Generalized coverage problems: approximation through game design

Dario Paccagnan

Joint work with J. R. Marden (UCSB)

Combinatorial resource allocation

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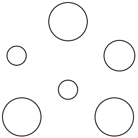

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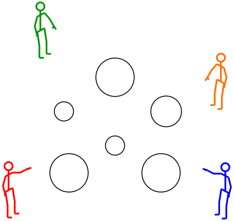

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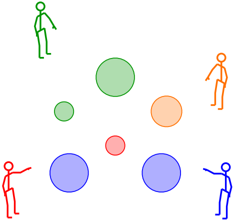

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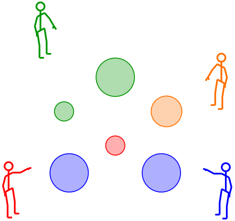

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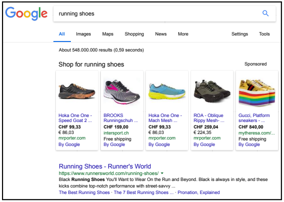

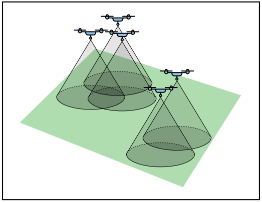

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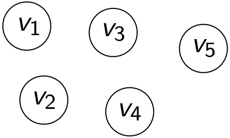

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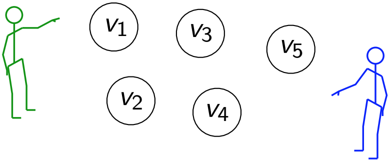

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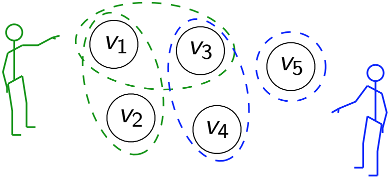

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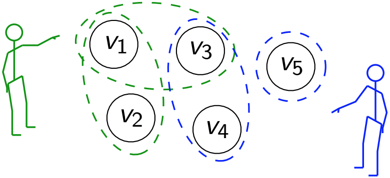

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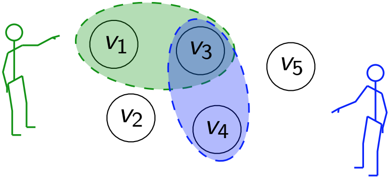

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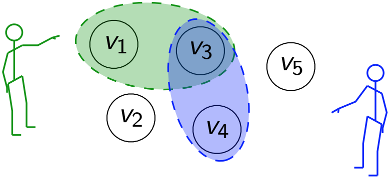

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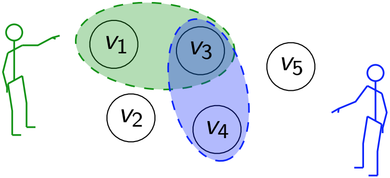

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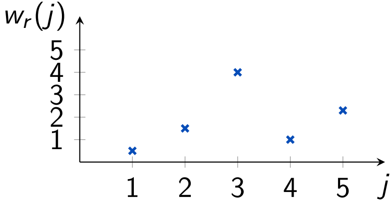

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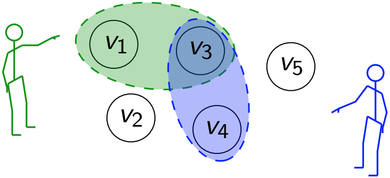

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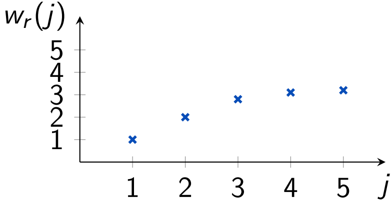

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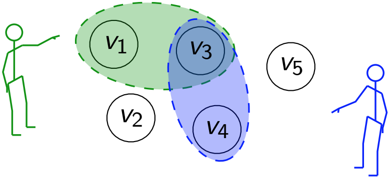

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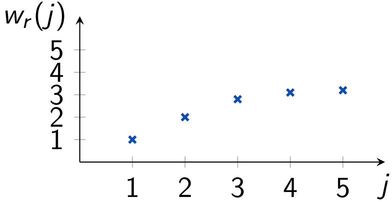
- ▷ to ease the presentation $w_r(j) = w(j)$



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[Che04] C. Chekuri et al. "Maximum Coverage Problem with Group Budget Constraints and Applications", APPROX 04

[Gair09] M. Gairing, "Covering Games: Approximation through Non-cooperation", WINE 09

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[Svir17] M. Sviridenko, J. Vondrák, J. Ward, "Optimal Approximation for Submodular and Supermodular Optimization with Bounded Curvature", MOR 17

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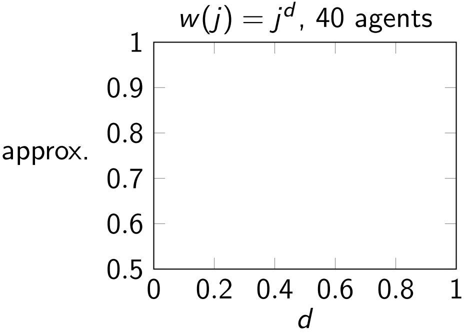

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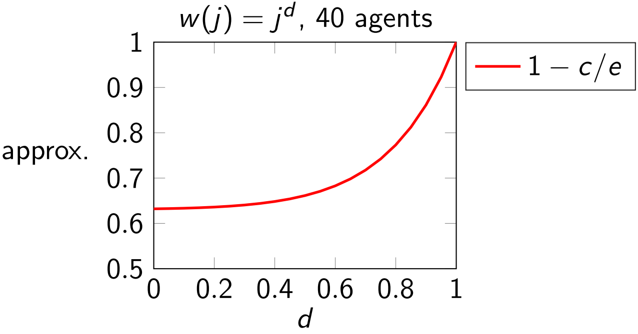

