The Risks and Rewards of Conditioning Noncooperative Designs to Additional Information

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Abstract— A fundamental challenge in multiagent systems is to design local control algorithms to ensure a desirable collective behaviour. The information available to the agents, gathered either through communication or sensing, defines the structure of the admissible control laws and naturally restricts the achievable performance. Hence, it is fundamental to identify what piece of information can be used to produce a significant performance enhancement. This paper studies, within a class of resource allocation problems, the case when such information is uncertain or inaccessible and pinpoints a fundamental risk-reward tradeoff faced by the system designer.

I. Introduction

Several social and engineering systems can be thought of as a collection of multiple subsystems or agents, each taking local decisions in response to available information. A central goal in this field is to design control algorithms for the individual subsystems to ensure that the collective behaviour is desirable with respect to a global objective. Achieving this goal is particularly challenging because of the restriction on the information available to each agent and to the large scale of typical systems. Examples include, but are not limited to power grid networks [1], charging of electric vehicles [2], transportation network [3], task assignment problems [4], sensor allocation [5], robotic networks [6]. A considerable bulk of the research has focused on the design of local control algorithms in a framework where the information at agents' disposal is itself a fixed datum of the problem. A non exhaustive list includes [7], [8] and references therein. Understanding the impact of information availability on the achievable performances is a seemingly important but less tracked problem [9], [10], [11].

Of particular interest is to recognise what supplementary piece of information would coordinate agents to improve the system performance and how to incorporate this additional knowledge into a control algorithm. For example, [12] shows how giving to certain agents full knowledge of the constraint sets of some others, improves the system performance, relatively to a class of allocation problems. It is important to highlight that providing each agent with *all* the information available to the system is clearly beneficial, but

not necessarily desirable. Therefore, the previous question has to be understood within this context. Ideally, one is interested in a piece of information that gives a significant performance enhancement, and is simple to obtain.

Due to the large scale and spatial distribution of multi-agent systems, uncertainty plays an important role and needs to be managed while designing control algorithms. Power grid networks [1], demand-response methods [2] and transportation networks [3] are only a few examples that show the importance of modelling uncertainty in a distributed system. Following this observation, one would like to understand what is the risk associated with injecting an additional piece of *incorrect* information into previously designed control algorithms.

The paper proceeds by considering covering problems [13], [14], a class of resource allocation problems where agents are assigned to resources in order to maximise the total value of covered items. Examples include vehicle-target assignment problems [15], sensor allocation [5], task assignment [16], among others. Due to the inherent limitations in sensing and communication, in all these applications the control algorithms are required to rely only on local information. Thus, we model distributed covering problems as strategic-form games, where the system operator has the ability to assign local objective functions to each agent. As a matter of fact, Game Theory lends itself to analyse distributed systems where individual agents adjust their behaviour in response to partial information, as shown in [17], [9]. The overarching goal of the system operator is to *design* local utilities in order to render the equilibria of the game as efficient as possible. Agents can then be guided towards an equilibrium of such game by means of existing distributed algorithms [4], [18].

Within this framework, [10] shows that the maximum number of players that can simultaneously select a resource (cardinality) constitutes a valuable piece of information. More precisely, when the system operator is aware of the cardinality of the problem, he can tailor agents' utility functions to improve the overall performance. Nevertheless, the knowledge of the exact cardinality is in many applications not available or may require excessive communication to be determined. Building on this, we study the problem of optimally designing the utility functions in the case when the true cardinality is not known, but only an upper bound is available.[1] We further perform a risk-reward analysis in the case when the information on the cardinality of the game is uncertain. When the goal is to guard the system

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[1]A simple bound is given by the number of agents.

against the worst case performances, the right choice is to design the utilities as if the true cardinality was the given upper bound. Different designs will offer potential benefits, but come with a certain degree of risk. These results are presented in Theorem 2.

The remaining of the paper is organised as follows. The next section introduces the covering problem, its formulation as a strategic game and the metric used to measure the system-level performance. Section III studies the utility design problem when a sole upper bound on the cardinality is available and presents the risk-reward tradeoff associated with the use of uncertain information. Future directions and conclusions follow.

Notation

For any two positive integers $p \leq q$, denote $[p] = \{1, ..., p\}$ and $[p, q] = \{p, ..., q\}$; given $(a^1, \ldots, a^n)$, denote $a^{-i} = (a^1, \ldots, a^{i-1}, a^{i+1}, \ldots, a^n)$. We use $\mathbb{N}$ and $\mathbb{R}_{\geq 0}$ to denote the set of natural numbers (excluding zero) and the set of non-negative real numbers, respectively.

II. Distributed covering via game theory

In this section we present the covering problem and the associated covering game. We further define the performance metric used throughout the paper and recap previous results.

A. Model

Let us consider the problem of assigning a collection of agents $N = \{1, \ldots, n\}$ to a finite set of resources $\mathcal{R} = \{r_1, \ldots, r_m\}$ with the goal of maximising the value of covered resources. The feasible allocations for each agent $i \in N$ are the elements of the action set $\mathcal{A}^i \subseteq 2^{\mathcal{R}}$, while every resource $r \in \mathcal{R}$ is associated with a non-negative value $v_r \geq 0$. The welfare of an allocation $a = (a^1, \ldots, a^n) \in \mathcal{A}^1 \times \cdots \times \mathcal{A}^n$ is measured by the total value of covered resources

$$W(a) := \sum_{r \,:\, |a|_r \geq 1} v_r \,,$$

where $|a|_r$ denotes the number of agents that choose resource r in allocation a. The *covering problem* $C = \{N, \mathcal{R}, \{\mathcal{A}^i\}_{i \in N}, \{v_r\}_{r \in \mathcal{R}}\}$ consists in finding an optimal allocation, that is an assignment $a_o \in \arg\max_{a \in \mathcal{A}} W(a)$. We associate to every covering problem C its cardinality

$$k := \max_{\substack{r \in \mathcal{R} \\ a \in \mathcal{A}}} |a|_r \tag{1}$$

representing the maximum number of players that can concurrently select the same resource.

