An Algebraic Proof of Cut Elimination

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January 25, 2010

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- These slides are adapted from the talk given by prof. Ono at the Logic Summer School, ANU, December 2004.

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- We introduce *Gentzen structures* for the sequent system $\mathbf{FL_{ew}}$ without cut. $\mathbf{FL_{ew}}$ is intuitionistic logic without the contraction rule.

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- This method works for a variety of sequent systems of nonclassical (substructural, modal) logic, both in the propositional and predicate case.
- In the process we show that the quasi-completion is a generalization of the MacNeille completion.
- Moreover, the finite model property is obtained for many cases by modifying our completeness proof.

The sequent calculus $\mathbf{FL_{ew}}$

The sequent calculus $\mathbf{FL_{ew}}$ is obtained from intuitionistic logic **LJ** by deleting the contraction rule.

Initial sequents: 1) $\alpha \Rightarrow \alpha$, 2) $0 \Rightarrow$, 3) $\Rightarrow 1$.

Logical rules:

$$\frac{\Gamma \Rightarrow \delta}{1, \Gamma \Rightarrow \delta}\ (1 \Rightarrow) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0}\ (\Rightarrow 0)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \beta, \Sigma \Rightarrow \delta}{\alpha \rightarrow \beta, \Gamma, \Sigma \Rightarrow \delta}\ (\rightarrow\Rightarrow) \qquad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \rightarrow \beta}\ (\Rightarrow\rightarrow)$$

$$\frac{\alpha, \Gamma \Rightarrow \delta}{\alpha \wedge \beta, \Gamma \Rightarrow \delta}\ (\wedge 1 \Rightarrow) \qquad \frac{\beta, \Gamma \Rightarrow \delta}{\alpha \wedge \beta, \Gamma \Rightarrow \delta}\ (\wedge 2 \Rightarrow) \qquad \frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\ (\Rightarrow \wedge)$$

$$\frac{\alpha, \Gamma \Rightarrow \delta \quad \beta, \Gamma \Rightarrow \delta}{\alpha \vee \beta, \Gamma \Rightarrow \delta}\ (\vee \Rightarrow) \qquad \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta}\ (\Rightarrow \vee 1) \qquad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta}\ (\Rightarrow \vee 2)$$

$$\frac{\alpha, \beta, \Gamma \Rightarrow \delta}{\alpha \cdot \beta, \Gamma \Rightarrow \delta}\ (\cdot \Rightarrow) \qquad \frac{\Gamma \Rightarrow \alpha \quad \Sigma \Rightarrow \beta}{\Gamma, \Sigma \Rightarrow \alpha \cdot \beta}\ (\Rightarrow \cdot)$$

The sequent calculus $\mathbf{FL_{ew}}$

Structural rules:

$$\frac{\Gamma \Rightarrow \delta}{\alpha, \Gamma \Rightarrow \delta}\ (w \Rightarrow) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \alpha}\ (\Rightarrow w)$$

$$\frac{\Gamma, \alpha, \beta, \Sigma \Rightarrow \delta}{\Gamma, \beta, \alpha, \Sigma \Rightarrow \delta}\ (e \Rightarrow) \qquad \frac{\Gamma \Rightarrow \alpha \quad \alpha, \Sigma \Rightarrow \delta}{\Gamma, \Sigma \Rightarrow \delta}\ (cut)$$

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Theorem (Cut elimination [4])

If a sequent $\Gamma \Rightarrow \delta$ is provable in $\mathbf{FL_{ew}}$ then it is provable in $\mathbf{FL_{ew}}$ without using the cut rule.

$\mathbf{FL_{ew}}^{-}$ denotes the sequent system obtained from $\mathbf{FL_{ew}}$ by deleting the cut rule.

$\mathbf{FL_{ew}}$-algebras

Definition

A structure $\mathbf{P} = \langle P, \wedge, \vee, \cdot, \rightarrow, 0, 1 \rangle$ is a $\mathbf{FL_{ew}}$-algebra if:

1. $\langle P, \wedge, \vee, 0, 1 \rangle$ is a bounded lattice,
2. $\langle P, \cdot, 1 \rangle$ is a commutative monoid with the unit 1,
3. $a \cdot b \leq c$ iff $a \leq (b \rightarrow c)$ (law of residuation).

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The assignment h can be lifted to the set of all formulas.

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Definition

A sequent $\alpha_1, \ldots, \alpha_n \Rightarrow \beta$ is valid on an $\mathbf{FL_{ew}}$-algebra $\mathbf{P}$ iff $h(\alpha_1) \cdot \ldots \cdot h(\alpha_n) \leq h(\beta)$ holds in $\mathbf{P}$ for any assignment h.