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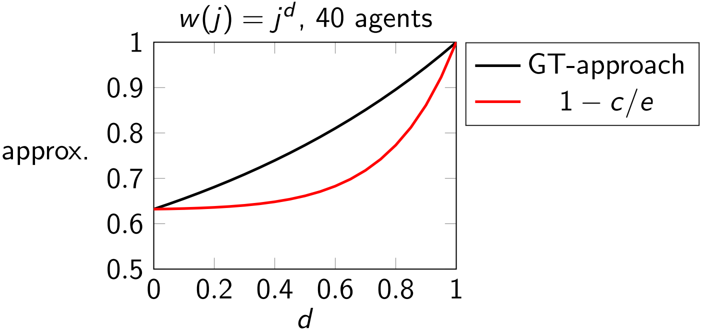

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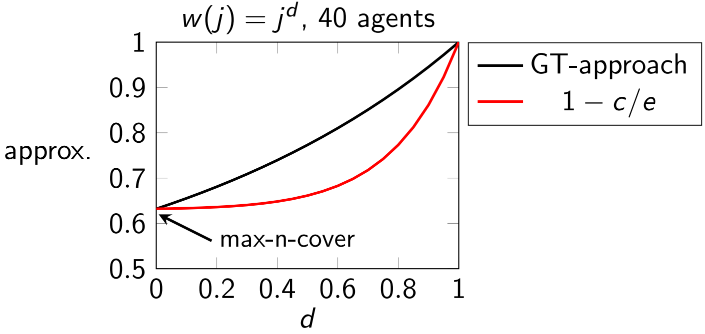

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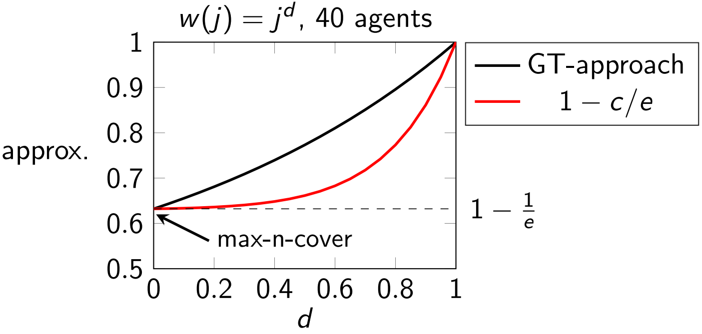

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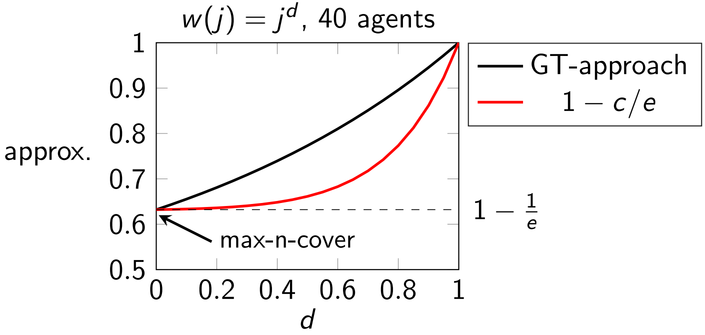

[Pac18a] DP, R.Chandan, J. Marden "Distributed resource allocation through utility design -Part I: optimizing the performance certificates via the price of anarchy", ArXiv 2018

[Pac18b] DP, J. Marden "- Part II: applications to submodular, supermodular and set covering problems", ArXiv 2018

Outline

The game-theoretic approach

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$$\max W(a)$$

- ▷ Distributed algorithm
- ▷ Good approximation
- ▷ Polytime

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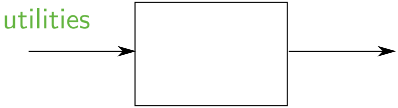

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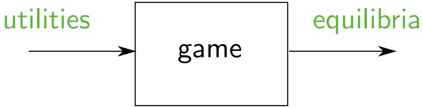

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Given instance I, fix f

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$$u_i(a_i, a_{-i}) = \sum_{r \in a_i} v_r w(|a|_r) f(|a|_r) \qquad f : \mathbb{N} \to \mathbb{R}_{\geq 0} \quad (\textit{distributed})$$

How to design f? Maximize worst-case performance

Given instance I, fix $f \to$ game $\mathrm{G}_f = \{\mathrm{I}, f\}$

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$\boxed{\mathrm{PoA}(f) \text{ is the approx. ratio of any equilibrium-computing algorithm}}$

Utility design reduces to

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$$\text{Utility design reduces to} \begin{cases} \boxed{\text{Given } f\text{, characterize or bound } \mathrm{PoA}(f)} \\ \text{Find } f \text{ with highest } \mathrm{PoA}(f) \end{cases}$$

Characterizing the price of anarchy

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The quantity we wish to compute: $\mathrm{PoA}(f) = \inf_{G \in \mathcal{G}} \left(\frac{\min_{a \in NE(G)} W(a)}{W(a_{opt})} \right)$

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Theorem (Characterization of $\mathrm{PoA}(f)$) [Pac18a],[Pac18b]

$\mathrm{PoA}(f)$ *is the solution to a tractable LP in 2 variables,* $\mathcal{O}(n^2)$ *constraints*

▷ *LP involves all the components* $w(j)$ *and* $f(j)$

Proof Sketch - Part 1/4

$\boxed{\textbf{Idea}\text{: transform the definition of } \mathrm{PoA} \text{ itself into a LP}}$

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Four steps towards the goal:

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1. $\mathrm{PoA}(f)$ is the same of the price of anarchy over a reduced class of games where each agent has **only two feasible allocations**

