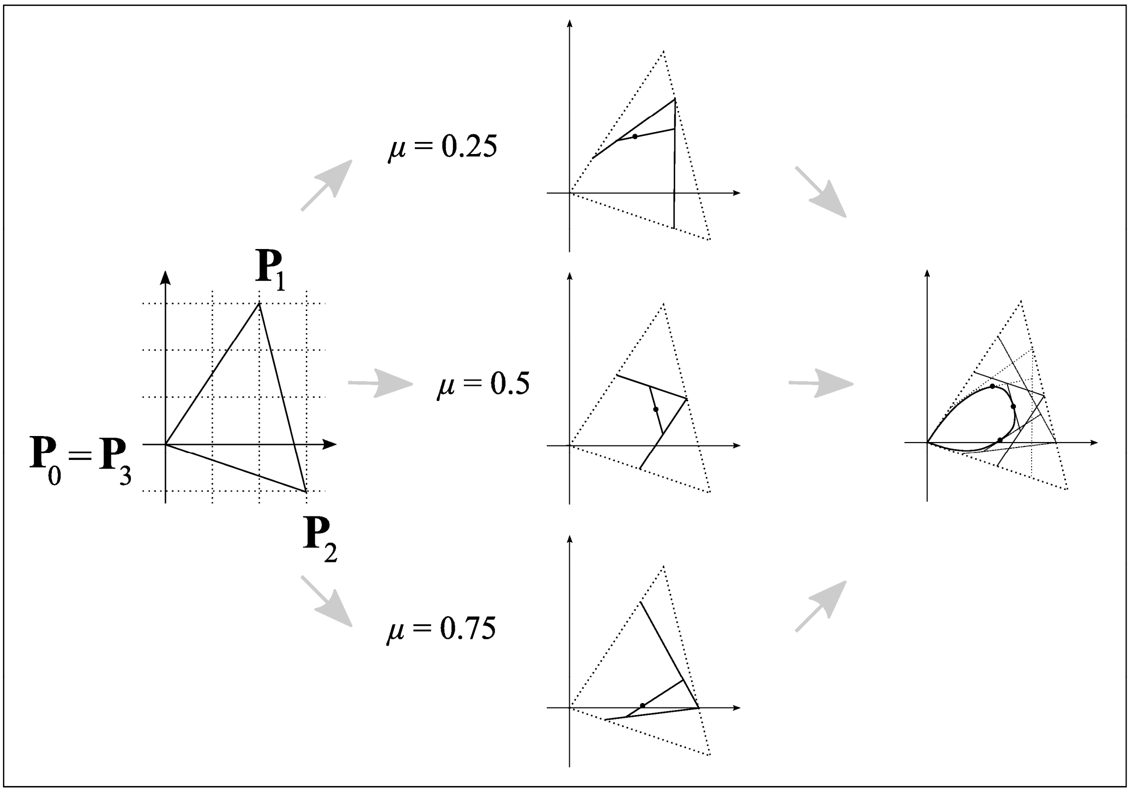
Tutorial 6: Spline Curves and Surfaces - Solutions

Q1.

a. The de Casteljau construction is illustrated below with choices for the parameter taken at 0.25, 0.5 and 0.75.



b. To find the patch coefficients, note that the Bezier Curve equation is

$$\mathbf{P}(\mu) = \mathbf{P}_0\,(1-\mu)^3 + 3\mathbf{P}_1\,\mu(1-\mu)^2 + 3\mathbf{P}_2\,\mu^2(1-\mu) + \mathbf{P}_3\,\mu^3$$

if we multiply out we get:

$$\mathbf{P}(\mu) = \mathbf{a}_3\mu^3 + \mathbf{a}_2\mu^2 + \mathbf{a}_1\mu + \mathbf{a}_0$$

where

$$\mathbf{a}_3 = \mathbf{P}_3 - 3\mathbf{P}_2 + 3\mathbf{P}_1 - \mathbf{P}_0$$
$$\mathbf{a}_2 = 3\mathbf{P}_2 - 6\mathbf{P}_1 + 3\mathbf{P}_0$$
$$\mathbf{a}_1 = 3\mathbf{P}_1 - 3\mathbf{P}_0$$
$$\mathbf{a}_0 = \mathbf{P}_0$$