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Theorem (Completeness of $\mathbf{FL_{ew}}$)

A sequent $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$ is provable in $\mathbf{FL_{ew}}$ iff it is valid on every $\mathbf{FL_{ew}}$-algebra.

Gentzen structures for $\mathbf{FL_{ew}}$

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A *Gentzen structure* for $\mathbf{FL_{ew}}$ is a tuple $\mathbf{Q} = \langle Q, \preceq, \wedge, \vee, \cdot, \rightarrow, 0, 1\rangle$ such that $0, 1 \in Q$, $\wedge, \vee, \cdot, \rightarrow$ are binary operations on Q, and $\preceq$ is a subset of $Q^* \times (Q \cup \{\varepsilon\})$ that satisfies the following conditions:

- $a \preceq a$ and $0 \preceq c$ and $\varepsilon \preceq 1$
- $x \preceq c$ implies $dx \preceq c$
- $x \preceq a$ and $by \preceq c$ imply $(a \rightarrow b)xy \preceq c$
- $ax \preceq b$ implies $x \preceq a \rightarrow b$
- $ax \preceq c$ and $bx \preceq c$ imply $(a \vee b)x \preceq c$
- $x \preceq a$ implies $x \preceq a \vee b$
- $x \preceq b$ implies $x \preceq a \vee b$
- $ax \preceq c$ implies $(a \wedge b)x \preceq c$
- $bx \preceq c$ implies $(a \wedge b)x \preceq c$
- $x \preceq a$ and $x \preceq b$ imply $x \preceq a \wedge b$
- $abx \preceq c$ implies $(a \cdot b)x \preceq c$
- $x \preceq a$ and $y \preceq b$ imply $xy \preceq a \cdot b$

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The system $\mathbf{FL_{ew}}^-$ is complete with respect to the class of Gentzen structures.

Theorem

A sequent $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$ is provable in $\mathbf{FL_{ew}}^-$ iff it is valid on every Gentzen structure.

Gentzen structures and $\mathbf{FL_{ew}}$-algebras

- Each $\mathbf{FL_{ew}}$-algebra can be seen as a Gentzen structure if $\preceq$ is defined by

$$\langle a_1, \ldots, a_m \rangle \preceq c \quad \textit{iff} \quad (a_1 \cdot \ldots \cdot a_m) \leq c$$

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- Also, let $\mathbf{Q}$ be any Gentzen structure with a strongly transitive $\preceq$:

$$x \preceq a \textit{ and } ay \preceq c \textit{ imply } xy \preceq c$$

If the restriction $\preceq_0$ of $\preceq$ to $Q \times Q$ is moreover antisymmetric, then $\mathbf{Q}$ is a $\mathbf{FL_{ew}}$-algebra with the lattice order $\preceq_0$.

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  If the restriction $\preceq_0$ of $\preceq$ to $Q \times Q$ is moreover antisymmetric, then $\mathbf{Q}$ is a $\mathbf{FL_{ew}}$-algebra with the lattice order $\preceq_0$.

- In conclusion, we can say that any Gentzen structure with a strongly transitive relation can be identified with a $\mathbf{FL_{ew}}$-algebra, and vice versa.

Cut elimination

To prove cut elimination for $\mathbf{FL_{ew}}$ it is enough to show the following result:

Lemma

if $\langle g(\alpha_1), \ldots, g(\alpha_n) \rangle \preceq g(\beta)$ *fails for some* g *in a Gentzen structure* $\mathbf{Q}$ *then* $h(\alpha_1) \cdot \ldots \cdot h(\alpha_n) \leq h(\beta)$ *fails for some* h *in an* $\mathbf{FL_{ew}}$*-algebra* $\mathbf{P}$.

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How do we get such an $\mathbf{FL_{ew}}$-algebra $\mathbf{P}$ from a given Gentzen structure $\mathbf{Q}$? Moreover, $\mathbf{Q}$ must be embedded into the $\mathbf{FL_{ew}}$-algebra $\mathbf{P}$.

1. We give a uniform way of constructing such a $\mathbf{P}$ called the *quasi-completion* of $\mathbf{Q}$;
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When $\mathbf{Q}$ is a $\mathbf{FL_{ew}}$-algebra, $\mathbf{P}$ is a MacNeille completion of $\mathbf{Q}$ and the quasi-embedding becomes a complete embedding.

Closure operators

Let $\mathbf{M} = \langle M, \cdot, 1 \rangle$ be a commutative monoid.
A unary function C on $\wp(M)$ is a *closure operator* if for all $X, Y \in \wp(M)$:

1. $X \subseteq C(X)$
2. $C(C(X)) \subseteq C(X)$
3. $X \subseteq Y$ implies $C(X) \subseteq C(Y)$
4. $C(X) * C(Y) \subseteq C(X * Y)$, where $W * Z = \{w \cdot z \mid w \in W \text{ and } z \in Z\}$.