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$$\mathrm{PoA}(f) = \inf_{G \in \mathcal{G}} \left(\frac{\min_{a \in NE(G)} W(a)}{\max_{a \in \mathcal{A}} W(a)} \right)$$

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$$\mathrm{PoA}(f) = \inf_{G \in \tilde{\mathcal{G}}} \left(\frac{W(e)}{W(o)} \right) \quad \text{s.t.} \quad u_i(e) \geq u_i(o_i, e_{-i}) \quad \forall i$$

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2. Relax the previous program

$$\mathrm{PoA}(f) = \inf_{G \in \tilde{\mathcal{G}}} \frac{W(e)}{W(o)}$$
$$\text{s.t.} \quad \sum_i u_i(e) \geq \sum_i u_i(o_i, e_{-i})$$

Proof Sketch - Part 2/4

3. How to describe an instance? Need to describe $W(a)$, $u_i(a)$ on $\tilde{\mathcal{A}}$

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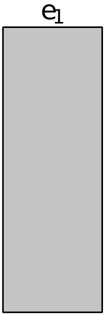

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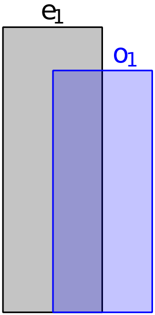

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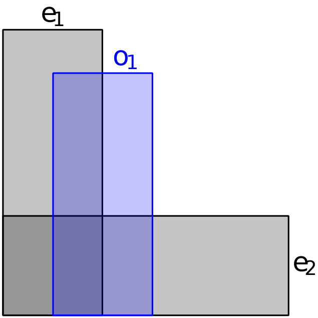

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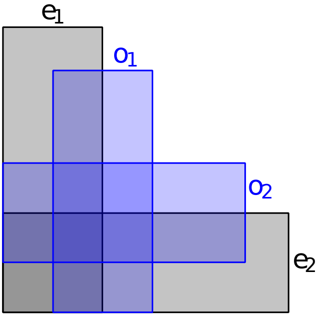

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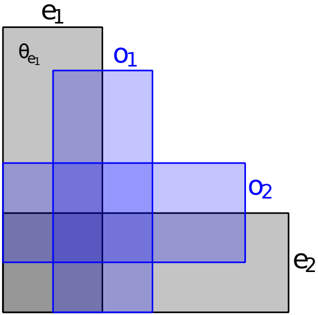

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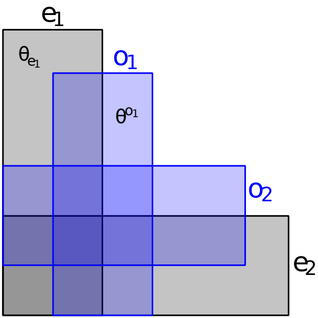

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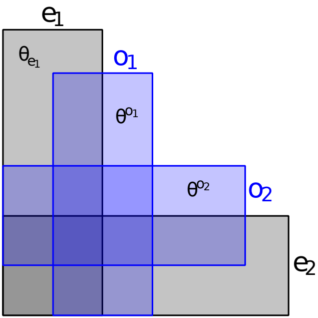

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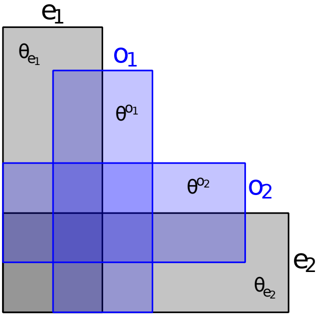

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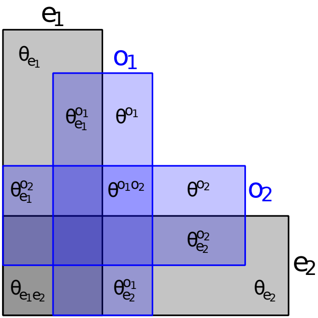

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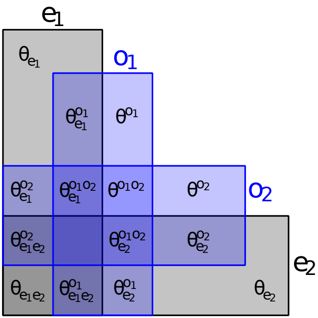

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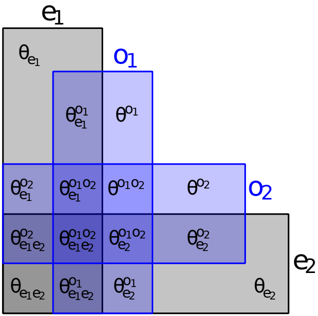

Proof Sketch - Part 2/4

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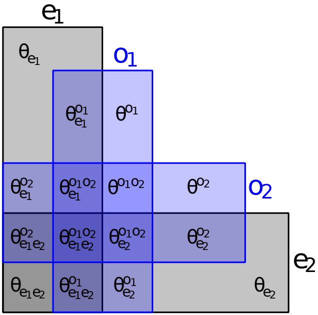
Variables θ allow to compute $W(a)$, $u_i(a)$ in **all allocations**



Proof Sketch - Part 2/4

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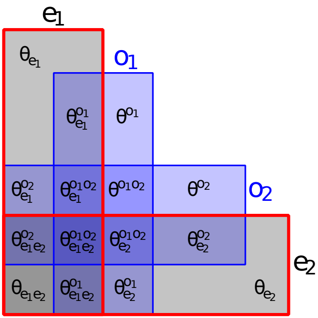

Variables θ allow to compute $W(a)$, $u_i(a)$ in **all allocations**, e.g.