We know the coordinates of $\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$ so we can substitute these in the above to obtain

$\mathbf{a_0}$ = (0, 0), $\mathbf{a_1}$ = (6, 9) $\mathbf{a_2}$ = (-3, -21) and $\mathbf{a_3}$ = (-3, 12)

c. Differentiating the spline gives

$$\mathbf{P}'(\mu) = 3\mathbf{a}_3\mu^2 + 2\mathbf{a}_2\mu + \mathbf{a}_1$$

and at $\mathbf{P}_3$ we have $\mu = 1$, therefore we have

$$\mathbf{P}'(1) = 3\mathbf{a}_3 + 2\mathbf{a}_2 + \mathbf{a}_1 = (-9, 3)$$

We also have

$$\mathbf{P}_3 - \mathbf{P}_2 = (-3, 1) \Rightarrow 3 \times (\mathbf{P}_3 - \mathbf{P}_2) = (-9, 3) = \mathbf{P}'(1)$$

So $\mathbf{P}'(1)$ is in the same direction as $\mathbf{P}_3 - \mathbf{P}_2$

Q2.

a. The array of points given in the question is

		μ			
		-1	0	1	2
ν	-1	(0, 0, 0)	(0, 10, 5)	(0, 20, 10)	(0, 30, 20)
	0	(10, 0, 5)	(10, 10, 20)	(10, 25, 30)	(15, 35, 40)
	1	(20, 0, 10)	(20, 12, 40)	(20, 30, 50)	(25, 40, 30)
	2	(30, 0, 5)	(35, 15, 30)	(40, 35, 40)	(50, 50, 20)

Considering $\mathbf{P}(\mu,0)$ for now, we need to identify the points $\mathbf{P}_i$ and $\mathbf{P}_{i+1}$ and the directions $\mathbf{P'}_i$ and $\mathbf{P'}_{i+1}$. Reading along the row $\nu = 0$, the points $\mathbf{P}_i$ and $\mathbf{P}_{i+1}$ can be taken directly:

$\mathbf{P}_i$ = (10, 10, 20) and $\mathbf{P}_{i+1}$ (10, 25, 30)

Using central differences, we can estimate $\mathbf{P'}_i$ and $\mathbf{P'}_{i+1}$ as follows:

$$\mathbf{P'}_i = \frac{1}{2}(\mathbf{P}_{i+1} - \mathbf{P}_{i-1}) \quad \text{and} \quad \mathbf{P'}_{i+1} = \frac{1}{2}(\mathbf{P}_{i+2} - \mathbf{P}_i)$$

$$\Rightarrow \mathbf{P'}_i = \frac{(10,25,30)-(10,0,5)}{2} = (0,12.5,12.5)$$

$$\text{and} \quad \mathbf{P'}_{i+1} = \frac{(15,35,40)-(10,10,20)}{2} = (2.5,12.5,10)$$

Repeating this process for the other bounding curves gives the following point and direction values

	$\mathbf{P}_i$	$\mathbf{P'}_i$	$\mathbf{P}_{i+1}$	$\mathbf{P'}_{i+1}$
$\mathbf{P}(\mu,0)$	(10, 10, 20)	(0, 12.5, 12.5)	(10, 25, 30)	(2.5, 12.5, 10)
$\mathbf{P}(\mu,1)$	(20, 12, 40)	(0, 15, 20)	(20, 30, 50)	(2.5, 14, -5)
$\mathbf{P}(0,\nu)$	(10, 10, 20)	(10, 1, 17.5)	(20, 12, 40)	(12.5, 2.5, 5)
$\mathbf{P}(1,\nu)$	(10, 25, 30)	(10, 5, 20)	(20, 30, 50)	(15, 5, 5)