Instead of directly specifying a distributed algorithm, we shift the focus to the design of local utility functions for each agent, as proposed first in the framework of distributed welfare games by [19], [4] and successively by [10]. We consider utility functions of the form

$$u^i(a^i, a^{-i}) := \sum_{r \in a^i} v_r f(|a|_r), \qquad i \in N \,. \tag{2}$$

The function $f : [k] \to \mathbb{R}_{\geq 0}$ constitutes our design choice and is called *distribution rule* as it represents the fractional benefit an agent receive from each resource he selects.

The advantages of using utilities of the form (2) are twofold. First, $u^i(a^i, a^{-i})$ is local as it depends only on the resources agent i selects, their value and the number of agents that selects the same resources. Second, (2) allows to construct a distribution rule irrespective of $\{\mathcal{A}^i\}_{i \in N}$, so that the final design is scalable and applies to different choices of the action sets.

Given a covering problem C with cardinality k and a distribution rule $f : [k] \to \mathbb{R}_{\geq 0}$, we consider the associated *covering game* $G := \{C, f\} = \{N, \mathcal{R}, \{\mathcal{A}^i\}_{i \in N}, \{v_r\}_{r \in \mathcal{R}}, f\}$, where $\mathcal{A}^i$ is the set of feasible allocations and the utility of player $i \in N$ is as in equation (2). We *do not* aim at designing f using information on the specific instance of covering problem at hand, as such information is often not available to the system designer. We rather construct a distribution rule that behaves well for a large class of problems. Hence, we consider the set of covering problems for which the cardinality is smaller or equal to $k \in \mathbb{N}$ (with slight abuse of notation in the use of k). Given a distribution rule $f : [k] \to \mathbb{R}_{\geq 0}$, we define the set of associated games as

$$\mathcal{G}_f^k := \left\{ G \; : \; \max_{r \in \mathcal{R}, a \in \mathcal{A}} |a|_r \leq k \right\} .$$

Our objective is to design $f : [k] \to \mathbb{R}_{\geq 0}$ so that the efficiency of all the equilibria of games in the class $\mathcal{G}_f^k$ is as high as possible. Note that for fixed f, any game $G \in \mathcal{G}_f^k$ is potential [19]. Hence existence of equilibria is guaranteed and distributed algorithms, such as the best response scheme, converge to them [20]. Throughout the paper, we focus on pure Nash equilibria [21], which we will refer to in the following just as equilibria.

Definition 1 (Pure Nash equilibrium)**.** *Given a game G, an allocation $a_e \in \mathcal{A}$ is a pure Nash equilibrium iff $u^i(a_e^i, a_e^{-i}) \geq u^i(a^i, a_e^{-i})$ for all deviations $a^i \in \mathcal{A}^i$ and for all players $i \in N$. In the following we use $NE(G)$ to denote the set of Nash equilibria of G.*

For a given distribution rule, we evaluate the efficiency of the Nash equilibria of games in $\mathcal{G}_f^k$, adapting the concept of Price of Anarchy from [22] as

$$\mathrm{PoA}(\mathcal{G}_f^k) := \inf_{G \in \mathcal{G}_f^k} \left\{ \frac{\min_{a \in \mathrm{NE}(G)} W(a)}{W(a_o)} \right\} \leq 1 \,. \tag{3}$$

In essence, the quantity $\mathrm{PoA}(\mathcal{G}_f^k)$ bounds the inefficiency of all the equilibria over all games in $\mathcal{G}_f^k$. The higher the price of anarchy, the better the performance guarantees we can provide.

B. Related Work and Performance Guarantees

The problem of designing a distribution rule so as to maximise $\mathrm{PoA}(\mathcal{G}_f^k)$ has been studied in [10] and [12]. Both works impose a natural constraint on the admissible f, requiring $f(1) = 1$ and $f : [k] \to \mathbb{R}_{\geq 0}$ to be non-increasing. The optimal distribution rule is explicitly derived in the

former work, while the latter shows how $\text{PoA}(\mathcal{G}_f^k)$ is fully characterised by a single scalar quantity χ_f^k, measuring how fast the distribution rule f decreases. We intend to build upon these results, which are summarised in the following theorem.

Given k and a distribution rule f, we define χ_f^k as

$$\chi_f^k := \min_{x \geq 0} x$$
$$\text{s.t.} \quad jf(j) - f(j+1) \leq x \quad j \in [k-1], \quad (4)$$
$$(k-1)f(k) \leq x.$$

Theorem 1 ([10], [12]). *Consider a non-increasing distribution rule $f : [k] \to \mathbb{R}_{\geq 0}$, with $f(1) = 1$.*

i) *The price of anarchy over the class $\mathcal{G}_f^k$ is*

$$\text{PoA}(\mathcal{G}_f^k) = \frac{1}{1 + \chi_f^k}.$$

ii) *The price of anarchy over the class $\mathcal{G}_f^k$ is maximised for*

$$f_k^\star(j) = (j-1)! \frac{\frac{1}{(k-1)(k-1)!} + \sum_{i=j}^{k-1} \frac{1}{i!}}{\frac{1}{(k-1)(k-1)!} + \sum_{i=1}^{k-1} \frac{1}{i!}}, \quad (5)$$

with corresponding

$$\chi_{f_k^\star}^k = (k-1)f_k^\star(k). \quad (6)$$

iii) *The optimal price of anarchy is a decreasing function of the cardinality k*

$$\text{PoA}(\mathcal{G}_{f_k^\star}^k) = 1 - \frac{1}{\frac{1}{(k-1)(k-1)!} + \sum_{i=1}^{k-1} \frac{1}{i!}}.$$

III. THE CASE OF UNKNOWN CARDINALITY: A RISK-REWARD TRADEOFF

When the cardinality of the covering problem we intend to solve is known, Theorem 1 gives a conclusive answer on which distribution rule agents should choose to achieve the best possible approximation. In spite of that, the knowledge of the exact cardinality is in many applications not available or may require excessive communications between the agents to be determined. Observe that a universal upper bound for such quantity can be easily computed as the number n of agents. Potentially tighter bounds can be derived for specific applications.

