Tutorial 4: Shading

Solutions

Q1. The normal to the triangle can be found by calculating the cross product of two (non-parallel) vectors in the same plane as the triangle.

First we find two vectors on the plane:

$$\mathbf{P}_2 - \mathbf{P}_1 = \begin{pmatrix} 15 \\ 25 \\ 25 \end{pmatrix} - \begin{pmatrix} -10 \\ 20 \\ 30 \end{pmatrix} = \begin{pmatrix} 25 \\ 5 \\ -5 \end{pmatrix} \qquad \mathbf{P}_2 - \mathbf{P}_3 = \begin{pmatrix} 15 \\ 25 \\ 25 \end{pmatrix} - \begin{pmatrix} 5 \\ -20 \\ 50 \end{pmatrix} = \begin{pmatrix} 10 \\ 45 \\ -25 \end{pmatrix}$$

Since we are not concerned with the magnitude of these vectors we can simplify the arithmetic by dividing both vectors by 5.

$$\frac{1}{5}\begin{pmatrix} 25 \\ 5 \\ -5 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix} \qquad \frac{1}{5}\begin{pmatrix} 10 \\ 45 \\ -25 \end{pmatrix} = \begin{pmatrix} 2 \\ 9 \\ -5 \end{pmatrix}$$

We can now find the normal **n** from the cross product:

$$(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

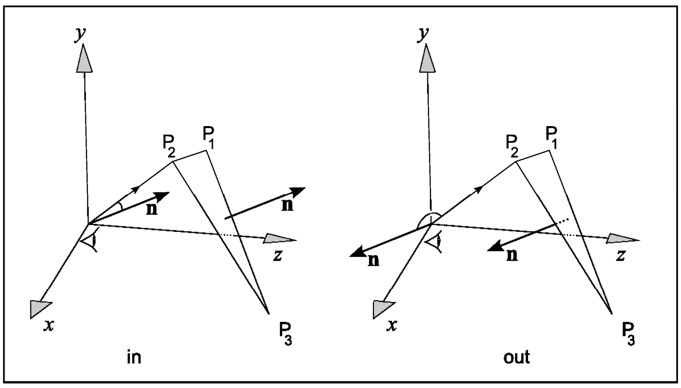
$$\text{So} \quad \mathbf{n} = \begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 9 \\ -5 \end{pmatrix} = \begin{pmatrix} 4 \\ 23 \\ 43 \end{pmatrix}$$

Now we need to decide whether the normal is an outer or an inner surface normal. Consider a vector from the origin to one of the vertices of the triangle. The angle between this vector and the normal can be used to determine whether the surface normal vector is inner or outer.

The two cases are illustrated below where the vertex $\mathbf{P}_2$ is used. For an inner surface normal, the angle is less than 90° and for an outer surface normal, it is greater than 90°.

$$\mathbf{n} \cdot \mathbf{OP}_2 = \begin{pmatrix} 4 \\ 23 \\ 43 \end{pmatrix} \cdot \begin{pmatrix} 15 \\ 25 \\ 25 \end{pmatrix} > 0$$

So the angle between them is $< 90^\circ$ and we have an inner normal as illustrated on the left. We can simply negate n to get an outer surface normal:



$$\begin{pmatrix} -4 \\ -23 \\ -43 \end{pmatrix}$$

Q2. Using Lambert's cosine law, the brightest point in the triangle will be where the angle between the surface normal and the light direction is closest to zero.

First we find the equation of a line that goes through the light source (at (-2, -40, -50)) and that is perpendicular to the plane containing the triangle. We can use one of the normal vectors found in the previous question to write the equation of the line in parametric form:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ -40 \\ -50 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 23 \\ 43 \end{pmatrix}$$

The equation of the plane containing the triangle can be written as:

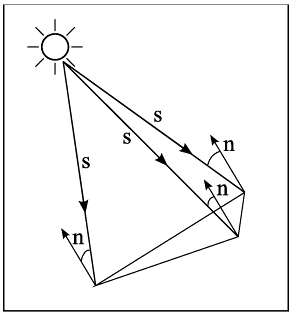
$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{OP}_1) = 0$$
$$\Rightarrow \mathbf{n} \cdot \mathbf{x} - \mathbf{n} \cdot \mathbf{OP}_1 = 0$$
$$\Rightarrow \begin{pmatrix} 4 \\ 23 \\ 43 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 4 \\ 23 \\ 43 \end{pmatrix} \cdot \begin{pmatrix} -10 \\ 20 \\ 30 \end{pmatrix} = 0$$
$$\Rightarrow 4x + 23y + 43z - 1710 = 0$$

We now find the intersection of the line and the plane. First find μ by substituting for x, y and z:

$$4(-2 + 4\mu) + 23(-40 + 23\mu) + 43(-50 + 43\mu) - 1710 = 0 \Rightarrow \mu = 2$$

Substituting back into the line equation gives the intersection as (6, 6, 36). This point can be shown to be inside the triangle, and therefore is the brightest point. If it were not inside the triangle it would be necessary to find the minimum distance of the point of intersection to the triangle. (quite a lot of calculation).