$$
\begin{aligned}
&W(e_1, e_2) \\
&= (\theta_{e_1} + \theta_{e_1}^{o_1} + \theta_{e_1}^{o_2} + \theta_{e_1}^{o_1 o_2})w(1) \\
&+ (\theta_{e_2} + \theta_{e_2}^{o_2} + \theta_{e_2}^{o_1} + \theta_{e_2}^{o_1 o_2})w(1) \\
&+ (\theta_{e_1 e_2} + \theta_{e_1 e_2}^{o_1} + \theta_{e_1 e_2}^{o_2} + \theta_{e_1 e_2}^{o_1 o_2})w(2)
\end{aligned}
$$

Proof Sketch - Part 2/4

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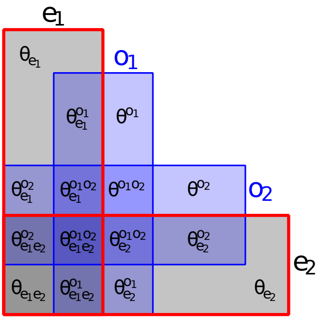

Variables θ allow to compute $W(a)$, $u_i(a)$ in **all allocations**, e.g.

$$W(e_1, e_2) = (\theta_{e_1} + \theta_{e_1}^{o_1} + \theta_{e_1}^{o_2} + \theta_{e_1}^{o_1 o_2})w(1) + (\theta_{e_2} + \theta_{e_2}^{o_2} + \theta_{e_2}^{o_1} + \theta_{e_2}^{o_1 o_2})w(1) + (\theta_{e_1 e_2} + \theta_{e_1 e_2}^{o_1} + \theta_{e_1 e_2}^{o_2} + \theta_{e_1 e_2}^{o_1 o_2})w(2)$$

$$u_1(e_1, e_2) = (\theta_{e_1} + \theta_{e_1}^{o_1} + \theta_{e_1}^{o_2} + \theta_{e_1}^{o_1 o_2})w(1)f(1) + (\theta_{e_1 e_2} + \theta_{e_1 e_2}^{o_1} + \theta_{e_1 e_2}^{o_1 o_2} + \theta_{e_1 e_2}^{o_2})w(2)f(2)$$

Proof Sketch - Part 2/4

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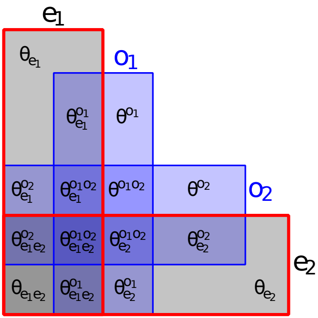

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Issue: #weights is exponential!

Proof Sketch - Part 3/4

4. use **reduced variables** for $\mathbf{W}(\mathbf{e})$, $\mathbf{W}(\mathbf{o})$, $\sum_{\mathbf{i}} \mathbf{u}_{\mathbf{i}}(\mathbf{e}) - \mathbf{u}_{\mathbf{i}}(\mathbf{o}_{\mathbf{i}}, \mathbf{e}_{-\mathbf{i}})$
   $\rightarrow$ define $\theta(a, x, b) \in \mathbb{R}_{\geq 0}$ for $1 \leq a + x + b \leq n, a, x, b \in \{1, \ldots, n\}$

Proof Sketch - Part 3/4

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$$W(e) = \sum_{a,x,b} \mathbb{1}_{\{a+x \geq 1\}} w(a + x)\theta(a, x, b)$$

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Proof Sketch - Part 3/4

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$$\text{equil.} = \sum_{a,x,b} [af(a+x)w(a+x) - bf(a+x+1)w(a+x+1)]\theta(a, x, b)$$

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The program becomes

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The program becomes $\mathrm{PoA}(f) = \inf_{\theta(a,x,b)\geq 0} \dfrac{W(e)}{W(o)}$

$$\text{s.t.} \quad \sum_i u_i(e) - u_i(o_i, e_{-i}) \geq 0$$

Proof Sketch - Part 3/4

4. use **reduced variables** for $\mathbf{W(e)}$, $\mathbf{W(o)}$, $\sum_{\mathbf{i}} \mathbf{u_i(e) - u_i(o_i, e_{-i})}$
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$$W(e) = \sum_{a,x,b} \mathbb{1}_{\{a+x\geq 1\}} w(a+x)\theta(a,x,b)$$

$$W(o) = \sum_{a,x,b} \mathbb{1}_{\{b+x\geq 1\}} w(b+x)\theta(a,x,b)$$

$$\text{equil.} = \sum_{a,x,b} [af(a+x)w(a+x) - bf(a+x+1)w(a+x+1)]\theta(a,x,b)$$

The program becomes $\mathrm{PoA}(f) = \inf_{\theta(a,x,b)\geq 0} \dfrac{1}{W(o)}$

$$\text{s.t.} \quad \sum_i u_i(e) - u_i(o_i, e_{-i}) \geq 0$$

$$W(e) = 1$$

Proof Sketch - Part 4/4: Primal LP

$$\mathrm{PoA}(f) = \frac{1}{W^\star}$$

$$W^\star = \sup_{\theta(a,x,b)} \sum_{a,x,b} \mathbb{1}_{\{b+x\geq 1\}} w(b+x)\theta(a,x,b)$$

$$\text{s.t.} \sum_{a,x,b} [af(a+x)w(a+x) - bf(a+x+1)w(a+x+1)]\theta(a,x,b) \geq 0$$

$$\sum_{a,x,b} \mathbb{1}_{\{a+x\geq 1\}} w(a+x)\theta(a,x,b) = 1$$

$$\theta(a,x,b) \geq 0 \quad \forall (a,x,b) \in \mathcal{I}\,.$$

Dual LP

Dual LP

$$\mathrm{PoA}(f) = \frac{1}{W^\star}$$

$$W^\star = \inf_{\lambda \in \mathbb{R}_{\geq 0},\, \mu \in \mathbb{R}} \ \mu$$

$$\begin{aligned}\text{s.t.}\quad & \mathbb{1}_{\{b+x\geq 1\}} w(b+x) - \mu \mathbb{1}_{\{a+x \geq 1\}} w(a+x) + \\ & + \lambda[af(a+x)w(a+x) - bf(a+x+1)w(a+x+1)] \leq 0 \\ & \qquad\qquad\qquad\qquad\qquad\qquad \forall (a, x, b) \in \mathcal{I}\end{aligned}$$