Now we need to find the constant vectors $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$. Consider again, the bounding curve $\mathbf{P}(\mu,0)$. The constants for this curve can be found using the equation:

$$\begin{pmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & -2 & 3 & -1 \\ 2 & 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{P}_i \\ \mathbf{P'}_i \\ \mathbf{P}_{i+1} \\ \mathbf{P'}_{i+1} \end{pmatrix}$$

Where the expressions that appear as column vectors are in fact matrices obtained by writing in the row form expressions of $\mathbf{P}_i$, $\mathbf{P'}_i$, $\mathbf{P}_{i+1}$, $\mathbf{P'}_{i+1}$ and $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$. Using the information in the above table for $\mathbf{P}(\mu,0)$ gives

$$\begin{pmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & -2 & 3 & -1 \\ 2 & 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 10 & 10 & 20 \\ 0 & 12.5 & 12.5 \\ 10 & 25 & 30 \\ 2.5 & 12.5 & 10 \end{pmatrix} = \begin{pmatrix} 10 & 10 & 20 \\ 0 & 12.5 & 12.5 \\ -2.5 & 7.5 & -5 \\ 2.5 & -5 & 2.5 \end{pmatrix}$$

where the rows of the final matrix give the vectors $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$.

Contd.

Repeating this for all four bounding curves, we can obtain the constant vectors for each one. These are shown in the table below:

	$\mathbf{a}_0$	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$
$\mathbf{P}(\mu,0)$	(10, 10, 20)	(0, 12.5, 12.5)	(-2.5, 7.5, -5)	(2.5, -5, 2.5)
$\mathbf{P}(\mu,1)$	(20, 12, 40)	(0, 15, 20)	(-2.5, 10, -5)	(2.5, -7, -5)
$\mathbf{P}(0,\nu)$	(10, 10, 20)	(10, 1, 17.5)	(-2.5, 1.5, 20)	(2.5, -0.5, -17.5)
$\mathbf{P}(1,\nu)$	(10, 25, 30)	(10, 5, 20)	(-5, 0, 15)	(5, 0, -15)

b. At the mid-point $\mu = \nu = 1/2$, and each bounding curve evaluates to an expression of the form

$$\mathbf{P} = \frac{1}{8}\mathbf{a}_3 + \frac{1}{4}\mathbf{a}_2 + \frac{1}{2}\mathbf{a}_1 + \mathbf{a}_0$$

Where the values of $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ vary according to which bounding curve is chosen. Evaluating each bounding curve gives:

$$\mathbf{P}(0.5,0) = \frac{1}{8}(77.5, 140, 202)$$

$$\mathbf{P}(0.5,1) = \frac{1}{8}(157, 169, 385)$$

$$\mathbf{P}(0,0.5) = \frac{1}{8}(117.5, 86.5, 287.5)$$

$$\mathbf{P}(1,0.5) = \frac{1}{8}(115, 220, 335)$$

The formula

$$\begin{aligned}\mathbf{P}(\mu,\nu) \quad &= \quad \mathbf{P}(\mu,0)(1-\nu) + \mathbf{P}(\mu,1)\nu + \mathbf{P}(0,\nu)(1-\mu) + \mathbf{P}(1,\nu)\mu \\ &\quad - \mathbf{P}(0,0)(1-\mu)(1-\nu) - \mathbf{P}(0,1)(1-\mu)\nu - \mathbf{P}(1,0)\mu(1-\nu) - \mathbf{P}(1,1)\mu\nu\end{aligned}$$

then becomes

$$\begin{aligned}\mathbf{P}(0.5,0.5) \quad &= \quad 0.5 \times \left[\mathbf{P}(0.5,0) + \mathbf{P}(0.5,1) + \mathbf{P}(0,0.5) + \mathbf{P}(1,0.5)\right] \\ &\quad - 0.25 \times \left[\mathbf{P}(0,0) + \mathbf{P}(0,1) + \mathbf{P}(1,0) + \mathbf{P}(1,1)\right]\end{aligned}$$

$$\Rightarrow \mathbf{P}(0.5,0.5) \approx (29.22, 38.47, 75.63) - (15, 19.25, 35) \approx (14.2, 19.2, 40.6)$$