Motivated by this observation, we study in the following the problem of designing a distribution rule when the true cardinality k is not known, but an upper bound $k \leq \bar{k}$ is available. Our objective is to design a distribution rule $f : [\bar{k}] \to \mathbb{R}_{\geq 0}$ with the best performance guarantees possible with the sole knowledge of $\bar{k}$. Two natural questions arise:

1) How should we select the distribution rule?
2) What performance can we guarantee?

We will show how selecting $f_{\bar{k}}^\star$ guards us against the worst case performance but will not guarantee the same efficiency of $f_k^\star$ in (5). We will then present the potential benefits and risks associated with a more aggressive choice.

A. A safe and a risky distribution

A natural choice when an upper bound on the cardinality is available consists in designing the distribution rule exactly at the upper bound. A different choice might entail designing the distribution rule as if the cardinality was lower than the upper bound, and then to optimally fill the tail. The latter suggestion is inspired by the observation that the price of anarchy is higher for lower caridinality k, when $f_k^\star$ is used. In the following we define and compare these distributions, called respectively safe and risky distribution rule.

The *safe* distribution rule, denoted with $\bar{f}$, is the distribution rule obtained as if the true cardinality was exactly $\bar{k}$, i.e.

$$\bar{f} := f_{\bar{k}}^\star, \quad (7)$$

where $f_{\bar{k}}^\star$ is defined in equation (5).

The *risky* distribution rule, denoted with $\hat{f}_p$ is a potentially *rewarding* alternative to $\bar{f}$. More precisely, $\hat{f}_p$ is a family of distributions parametrised by $p \in [\bar{k}]$. It is constructed as if the true cardinality was $p \leq \bar{k}$, that is fixing the first entries to $\hat{f}_p(j) = f_p^\star(j)$ for $j \in [p]$. The tail entries corresponding to $j \in [p+1, \bar{k}]$ are chosen to mitigate the risk taken. Formally, we define the family $\hat{f}_p$ as a solution of the following optimisation program

$$\hat{f}_p \in \arg\max_{f \in \mathcal{F}} \text{PoA}(\mathcal{G}_f^{\bar{k}})$$
$$\text{s.t.} \quad f(j) = f_p^\star(j) \quad \forall j \in [p], \quad (8)$$

where $\mathcal{F} = \{f : [\bar{k}] \to \mathbb{R}_{\geq 0} \text{ with } f(1) = 1, \ f(j+1) \leq f(j), \ \forall j \in [\bar{k}-1]\}$ is the set of admissible distributions and $p \in [\bar{k}]$. In essence, we are operating as if the true cardinality was $p \leq \bar{k}$ and using the tail entries of the distribution to balance the risk taken.

The next proposition characterises explicitly $\hat{f}_p$.

Proposition 1. *The family $\hat{f}_p$ takes the form*

$$\hat{f}_p(j) = \begin{cases} f_p^\star(j) & j \in [p] \\ \frac{(j-1)!}{(p-1)!} f_p^\star(p) - \chi_{\hat{f}_p}^{\bar{k}} \left(\sum_{h=1}^{j-1-p} \frac{(j-1)!}{(j-h-1)!} + 1 \right) & j \in [p+1, \bar{k}] \end{cases}$$
$$(9)$$

where $\chi_{\hat{f}_p}^{\bar{k}}$ defined in equation (4) is given by

$$\chi_{\hat{f}_p}^{\bar{k}} = \frac{(\bar{k}-1)(\bar{k}-1)!}{\bar{k} + (\bar{k}-1)\sum_{h=1}^{\bar{k}-1-p} \frac{(\bar{k}-1)!}{(\bar{k}-h-1)!}} \frac{f_p^\star(p)}{(p-1)!}, \quad (10)$$

The proof uses (4) and the result i) of Theorem 1. It is based on a relaxation argument and can be found in the appendix.

B. Performance comparison

In this section we compare the performance of the risky and safe distribution rule based on the metric introduced in (3). Theorem 2 constitutes the main result of this section.

Theorem 2. *Consider a covering game with cardinality k upper bounded by $\bar{k}$.*

i) *The safe distribution $\bar{f}$ has performance*

$$\text{PoA}(\mathcal{G}_{\bar{f}}^k) = \text{PoA}(\mathcal{G}_{\bar{f}}^{\bar{k}}) = \frac{1}{1 + \chi_{f_{\bar{k}}^\star}^{\bar{k}}}.$$

Such performance is strictly worse than the one achieved by the optimal distribution $f_k^\star$ if $k < \bar{k}$ and equal if $k = \bar{k}$.

ii) *For $p \in [k-1]$ the risky distribution $\hat{f}_p$ has performance*

$$PoA(\mathcal{G}_{\hat{f}_p}^k) = PoA(\mathcal{G}_{\hat{f}_p}^{\bar{k}}) = \frac{1}{1+\chi_{\hat{f}_p}^{\bar{k}}} \,.$$

Such performance is strictly worse than the one achieved by the safe distribution $\bar{f}$.

iii) *For $p \in [k, \bar{k}]$ the risky distribution $\hat{f}_p$ has performance*

$$PoA(\mathcal{G}_{\hat{f}_p}^k) = PoA(\mathcal{G}_{f_p^\star}^p) = \frac{1}{1+\chi_{f_p^\star}^p} \,.$$

Such performance is strictly better than the one achieved by the safe distribution $\bar{f}$ if $p \in [k, \bar{k}-1]$ and equal if $p = \bar{k}$.