Q3. The brightest vertex for interpolation shading will be the one where the angle between the normal and the vector to the light source is smallest. This means we are looking for the vertex with the maximum value of $\dfrac{\mathbf{n} \cdot \mathbf{s}}{|\mathbf{n}||\mathbf{s}|}$ which is the cosine of this angle.



$|\mathbf{n}|$ is constant, so we need only find the point where $\dfrac{\mathbf{n} \cdot \mathbf{s}}{|\mathbf{s}|}$ is maximum.

Point	Coordinates	s	n.s	$\lvert\mathbf{s}\rvert$	$\dfrac{\mathbf{n} \cdot \mathbf{s}}{\lvert\mathbf{s}\rvert}$
$\mathbf{P}_1$	(-10, 20, 30)	(8, -60, -80)	4788	100.3	47.73
$\mathbf{P}_2$	(15, 25, 25)	(-17, -65, -75)	4788	101.7	47.55
$\mathbf{P}_3$	(5, -20, 50)	(-7, -20, -100)	4788	102.2	46.84

So point $\mathbf{P}_1$ will be the brightest. Just.

Q4. No, using the inverse square law will not change the result. Even if we use it, $\mathbf{P}_1$ will remain the brightest since it is the closest to the light source (see values of $|\mathbf{s}|$ above).

Q5. We have already found the normal to the triangle $\mathbf{P}_1\ \mathbf{P}_2\ \mathbf{P}_3$. We need to find the normal vectors to the other two triangles: $\mathbf{P}_1\ \mathbf{P}_2\ \mathbf{P}_4$ and $\mathbf{P}_1\ \mathbf{P}_3\ \mathbf{P}_4$.

For the triangle $\mathbf{P}_1\ \mathbf{P}_2\ \mathbf{P}_4$ we have a normal vector given by the cross product:

$$\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} -8 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -7 \\ 8 \end{pmatrix}$$

As in Q1, the difference vectors (e.g. $\mathbf{P}_4 - \mathbf{P}_1$) were scaled down to make the arithmetic easier.

We can show that this is an inner surface normal (See Q1 again) so we negate to get an outer surface normal $(-3, 7, -8)^T$.

For the triangle $\mathbf{P}_1\ \mathbf{P}_3\ \mathbf{P}_4$ we have a normal vector given by the cross product:

$$\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} -6 \\ 9 \\ -2 \end{pmatrix} = \begin{pmatrix} -20 \\ -18 \\ -21 \end{pmatrix}$$

which is an outer surface normal.

Now we have outer normal vectors for all three triangles adjacent to $\mathbf{P}_1$:

Triangle	$\mathbf{P}_1\ \mathbf{P}_2\ \mathbf{P}_3$	$\mathbf{P}_1\ \mathbf{P}_2\ \mathbf{P}_4$	$\mathbf{P}_1\ \mathbf{P}_3\ \mathbf{P}_4$
Normal	$\begin{pmatrix} -4 \\ -23 \\ -43 \end{pmatrix}$	$\begin{pmatrix} -3 \\ 7 \\ -8 \end{pmatrix}$	$\begin{pmatrix} -20 \\ -18 \\ -21 \end{pmatrix}$

These need to be converted to unit normal vectors before using in Gouraud or Phong shading. This gives the following:

Triangle	$\mathbf{P}_1\ \mathbf{P}_2\ \mathbf{P}_3$	$\mathbf{P}_1\ \mathbf{P}_2\ \mathbf{P}_4$	$\mathbf{P}_1\ \mathbf{P}_3\ \mathbf{P}_4$
Unit Normal	$\begin{pmatrix} -0.08 \\ -0.47 \\ -0.88 \end{pmatrix}$	$\begin{pmatrix} -0.27 \\ 0.63 \\ -0.72 \end{pmatrix}$	$\begin{pmatrix} -0.59 \\ -0.53 \\ -0.62 \end{pmatrix}$

To find the normal used by Gouraud and Phong shading we simply average these to get $(-0.31, -0.12, -0.74)^T$ and then normalise again to find the unit vector $(-0.38, -0.15, -0.91)^T$.