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Let $C(\wp(M))$ be the set of all C-closed subsets, define operations $\cup_C, *_C$ and $\Rightarrow$ on $C(\wp(M))$ as follows:

- $X \cup_C Y = C(X \cup Y)$
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Lemma

The tuple $\mathbf{C_M} = \langle C(\wp(M)), \cap, \cup_C, *_C, \Rightarrow, C(\emptyset), C(\{1\})\rangle$ *is a* $\mathbf{FL_e}$*-algebra, not necessarily integral.*

Quasi-completions

- Let $\mathbf{Q}$ be a Gentzen structure, for $x \in Q^*$ and $a \in Q \cup \{\varepsilon\}$ define

$$[x; a] \quad = \quad \{w \in Q^* \mid xw \preceq a\}$$

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- Now define a function C on $\wp(Q^*)$ by

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- The function C is a closure operator such that $C(\{\varepsilon\}) = Q^* = C(\{1\})$. Thus, $\mathbf{C}_{Q^*}$ is a $\mathbf{FL_{ew}}$-algebra, which is called the *quasi-completion* of **Q**.

Quasi-embeddings

- To show that the Gentzen structure $\mathbf{Q}$ is quasi-embeddable into $\mathbf{C}_{Q^*}$ we define a *quasi-embedding* $k : Q \to C(\wp(Q^*))$ as

$$k(a) \quad = \quad [\varepsilon; a] = \{w \in Q^* \mid w \preceq a\}$$

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- Then we can prove the following.

Lemma

Suppose that $a, b \in Q$ and that U and V are arbitrary C-closed subsets of Q^* such that $a \in U \subseteq k(a)$ and $b \in V \subseteq k(b)$, then for each $\star \in \{\wedge, \vee, \cdot, \to\}$:

$$a \star b \in U \star_C V \subseteq k(a \star b),$$

where $\star_C$ denotes $\cap, \cup_C, *_C$ and $\Rightarrow$ respectively.

Proof of cut elimination - concluded

- Suppose that $\langle g(\alpha_1), \ldots, g(\alpha_n) \rangle \preceq g(\beta)$ does not hold in $\mathbf{Q}$ by an assignment g.

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- Define an assignment h on $\mathbf{C}_{Q^*}$ as $h(q) = k(g(q))$ for each proposition q.

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- By induction on the length of a formula ϕ we can show that:

$$g(\phi) \in h(\phi) \subseteq k(g(\phi))$$

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- Then in particular $h(\alpha_1) *_C \ldots *_C h(\alpha_n) \subseteq h(\beta)$, and by the results above,

$$\langle g(\alpha_1), \ldots, g(\alpha_n)\rangle \in h(\alpha_1) *_C \ldots *_C h(\alpha_n) \subseteq h(\beta) \subseteq k(g(\beta)) = \{w \mid w \preceq g(\beta)\}$$

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- But this implies $\langle g(\alpha_1), \dots, g(\alpha_n)\rangle \preceq g(\beta)$, which is a contradiction. Thus, $\alpha_1, \dots \alpha_n \Rightarrow \beta$ is not valid in $\mathbf{C}_{Q^*}$.

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- This completes the proof of cut elimination for $\mathbf{FL_{ew}}$.

MacNeille- and Quasi-completions

- Let **P** be a $\mathbf{FL_{ew}}$-algebra and define

$$C(X) \quad = \quad \bigcap\{[x; a] \mid X \subseteq [x; a] \text{ for } x \in P^* \text{ and } a \in P \cup \{\varepsilon\}\}$$

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- Then we can show that

$$C(X) \quad = \quad (X^{\rightarrow})^{\leftarrow} = \{a \mid a \leq b \textit{ for all } b \textit{ such that } b \geq c \textit{ for all } c \in X\}$$

  and therefore the quasi-completion $\mathbf{C}_{\mathbf{P}^*}$ of **P** is isomorphic to the MacNeille completion of **P**.

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  and therefore the quasi-completion $\mathbf{C_{P^*}}$ of $\mathbf{P}$ is isomorphic to the MacNeille completion of $\mathbf{P}$.
- Further, since $\leq$ is strongly transitive, $a \star b \in k(a) \star_C k(b)$ implies that $k(a \star b) = k(a) \star_C k(b)$.