The proof can be found in the Appendix.

Remark. Claim i) in Theorem 2 shows that the performance of the distribution $\bar{f}$ on the class of games with cardinality bounded by k is independent on k, for any $k \le \bar{k}$, such performance is governed by $PoA(\mathcal{G}_{\bar{f}}^{\bar{k}})$. Claims ii) and iii) in Theorem 2 certifies that the distribution $\hat{f}_p$ performs worse than $\bar{f}$ for $p < k$, and better for $p \ge k$.
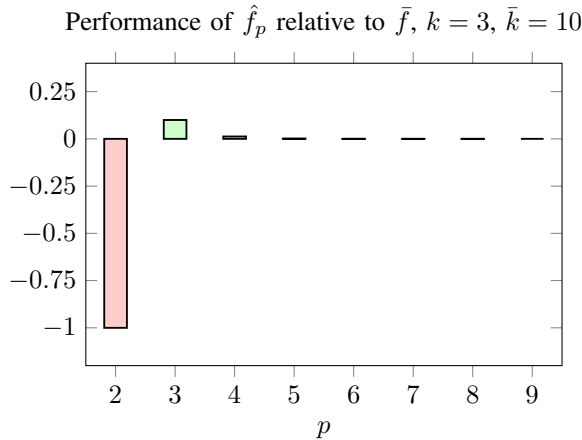

Performance of $\hat{f}_p$ relative to $\bar{f}$, $k = 3$, $\bar{k} = 10$

Fig. 1: The bars represent the difference $PoA(\mathcal{G}_{\hat{f}_p}^k) - PoA(\mathcal{G}_{\bar{f}}^k)$, normalized by its largest value. As such, it describes the normalized difference in performance between $\hat{f}_p$ and $\bar{f}$ for various value of $1 < p < \bar{k} = 10$.

In Figure 1 we compare the performance of $\hat{f}_p$ with the performance of $\bar{f}$. It is important to note that the performance degradation (incurred whenever $p < k$) *dominate* significantly the potential gains (achieved when $p \ge k$). A similar trend is obtained for any other $\bar{k}$. This motivates the interest in future work where we intend to understand if it is possible to dynamically adjust the distribution rule used to obtain the benefits of $\hat{f}$ at no risk.

IV. CONCLUSIONS AND FUTURE WORK

In this work we studied how additional information impacts the optimal design of local utility functions, when the goal is to improve the overall efficiency of the system. Focusing on covering problems, we studied the case when such additional information is uncertain or not available. Within this setup, we highlighted an inherent tradeoff between potential risks and rewards associated with committing to different distribution rules.

Theorem 2 has demonstrated how $\bar{f}$ guards against worst case performance while $\hat{f}_p$ could give potential benefits, but comes with a certain degree of risk. It is therefore interesting to understand, wether this tradeoff can be completely eradicated by allowing the distribution rule to be dynamically updated. In particular, when the cardinality is not available at all, can the agents *learn* it while simultaneously adjusting their behaviour to guarantee an improved performance?

APPENDIX I
PROOF OF PROPOSITION 1

Thanks to result i) in Theorem 1, $PoA(\mathcal{G}_f^{\bar{k}}) = \frac{1}{1+\chi_f^{\bar{k}}}$, with $\chi_f^{\bar{k}}$ implicitly defined by (4). Hence maximising $PoA(\mathcal{G}_f^{\bar{k}})$ is equivalent to minimising $\chi_f^{\bar{k}}$ and $\hat{f}_p$ can be computed by the following linear program (LP) in the unknowns x, $\{f(j)\}_{j=1}^{\bar{k}}$

$$\begin{aligned}
\min_{x \ge 0, \, f \in \mathcal{F}} \quad & x \\
\text{s.t.} \quad & jf(j) - f(j+1) \le x \quad j \in [\bar{k}-1], \\
& (\bar{k}-1)f(\bar{k}) \le x, \\
& f(j) = f_p^\star(j) \qquad\qquad j \in [p].
\end{aligned} \tag{11}$$

We remove the constraints $x \ge 0$, $f \in \mathcal{F}$ as well as $jf(j) - f(j+1) \le x$ for $j \in [p-1]$ and introduce the following relaxed linear program

$$\begin{aligned}
\min_{x, \, f} \quad & x \\
\text{s.t.} \quad & jf(j) - f(j+1) \le x \quad j \in [p, \bar{k}-1], \\
& (\bar{k}-1)f(\bar{k}) \le x, \\
& f(j) = f_p^\star(j) \qquad\qquad j \in [p].
\end{aligned} \tag{12}$$

The proof is divided in two subproofs:

i) We show that a solution to the relaxed program (12) is given by (9) and (10).

ii) We show that the solution to the relaxed program obtained in i) is feasible for the original problem too.

Proof. i) The proof proceeds by showing that a solution of (12) can be obtained transforming all the inequality constraint into equalities. This will produce the expressions (9) and (10).

