d}_1 = \mathbf{P}_2 - \mathbf{P}_1$$
$$\mathbf{d}_2 = \mathbf{P}_3 - \mathbf{P}_1$$

Putting it all together, a general point $\mathbf{P}$ can be expressed by the parametric plane equation:

$$\mathbf{P} = \mathbf{P}_1 + \mu\,\mathbf{d}_1 + \nu\,\mathbf{d}_2$$

We can take the dot product of both sides with the normal vector $\mathbf{n}$:

$$\mathbf{P} \cdot \mathbf{n} = \mathbf{P}_1 \cdot \mathbf{n} + \mu\,\mathbf{d}_1 \cdot \mathbf{n} + \nu\,\mathbf{d}_2 \cdot \mathbf{n}$$

Because the vectors $\mathbf{d}_1$, $\mathbf{d}_2$ are parallel to the plane, they are perpendicular to the normal vector by definition. Their dot product with $\mathbf{n}$ is therefore zero which means that the last two terms on the right hand side vanish. So we obtain:

$$\mathbf{P} \cdot \mathbf{n} = \mathbf{P}_1 \cdot \mathbf{n}$$
$$\Rightarrow \quad \mathbf{P} \cdot \mathbf{n} - \mathbf{P}_1 \cdot \mathbf{n} = 0$$
$$\Rightarrow \quad (\mathbf{P} - \mathbf{P}_1) \cdot \mathbf{n} = 0$$

This demonstrates that the parametric plane equation is equivalent to the vector plane equation given in part 2.