Dual LP

$$\mathrm{PoA}(f) = \frac{1}{W^\star}$$

$$\begin{aligned} W^\star = \inf_{\lambda \in \mathbb{R}_{\geq 0},\, \mu \in \mathbb{R}} \quad & \mu \\ \text{s.t.} \quad & \mathbb{1}_{\{b+x\geq 1\}} w(b+x) - \mu \mathbb{1}_{\{a+x \geq 1\}} w(a+x) + \\ & +\lambda[af(a+x)w(a+x) - bf(a+x+1)w(a+x+1)] \leq 0 \\ & \qquad \forall (a,x,b) \in \partial\mathcal{I} \end{aligned}$$

Dual LP

$$\mathrm{PoA}(f) = \frac{1}{W^\star}$$

$$W^\star = \inf_{\lambda \in \mathbb{R}_{\geq 0},\, \mu \in \mathbb{R}} \; \mu$$

$$\text{s.t.} \quad \mathbb{1}_{\{b+x\geq 1\}} w(b+x) - \mu \mathbb{1}_{\{a+x\geq 1\}} w(a+x) +$$
$$+\lambda[af(a+x)w(a+x) - bf(a+x+1)w(a+x+1)] \leq 0$$
$$\forall (a, x, b) \in \partial\mathcal{I}$$

- ▷ 2 decision variables, $\mathcal{O}(n^2)$ constraints

Dual LP

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$$W^\star = \inf_{\lambda \in \mathbb{R}_{\geq 0},\, \mu \in \mathbb{R}} \ \mu$$

$$\text{s.t.} \quad \mathbb{1}_{\{b+x\geq 1\}} w(b+x) - \mu \mathbb{1}_{\{a+x\geq 1\}} w(a+x) +$$

$$+\lambda[af(a+x)w(a+x) - bf(a+x+1)w(a+x+1)] \leq 0$$

$$\forall (a,x,b) \in \partial\mathcal{I}$$

- 2 decision variables, $\mathcal{O}(n^2)$ constraints
- observe the special structure i.e. $\min_{\lambda,\mu} \mu$ subject to $\mu \geq \dots$

Dual LP

$$\mathrm{PoA}(f) = \frac{1}{W^\star}$$

$$W^\star = \inf_{\lambda \in \mathbb{R}_{\geq 0},\, \mu \in \mathbb{R}} \ \mu$$

$$\text{s.t.} \ \ \mathbb{1}_{\{b+x\geq 1\}} w(b+x) - \mu \mathbb{1}_{\{a+x\geq 1\}} w(a+x) +$$
$$+\lambda[af(a+x)w(a+x) - bf(a+x+1)w(a+x+1)] \leq 0$$
$$\forall (a,x,b) \in \partial\mathcal{I}$$

- 2 decision variables, $\mathcal{O}(n^2)$ constraints
- observe the special structure i.e. $\min_{\lambda,\mu} \mu$ subject to $\mu \geq \dots$
- gives PoA for e.g., $f_{\mathrm{sv}}(j) = 1/j$, $f_{\mathrm{mc}}(j) = 1 - w(j-1)/w(j)$

PoA: connection with existing literature

Covering Games: Approximation through Non-Cooperation ⋆

Martin Gairing

Department of Computer Science, University of Liverpool, U.K.
m.gairing@liverpool.ac.uk

Abstract. We propose approximation algorithms under game-theoretic considerations. We indroduce and study the *general covering problem* which is a natural generalization of the well-studied *max-n-cover problem*. In the general covering problem, we are given a universal set of weighted elements E and n collections of subsets of the elements. The task is to choose one subset from each collection such that the total weight of their union is as large as possible. In our game-theoretic setting, the choice in each collection is made by an independent player. For covering an element, the players receive a payoff defined by a non-increasing *utility sharing function*. This function defines the fraction that each covering player receives from the weight of the elements.
We show how to construct a utility sharing function such that every Nash Equilibrium approximates the optimal solution by a factor of $1 - \frac{1}{e}$. We also prove that any sequence of unilateral improving steps is polynomially bounded. This gives rise to a polynomial-time local search approximation algorithm whose approximation ratio is best possible.

PoA: connection with existing literature

Covering Games: Approximation through Non-Cooperation ⋆

Martin Gairing

IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 59, NO. 3, MARCH 2014 571

Generalized Efficiency Bounds in Distributed Resource Allocation

Jason R. Marden and Tim Roughgarden

***Abstract*—Game theory is emerging as a popular tool for distributed control of multiagent systems. To take advantage of these game theoretic tools, the interactions of the autonomous agents must be designed within a game-theoretic environment. A central component of this game-theoretic design is the assignment of a local utility function to each agent. One promising approach to utility design is assigning each agent a utility function according to the agent's *Shapley value*. This method frequently results in games that possess many desirable features, such as the existence of pure Nash equilibria with near-optimal efficiency. In this paper, we explore the relationship between the Shapley value utility design and the resulting efficiency of both pure Nash equilibria and coarse correlated equilibria. To study this relationship, we introduce a simple class of resource allocation problems. Within this class, we derive an explicit relationship between the structure of the resource allocation problem and the efficiency of the resulting equilibria. Lastly, we derive a bicriteria bound for this class of resource allocation problems—a bound on the value of the optimal allocation relative to the value of an equilibrium allocation with additional agents.**

in large-scale engineering systems, where a centralized control approach is undesirable or even infeasible. For example, a centralized control approach may be impossible for the aforementioned sensor allocation problem because of the complexity associated with a potentially large number of sensors, the vastness/uncertainty of the mission space, or potential stealth requirements that restrict communication capabilities. A more desirable control approach is to establish a distributed control algorithm that allows the sensors to allocate themselves effectively over the mission space without the need for global intervention [14], [15]. Such an algorithm would eliminate the need for centralized communication and introduce an inherent robustness to communication failures, sensor failures, and environmental uncertainties. While desirable, establishing such a distributed control algorithm comes with its share of challenges. Is it possible to characterize the global behavior that