Let us define $v_j = f(j)$ for $j \in [p+1, \bar{k}]$ and introduce the cost function $J(x, v_{p+1}, \ldots, v_{\bar{k}}) = x$. We further introduce the constraint functions $g_1(x, v_{p+1}) = -x - v_{p+1}$ and $g_i(x, v_{p+i-1}, v_{p+i}) = -x + j\, v_{p+i-1} - v_{p+i}$ for $i \in [2, \bar{k}-p]$ and $g_{\bar{k}-p+1}(x, v_{\bar{k}}) = -x + (\bar{k}-1)v_{\bar{k}}$. With these definitions the LP (12) is equivalent to the following where we have

removed the decision variables that are already determined

$$\min_{x, v_{p+1}, \ldots, v_{\bar{k}}} \quad J(x, v_{p+1}, \ldots, v_{\bar{k}}),$$

$$\text{s.t.} \quad g_1(x, v_{p+1}) \leq -p f_p^\star(p),$$
$$g_i(x, v_{p+i-1}, v_{p+i}) \leq 0 \qquad i \in [2, \bar{k} - p],$$
$$g_{\bar{k}-p+1}(x, v_{\bar{k}}) \leq 0.$$

Thanks to the convexity of the cost function and to the polytopic constraints, the Karush-Kuhn-Tucker conditions are necessary and sufficient for optimality [23]. Consequently, a feasible point $(x^\star, v_{k+1}^\star, \ldots, v_n^\star)$ is an optimiser iff there exists μ_i so that

$$\nabla J^\star + \sum_{i=1}^{\bar{k}-p+1} \mu_i \nabla g_i^\star = 0$$

$$g_i^\star \leq 0, \quad \mu_i \geq 0, \quad \mu_i g_i^\star = 0 \quad i \in [\bar{k} - p + 1]$$

where we used $\nabla J^\star$ to indicate $\nabla J(x^\star, v_{p+1}^\star, \ldots, v_{\bar{k}}^\star)$, and similarly for $g_i^\star$, $\nabla g_i^\star$. Observe that the distribution rule in (9) and the corresponding $\chi_{\hat{f}_p}^{\bar{k}}$ in (10) are the unique solution of the linear system $g_i^\star = 0$ for all $i \in [\bar{k} - p + 1]$, that is

$$\begin{cases} j\hat{f}_p(j) - \hat{f}_p(j+1) - \chi_f^{\bar{k}} = 0 & j \in [p, \bar{k}-1], \\ (\bar{k}-1)\hat{f}_p(\bar{k}) - \chi_f^{\bar{k}} = 0, & \\ \hat{f}_p(j) = f_p^\star(j) & j \in [p]. \end{cases} \quad (13)$$

Primal feasibility and complementarity slackness are hence naturally satisfied. We are only left to prove that there exists $\mu_i \geq 0$ such that $\nabla J^\star + \sum_{i=1}^{\bar{k}-p+1} \mu_i \nabla g_i^\star = 0$. We proceed by writing the stationarity conditions explicitly and show that this is indeed the case. Note that both the cost function and the constraints are linear so that their derivatives are constant functions

$$\nabla J^\star = (1, 0, \ldots, 0)$$
$$\nabla g_1^\star = (-1, -1, 0, \ldots, 0)$$
$$\nabla g_2^\star = (-1, p+1, -1, 0, \ldots, 0)$$
$$\vdots$$
$$\nabla g_{\bar{k}-p-1}^\star = (-1, 0, \ldots, 0, \bar{k}-2, -1, 0)$$
$$\nabla g_{\bar{k}-p}^\star = (-1, 0, \ldots, 0, \bar{k}-1, -1)$$
$$\nabla g_{\bar{k}-p+1}^\star = (-1, 0, \ldots, 0, \bar{k}-1)$$

Solving the stationarity condition in a recursive fashion starting from last component gives

$$\begin{cases} \mu_i = \mu_{\bar{k}-p+1} \frac{(\bar{k}-1)(\bar{k}-1)!}{(p+i-1)!} & i \in [\bar{k}-p] \\ \sum_{i=1}^{\bar{k}-p+1} \mu_i = 1. \end{cases}$$

Substituting the first equation into the second one and solving yields

$$\begin{cases} \mu_i = \left(\sum_{i=1}^{\bar{k}-p-1} \frac{(\bar{k}-1)(\bar{k}-1)!}{(p+i-1)!} \right)^{-1} \frac{(\bar{k}-1)(\bar{k}-1)!}{(p+i-1)!} & i \in [\bar{k}-p], \\ \mu_{\bar{k}-p+1} = \left(\sum_{i=1}^{\bar{k}-p-1} \frac{(\bar{k}-1)(\bar{k}-1)!}{(p+i-1)!} \right)^{-1}. \end{cases}$$

Since $\mu_i \geq 0$ for all $i \in [\bar{k} - p + 1]$, we conclude that (9) and (10) solve the relaxed program (12). $\square$

Proof. ii) The proof proceeds by showing that (9) and (10) satisfy the constraints removed when transforming the original program (11) into (12).

Using (10) and (9), it is trivial to verify that $\chi_{\hat{f}_p}^{\bar{k}} \geq 0$ and $\hat{f}_p(1) = 1$, $\hat{f}_p(j) \geq 0$. We proceed to prove that $\hat{f}_p$ is non increasing. Note that for $j \leq p$, $\hat{f}_p$ coincides with $f_p^\star$, which was proven to be non increasing in [10]. Further, from Lemma 1 we know that $j\hat{f}_p(j) - (j+1)\hat{f}_p(j+1) \geq 0$ for $j \in [p+1, \bar{k}-1]$. Thus

$$j\hat{f}_p(j) - j\hat{f}_p(j+1) \geq j\hat{f}_p(j) - (j+1)\hat{f}_p(j+1) \geq 0$$

for $j \in [p+1, \bar{k}-1]$, which guarantees that $\hat{f}_p$ is non increasing for $j \in [p, \bar{k}]$ too.

We are left to show that $j\hat{f}_p(j) - \hat{f}_p(j+1) \leq \chi_{\hat{f}_p}^{\bar{k}}$ for $j \in [p-1]$. Since $j \in [p-1]$, it holds that

$$j\hat{f}_p(j) - \hat{f}_p(j+1) = j f_p^\star(j) - f_p^\star(j+1).$$

Note that $j f_p^\star(j) - f_p^\star(j+1) \leq \chi_{f_p^\star}^p$ by definition of $\chi_{f_p^\star}^p$ in (4). Further, $\chi_{f_p^\star}^p \leq \chi_{f_{\bar{k}}^\star}^{\bar{k}}$ for any $p \leq \bar{k}$ since the price of anarchy is a monotonically decreasing function (Theorem 1). Finally, Lemma 2 shows that for any $p \leq \bar{k}$, it holds $\chi_{f_{\bar{k}}^\star}^{\bar{k}} \leq \chi_{\hat{f}_p}^{\bar{k}}$. Hence $j\hat{f}_p(j) - \hat{f}_p(j+1) \leq \chi_{\hat{f}_p}^{\bar{k}}$ for $j \in [p-1]$. It follows that $\hat{f}_p$ is feasible for the original problem (11).

Thanks to this, and to the fact that $\hat{f}_p$ is optimal for (12), we conclude that $\hat{f}_p$ is a solution of the original problem too. $\square$

PROOF OF THEOREM 2

Proof. i) Thanks to Theorem 1, the performance of $\bar{f}$ on the class of games with cardinality k can be computed as $\text{PoA}(\mathcal{G}_{\bar{f}}^k) = \frac{1}{1+\chi_{\bar{f}}^k}$. Recall from equation (7) that $\bar{f} = f_{\bar{k}}^\star$ and so $\text{PoA}(\mathcal{G}_{\bar{f}}^k) = \frac{1}{1+\chi_{f_{\bar{k}}^\star}^k}$. Since $k \leq \bar{k}$, we can apply part i) of Lemma 3 to $\chi_{f_{\bar{k}}^\star}^k$ and conclude

$$\text{PoA}(\mathcal{G}_{\bar{f}}^k) = \frac{1}{1+\chi_{f_{\bar{k}}^\star}^{\bar{k}}}.$$

This means that the performance of $\bar{f}$ on the set of games with cardinality k is the same performance of the distribution $f_{\bar{k}}^\star$ on the set of games with cardinality $\bar{k} \geq k$, and

$$\text{PoA}(\mathcal{G}_{\bar{f}}^k) = \text{PoA}(\mathcal{G}_{f_{\bar{k}}^\star}^{\bar{k}}) \leq \text{PoA}(\mathcal{G}_{f_k^\star}^k),$$

where the last inequality holds since $\text{PoA}(\mathcal{G}_{f_{\bar{k}}^\star}^k)$ is a decreasing function of k as seen in part iii) of Theorem 1. The inequality is tight if and only if $k = \bar{k}$.

ii) Thanks to Theorem 1, the performance of $\hat{f}_p$ on the class of games with cardinality k can be computed as $\text{PoA}(\mathcal{G}_{\hat{f}_p}^k) = \frac{1}{1+\chi_{\hat{f}_p}^k}$. Since $p \in [k-1]$, we apply part ii) of Lemma 3 to conclude that

$$\text{PoA}(\mathcal{G}_{\hat{f}_p}^k) = \frac{1}{1+\chi_{\hat{f}_p}^{\bar{k}}}.$$

Hence, for $p \in [\bar{k} - 1]$, the performance of $\hat{f}_p$ in the class of games with cardinality k is the same of the performance in the class of games with cardinality $\bar{k}$ i.e. $\mathrm{PoA}(\mathcal{G}^k_{\hat{f}_p}) = \mathrm{PoA}(\mathcal{G}^{\bar{k}}_{\hat{f}_p})$. Finally, by Lemma 2 we conclude that such performance is worse than what the safe distribution can offer

$$\mathrm{PoA}(\mathcal{G}^k_{\hat{f}_p}) = \frac{1}{1 + \chi^{\bar{k}}_{\hat{f}_p}} < \frac{1}{1 + \chi^{\bar{k}}_{f^\star_{\bar{k}}}} = \mathrm{PoA}(\mathcal{G}^k_{\hat{f}}).$$

iii) Since $k \leq p$, only the first k entries of $\hat{f}_p$ will determine the performance and these are identical to $f^\star_p$ by definition of $\hat{f}_p$. Hence $\mathrm{PoA}(\mathcal{G}^k_{\hat{f}_p}) = \mathrm{PoA}(\mathcal{G}^k_{f^\star_p}) = \frac{1}{1 + \chi^k_{f^\star_p}}$. Further $p \in [k, \bar{k}]$ and part i) of Lemma 3 applies

$$\mathrm{PoA}(\mathcal{G}^k_{\hat{f}_p}) = \frac{1}{1 + \chi^p_{f^\star_p}} = \mathrm{PoA}(\mathcal{G}^p_{f^\star_p}),$$

so that $\hat{f}_p$ has the same performance of $f^\star_p$. Using the fact that the optimal price of anarchy is a decreasing function, for any $p \in [k, \bar{k}]$ we get

$$\mathrm{PoA}(\mathcal{G}^k_{\hat{f}_p}) = \mathrm{PoA}(\mathcal{G}^p_{f^\star_p}) \geq \mathrm{PoA}(\mathcal{G}^{\bar{k}}_{f^\star_{\bar{k}}}) = \mathrm{PoA}(\mathcal{G}^k_{\hat{f}}).$$

The inequality is tight if and only if $p = \bar{k}$. $\qquad\square$

Lemma 1. *Let $p \in [k]$. The risky distribution $\hat{f}_p$ satisfies*

$$j\hat{f}_p(j) - (j+1)\hat{f}_p(j+1) \geq 0 \qquad j \in [p, \bar{k} - 1].$$

Proof. Recall that $\hat{f}_p$ is obtained from equation (13). Using $\chi^{\bar{k}}_{\hat{f}_p}$ from (10), one can reconstruct the tail entries of $\hat{f}_p(j)$ with the following backward recursion

$$j\hat{f}_p(j) - \hat{f}_p(j+1) = \chi^{\bar{k}}_{\hat{f}_p} \quad j \in [p, \bar{k} - 1],$$
$$(\bar{k} - 1)\hat{f}_p(\bar{k}) = \chi^{\bar{k}}_{\hat{f}_p}.$$