PoA: connection with existing literature

Covering Games: Approximation through Non-Cooperation *

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Generalized Efficiency Bounds in Distributed Resource Allocation

Jason R. Marden and Tim Roughgarden

Optimal Approximation for Submodular and Supermodular Optimization with Bounded Curvature

Maxim Sviridenko,[a] Jan Vondrák,[b] Justin Ward[c]

[a] Yahoo! Labs, New York, New York 10018; [b] Stanford University, Stanford, California 94305; [c] Ecole Polytechnique Federale de Lausanne, 1015 Lausanne, Switzerland

Contact: sviri@yahoo-inc.com (MS); jvondrak@stanford.edu (JV); justin.ward@epfl.ch (JW)

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OR/MS Subject Classification: Primary: analysis of algorithms; secondary: mathematics: functions

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Abstract. We design new approximation algorithms for the problems of optimizing submodular and supermodular functions subject to a single matroid constraint. Specifically, we consider the case in which we wish to maximize a monotone increasing submodular function or minimize a monotone decreasing supermodular function with a bounded total curvature c. Intuitively, the parameter c represents how nonlinear a function f is: when $c = 0$, f is linear, while for $c = 1$, f may be an arbitrary monotone increasing submodular function. For the case of submodular maximization with total curvature c, we obtain a $(1 - c/e)$-approximation—the first improvement over the greedy algorithm of of Conforti and Cornuéjols from 1984, which holds for a cardinality constraint, as well as a recent analogous result for an arbitrary matroid constraint.

Our approach is based on modifications of the continuous greedy algorithm and nonoblivious local search, and allows us to approximately maximize the sum of a nonnegative, monotone increasing submodular function and a (possibly negative) linear function. We show how to reduce both submodular maximization and supermodular minimization to this general problem when the objective function has bounded total curvature. We prove that the approximation results we obtain are the best possible in the value oracle model, even in the case of a cardinality constraint.

We define an extension of the notion of curvature to general monotone set functions and show a $(1 - c)$-approximation for maximization and a $1/(1 - c)$-approximation for minimization cases. Finally, we give two concrete applications of our results in the settings of maximum entropy sampling, and the column-subset selection problem.

Utility design reduces to

Given f, characterize or bound $\mathrm{PoA}(f)$

Find f with highest $\mathrm{PoA}(f)$

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Given f, characterize or bound $\mathrm{PoA}(f)$

$\boxed{\text{Find } f \text{ with highest } \mathrm{PoA}(f)}$

Optimal price of anarchy

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Corollary (Optimizing PoA) [Pac18a], [Pac18b]

Determining $f \in \mathbb{R}^n_{\geq 0}$ maximizing $\mathrm{PoA}(f)$ is a tractable linear program

Optimal price of anarchy

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Proof.

Optimal price of anarchy

Corollary (Optimizing PoA) [Pac18a], [Pac18b]

Determining $f \in \mathbb{R}^n_{\geq 0}$ maximizing $\mathrm{PoA}(f)$ is a tractable linear program

Proof.

$$\mathrm{PoA}(f) = \frac{1}{W^\star}$$

$$\begin{aligned} W^\star = \quad & \inf_{\lambda \in \mathbb{R}_{\geq 0},\, \mu \in \mathbb{R}} \ \mu \\ \text{s.t.} \quad & \mathbb{1}_{\{b+x\geq 1\}} w(b+x) - \mu \mathbb{1}_{\{a+x\geq 1\}} w(a+x) + \\ & + \lambda[af(a+x)w(a+x) - bf(a+x+1)w(a+x+1)] \leq 0 \\ & \qquad\qquad \forall (a,x,b) \in \partial\mathcal{I} \end{aligned}$$

Optimal price of anarchy

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Proof.

$$\mathrm{PoA}(f) = \frac{1}{W^\star}$$

$$W^\star = \min_{f \in \mathbb{R}^n_{\geq 0}} \inf_{\lambda \in \mathbb{R}_{\geq 0},\, \mu \in \mathbb{R}} \mu$$

$$\begin{aligned} \text{s.t.}\quad & \mathbb{1}_{\{b+x\geq 1\}} w(b+x) - \mu \mathbb{1}_{\{a+x\geq 1\}} w(a+x) + \\ & + \lambda[af(a+x)w(a+x) - bf(a+x+1)w(a+x+1)] \leq 0 \\ & \qquad\qquad\qquad\qquad \forall (a,x,b) \in \partial\mathcal{I} \end{aligned}$$

Optimal price of anarchy

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Proof.

$$\mathrm{PoA}(f) = \frac{1}{W^\star}$$

$$\begin{aligned}
W^\star = \min_{f \in \mathbb{R}^n_{\geq 0}} \; & \inf_{\lambda \in \mathbb{R}_{\geq 0},\, \mu \in \mathbb{R}} \; \mu \\
\text{s.t.} \;\; & \mathbb{1}_{\{b+x \geq 1\}} w(b+x) - \mu \mathbb{1}_{\{a+x \geq 1\}} w(a+x) + \\
& + \lambda[a f(a+x) w(a+x) - b f(a+x+1) w(a+x+1)] \leq 0 \\
& \qquad \forall (a, x, b) \in \partial \mathcal{I}
\end{aligned}$$

□

Back to the main result

Example:

- $\#$ agents ≤ 40

Back to the main result

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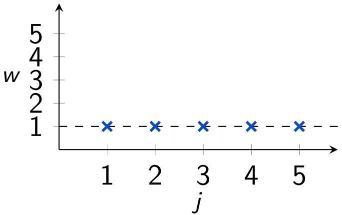
- $\#$ agents ≤ 40
- $w(j) = j^d$, $d = 0$



Back to the main result

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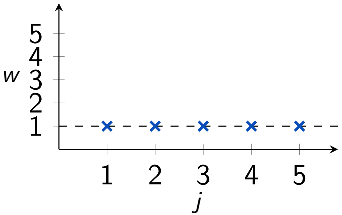
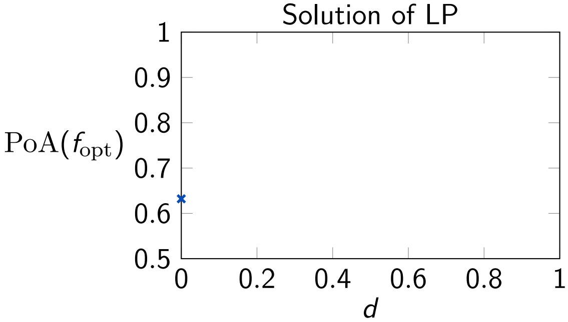
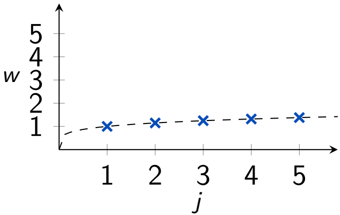


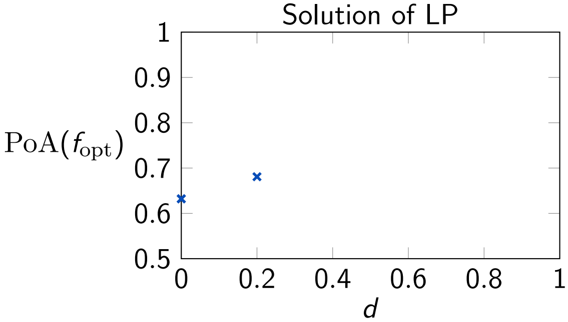


Back to the main result

Example:

- $\#$ agents ≤ 40
- $w(j) = j^d$, $d = 0.2$

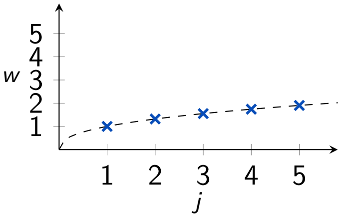


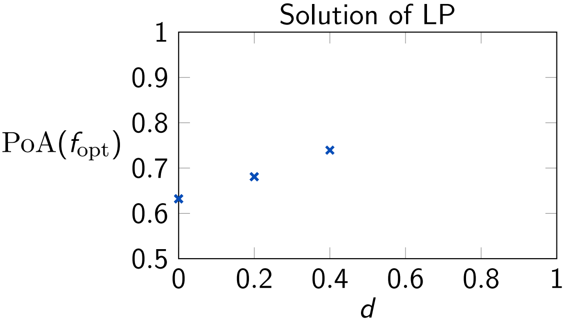


Back to the main result

Example:

- \# agents ≤ 40
- $w(j) = j^d$, $d = 0.4$

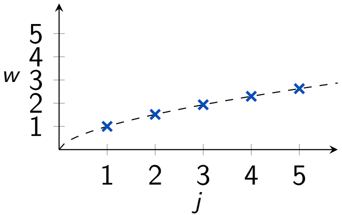


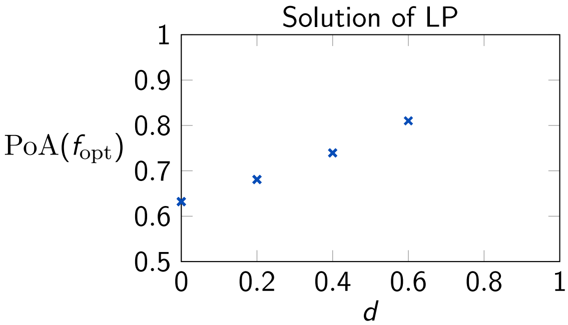


Back to the main result

Example:

- \# agents ≤ 40
- $w(j) = j^d$, $d = 0.6$

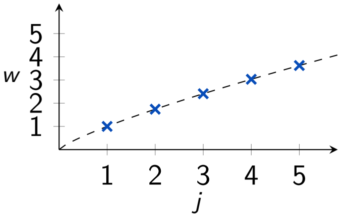


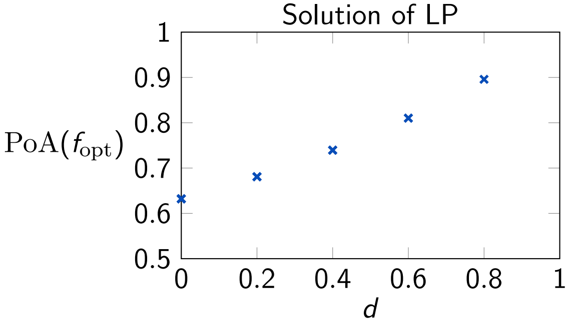


Back to the main result

Example:

- $\#$ agents ≤ 40
- $w(j) = j^d$, $d = 0.8$

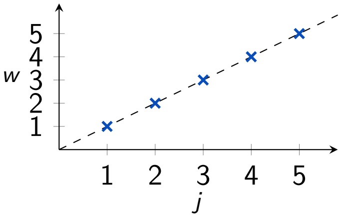


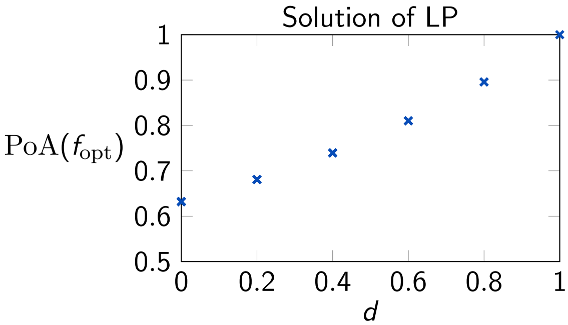


Back to the main result

Example:

- $\#$ agents ≤ 40
- $w(j) = j^d$, $d = 1$





Back to the main result

Example:

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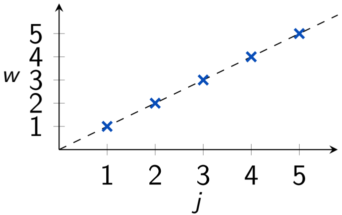
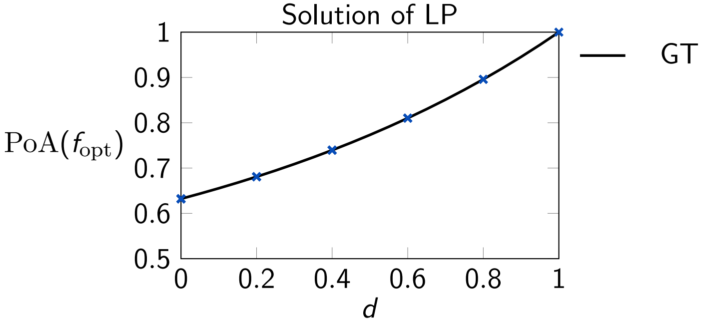
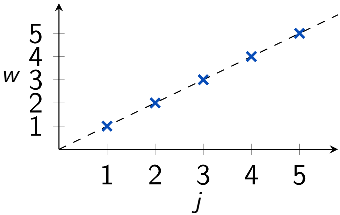


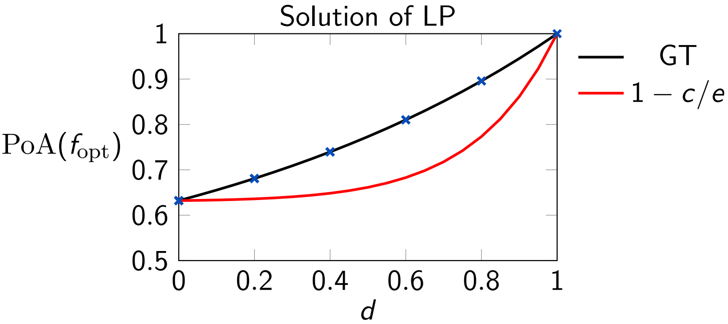


Back to the main result

Example:

- \# agents ≤ 40
- $w(j) = j^d$, $d = 1$





Comparison with other distributions, $\#agents \leq 20$

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- $f_{\mathrm{SV}}(j) = \frac{1}{j}$

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- $f_{\mathrm{SV}}(j) = \frac{1}{j}$
- $f_{\mathrm{MC}}(j) = 1 - \frac{w(j-1)}{w(j)}$

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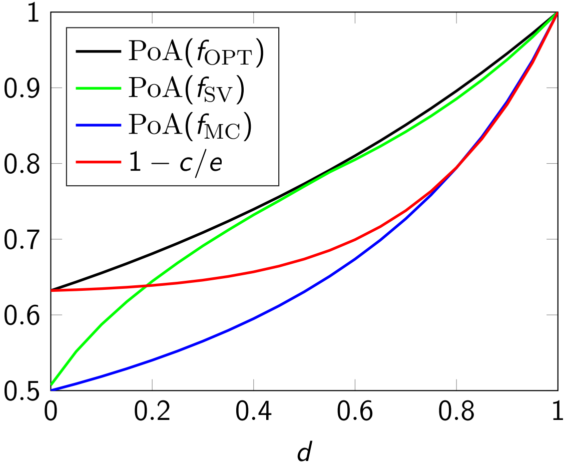

Conclusions and Outlook

Conclusions and Outlook

The problem: Generalized Multiagent Maximum Coverage

Conclusions and Outlook

The problem:	Generalized Multiagent Maximum Coverage
The approach:	Approximation through game theory

Conclusions and Outlook

The problem: Generalized Multiagent Maximum Coverage

The approach: Approximation through game theory

- Computing the exact price of anarchy
- Optimizing the price of anarchy

Conclusions and Outlook

The problem: Generalized Multiagent Maximum Coverage

The approach: Approximation through game theory

- Computing the exact price of anarchy
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The contribution: Distributed algorithms, improved performance

Conclusions and Outlook

The problem: Generalized Multiagent Maximum Coverage

The approach: Approximation through game theory

- Computing the exact price of anarchy
- Optimizing the price of anarchy

The contribution: Distributed algorithms, improved performance

Outlook: Extension to

- Coarse correlated equilibria
- More general W

Thank you

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Thank you

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[Pac18a] D. Paccagnan, R. Chandan and J.R. Marden. "Distributed resource allocation through utility design - Part I: optimizing the performance certificates via the price of anarchy". *ArXiv*, **2018**.

[Pac18b] D. Paccagnan and J.R. Marden. "Distributed resource allocation through utility design - Part II: applications to submodular, supermodular and set covering problems". *ArXiv*, **2018**.