Starting from $\hat{f}_p(\bar{k}) = \frac{\chi^{\bar{k}}_{\hat{f}_p}}{\bar{k} - 1}$, the first equation gives for $j \geq p$

$$j\hat{f}_p(j) = \chi^{\bar{k}}_{\hat{f}_p}\left(1 + \sum_{i=j+1}^{\bar{k}-1} \frac{j!}{i!} + \frac{j!}{(\bar{k} - 1)(\bar{k} - 1)!}\right)$$

hence

$$\frac{1}{\chi^{\bar{k}}_{\hat{f}_p}}(j\hat{f}_p(j) - (j+1)\hat{f}_p(j+1)) =$$

$$= \sum_{i=j+1}^{\bar{k}-1} \frac{j!}{i!} - \sum_{i=j+2}^{\bar{k}-1} \frac{(j+1)!}{i!} + \frac{j! - (j+1)!}{(\bar{k} - 1)(\bar{k} - 1)!} =$$

$$= \sum_{i=j+1}^{\bar{k}-2} \left(\frac{j!}{i!} - \frac{(j+1)!}{(i+1)!}\right) + \frac{j!}{(\bar{k} - 1)!}\left(1 + \frac{1}{\bar{k} - 1} - \frac{j+1}{\bar{k} - 1}\right).$$

Note that for $i > j$, one has

$$\frac{j!}{i!} = \frac{1}{i(i-1)\dots(j+1)} \quad \text{and so} \quad \frac{j!}{i!} - \frac{(j+1)!}{(i+1)!} > 0.$$

Further

$$1 + \frac{1}{\bar{k} - 1} + \frac{j+1}{\bar{k} - 1} = \frac{\bar{k} - j - 1}{\bar{k} - 1} \geq 0$$

since $j \leq \bar{k} - 1$ by assumption. Hence we conclude that

$$j\hat{f}_p(j) - (j+1)\hat{f}_p(j+1) \geq 0 \qquad j \in [p, \bar{k} - 1].$$

$\qquad\square$

Lemma 2. *For any $1 < p < \bar{k}$ it holds $\chi^{\bar{k}}_{f^\star_{\bar{k}}} < \chi^{\bar{k}}_{\hat{f}_p}$. If $p = 1$ or $p = \bar{k}$, it holds $\chi^{\bar{k}}_{f^\star_{\bar{k}}} = \chi^{\bar{k}}_{\hat{f}_p}$.*

Proof. If $p = 1$ or $p = \bar{k}$, trivially $\hat{f}_p = f^\star_{\bar{k}}$ and so $\chi^{\bar{k}}_{f^\star_{\bar{k}}} = \chi^{\bar{k}}_{\hat{f}_p}$. We are left to show that $\chi^{\bar{k}}_{f^\star_{\bar{k}}} < \chi^{\bar{k}}_{\hat{f}_p}$ when $1 < p < \bar{k}$. After some algebraic manipulation on the expressions of $\chi^{\bar{k}}_{f^\star_{\bar{k}}}$ in equation (6) and of $\chi^{\bar{k}}_{\hat{f}_p}$ in equation (10), we get

$$\chi^{\bar{k}}_{f^\star_{\bar{k}}} = \frac{(\bar{k} - 1)(\bar{k} - 1)!}{1 + (\bar{k} - 1)(\bar{k} - 1)!\sum_{i=1}^{\bar{k}-1} \frac{1}{i!}},$$

$$\chi^{\bar{k}}_{\hat{f}_p} = \frac{(\bar{k} - 1)(\bar{k} - 1)!}{\bar{k} + \sum_{h=1}^{\bar{k}-1-p} \frac{(\bar{k}-1)(\bar{k}-1)!}{(\bar{k}-h-1)!}} \frac{1}{1 + \sum_{h=1}^{p-1} \frac{(p-1)(p-1)!}{h!}}.$$

Instead of proving that $\chi^{\bar{k}}_{f^\star_{\bar{k}}} < \chi^{\bar{k}}_{\hat{f}_p}$, in the following we equivalently show that $\frac{1}{\chi^{\bar{k}}_{f^\star_{\bar{k}}}} > \frac{1}{\chi^{\bar{k}}_{\hat{f}_p}}$ i.e., that

$$1 + (\bar{k} - 1)(\bar{k} - 1)!\sum_{i=1}^{\bar{k}-1} \frac{1}{i!} = \bar{k} + (\bar{k} - 1)(\bar{k} - 1)!\sum_{i=1}^{\bar{k}-2} \frac{1}{i!} >$$

$$\left(\bar{k} + \sum_{h=1}^{\bar{k}-1-p} \frac{(\bar{k}-1)(\bar{k}-1)!}{(\bar{k}-h-1)!}\right)\left(1 + \sum_{h=1}^{p-1} \frac{(p-1)(p-1)!}{h!}\right).$$

The previous inequality can be rewritten as

$$(\bar{k} - 1)(\bar{k} - 1)!\left(\sum_{i=1}^{\bar{k}-2} \frac{1}{i!} - \sum_{h=1}^{\bar{k}-1-p} \frac{1}{(\bar{k}-h-1)!}\right) >$$

$$\left(\bar{k} + \sum_{h=1}^{\bar{k}-1-p} \frac{(\bar{k}-1)(\bar{k}-1)!}{(\bar{k}-h-1)!}\right)\left(\sum_{h=1}^{p-1} \frac{(p-1)(p-1)!}{h!}\right).$$

Since the left hand side is equal to $(\bar{k} - 1)(\bar{k} - 1)!\sum_{h=1}^{p-1} \frac{1}{h!}$, we can simplify the term $\sum_{h=1}^{p-1} \frac{1}{h!}$ to get

$$(\bar{k} - 1)(\bar{k} - 1)! > (p - 1)(p - 1)!\left(\bar{k} + \sum_{h=1}^{\bar{k}-1-p} \frac{(\bar{k}-1)(\bar{k}-1)!}{(\bar{k}-h-1)!}\right),$$

which is finally equivalent to

$$\frac{1}{(p - 1)(p - 1)!} > \frac{\bar{k}}{(\bar{k} - 1)(\bar{k} - 1)} + \sum_{h=1}^{\bar{k}-1-p} \frac{1}{(\bar{k}-h-1)!}. \tag{14}$$

We use *induction* to show that inequality (14) holds for $1 < p < \bar{k}$, as required. We start from $p = \bar{k} - 1$ and apply induction backwards until we reach $p = 2$.

- For $p = \bar{k} - 1$ and $p > 1$ inequality (14) reads as

$$\frac{1}{(p - 1)(p - 1)!} > \frac{p + 1}{p \cdot p!} \iff p^2 > p^2 - 1,$$

which is always satisfied.

- Let us assume the inequality holds for a generic $p \leq \bar{k} - 1$, we show that it holds also for $p-1$ (with $p > 1$). That is, we assume

$$\frac{1}{(p-1)(p-1)!} > \frac{\bar{k}}{(\bar{k}-1)(\bar{k}-1)} + \sum_{h=1}^{\bar{k}-1-p} \frac{1}{(\bar{k}-h-1)!}, \tag{15}$$

and want to show

$$\frac{1}{(p-2)(p-2)!} > \frac{\bar{k}}{(\bar{k}-1)(\bar{k}-1)} + \sum_{h=1}^{\bar{k}-p} \frac{1}{(\bar{k}-h-1)!}. \tag{16}$$

We can rewrite the right hand side of (16) and use (15) to upper bound it

$$\frac{\bar{k}}{(\bar{k}-1)(\bar{k}-1)} + \sum_{h=1}^{\bar{k}-p} \frac{1}{(\bar{k}-h-1)!} =$$
$$\frac{\bar{k}}{(\bar{k}-1)(\bar{k}-1)} + \sum_{h=1}^{\bar{k}-p-1} \frac{1}{(\bar{k}-h-1)!} + \frac{1}{(p-1)!} <$$
$$\frac{1}{(p-1)(p-1)!} + \frac{1}{(p-1)!} = \frac{p}{(p-1)(p-1)!} <$$
$$\frac{1}{(p-2)(p-2)!}. \tag{17}$$

The last inequality holds since it is equivalent to

$$\frac{p}{(p-1)^2} < \frac{1}{p-2} \iff p^2 - 2p < (p-1)^2,$$

which is always satisfied. Comparing the first and last term in (17) gives (16).

This completes the induction and thus the entire proof. $\square$

Lemma 3. i) *For any $l \in [m]$, $m \in \mathbb{N}$ it holds*

$$\chi^l_{f^\star_m} = \chi^m_{f^\star_m}.$$

ii) *For any $k \in [\bar{k}]$ and $p \in [k-1]$ it holds*

$$\chi^k_{\hat{f}_p} = \chi^{\bar{k}}_{\hat{f}_p}.$$

Proof. i) If $l = m$, the result holds trivially. Hence in the following we consider $l \in [m-1]$. By definition of $\chi^l_{f^\star_m}$ in (4)

$$\chi^l_{f^\star_m} := \min_{x \geq 0} x$$
$$\text{s.t.} \quad j f^\star_m(j) - f^\star_m(j+1) \leq x \quad j \in [l-1],$$
$$(l-1)f^\star_m(l) \leq x.$$

Note that $f^\star_m$ is derived in [10, Theorem 2] solving the following recursion

$$j f^\star_m(j) - f^\star_m(j+1) = \chi^m_{f^\star_m} \quad j \in [m-1]$$
$$(l-1)f^\star_m(l) = \chi^m_{f^\star_m}. \tag{18}$$

Since $m > l$, it follows that any feasible x from the LP above has to satisfy $x \geq \chi^m_{f^\star_m}$. In the following we show that setting $x = \chi^m_{f^\star_m}$, the constraint $(l-1)f^\star_m(l) \leq x$ is also

satisfied. This will be enough to conclude that $\chi^l_{f^\star_m} = \chi^m_{f^\star_m}$. Since $f^\star_m$ is non increasing, one has

$$(l-1)f^\star_m(l) - \chi^m_{f^\star_m} = lf^\star_m(l) - f^\star_m(l) - \chi^m_{f^\star_m} \leq$$
$$\leq lf^\star_m(l) - f^\star_m(l+1) - \chi^m_{f^\star_m} = 0,$$

where the equality holds applying (18) for $j = l \in [m-1]$.

ii) We intend to compute

$$\chi^k_{\hat{f}_p} := \min_{x \geq 0} x$$
$$\text{s.t.} \quad j \hat{f}_p(j) - \hat{f}_p(j+1) \leq x \quad j \in [k-1]$$
$$(k-1)\hat{f}_p(k) \leq x.$$

As shown before it is clear that for any feasible x, it must be $x \geq \chi^{\bar{k}}_{\hat{f}_p}$ due to how $\hat{f}_p(j)$ is recursively defined for $j > p$ in equation (13). Similarly to what shown before one can prove that $x = \chi^{\bar{k}}_{\hat{f}_p}$ will also satisfy the constraint $(k-1)\hat{f}_p(k) \leq x$. Hence $\chi^k_{\hat{f}_p} = \chi^{\bar{k}}_{\hat{f}_p}$ and the proof is concluded. $\square$

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