MacNeille- and Quasi-completions

- Let **P** be a $\mathbf{FL_{ew}}$-algebra and define

$$C(X) \quad = \quad \bigcap \{[x; a] \mid X \subseteq [x; a] \textit{ for } x \in P^* \textit{ and } a \in P \cup \{\varepsilon\}\}$$

- Then we can show that

$$C(X) \quad = \quad (X^{\rightarrow})^{\leftarrow} = \{a \mid a \leq b \textit{ for all } b \textit{ such that } b \geq c \textit{ for all } c \in X\}$$

  and therefore the quasi-completion $\mathbf{C_{P^*}}$ of **P** is isomorphic to the MacNeille completion of **P**.
- Further, since $\leq$ is strongly transitive, $a \star b \in k(a) \star_C k(b)$ implies that $k(a \star b) = k(a) \star_C k(b)$.
- Thus, the map k can be identified with the complete embedding of a $\mathbf{FL_{ew}}$-algebra **P** into its MacNeille completion.

Extensions to other systems

This algebraic proof of cut elimination can be extended to:

- the intuitionistic and classic systems **LJ** and **LK**.
- intuitionistic substructural systems:
  - propositional calculi $\mathbf{FL_e}$ and $\mathbf{FL_{ec}}$.
  - first-order calculi $\mathbf{QFL_{ew}}$, $\mathbf{QFL_e}$ and $\mathbf{QFL_{ec}}$.
- the classic substructural systems $\mathbf{CFL_{ew}}$, $\mathbf{CFL_e}$ and $\mathbf{CFL_{ec}}$.
- the propositional modal logics **K**, **T** and **S4**.

The Finite Model Property

The idea of the proof is based on [2, 3].

Lemma

Let $\mathbf{Q}$ *be a Gentzen structure for* $\mathbf{FL_{ew}}$ *such that the closed base* $\mathcal{B} = \{[x; a] \mid x \in Q^*, a \in Q \cup \{\varepsilon\}\}$ *is finite, then the quasi-completion* $\mathbf{C_{Q^*}}$ *of* $\mathbf{Q}$ *is also finite.*

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Now define a subset $\mathcal{P}_{(x,a)}$ of $Q^* \times (Q \cup \{\varepsilon\})$ such that:

1. $(x, a) \in \mathcal{P}_{(x,a)}$.
2. Suppose that $(w, b) \in \mathcal{P}_{(x,a)}$. If "$u \preceq c$ implies $w \preceq b$" is one of the conditions for $\preceq$ in $\mathbf{Q}$, then (u, c) is a member of $\mathcal{P}_{(x,a)}$. Similarly, if "$u \preceq c$ and $v \preceq d$ imply $w \preceq b$" is one of the conditions for $\preceq$.

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For a finite subset S of $Q^* \times (Q \cup \{\varepsilon\})$, let $\mathcal{P}_S$ be the union of $\mathcal{P}_{(x,a)}$ for $(x, a) \in S$. We say that the set S is *finitely based*, when $\mathcal{P}_S$ is finite.

The Finite Model Property

Now we show how to obtain a Gentzen structure for $\mathbf{FL_{ew}}$ such that the closed base $\mathcal{B}$ is finite.

Lemma

If $\mathbf{Q}$ *is a Gentzen structure for* $\mathbf{FL_{ew}}$ *and* S *is finitely based, then the relation* $\preceq^\star$ *such that for* $(w, b) \in \mathcal{P}_S$*,* $w \preceq^\star b$ *iff* $w \preceq b$*, and otherwise* $w \preceq^\star b$ *always holds, satisfies the following conditions:*

1. *the structure* $\mathbf{Q}^\star = \langle Q, \preceq^\star, \wedge, \vee, \cdot, \rightarrow, 0, 1 \rangle$ *is a Gentzen structure for* $\mathbf{FL_{ew}}$*,*
2. *the closed base* $\mathcal{B}$ *determined by* $\preceq^\star$ *is finite.*

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Now we show how to obtain a Gentzen structure for $\mathbf{FL_{ew}}$ such that the closed base $\mathcal{B}$ is finite.

Lemma

If $\mathbf{Q}$ is a Gentzen structure for $\mathbf{FL_{ew}}$ and S is finitely based, then the relation $\preceq^\star$ such that for $(w, b) \in \mathcal{P}_S$, $w \preceq^\star b$ iff $w \preceq b$, and otherwise $w \preceq^\star b$ always holds, satisfies the following conditions:

1. *the structure $\mathbf{Q}^\star = \langle Q, \preceq^\star, \wedge, \vee, \cdot, \rightarrow, 0, 1\rangle$ is a Gentzen structure for $\mathbf{FL_{ew}}$,*
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The finite model property for $\mathbf{FL_{ew}}$

- Suppose that $\mathbf{FL_{ew}} \nvdash \alpha_1, \ldots, \alpha_m \Rightarrow \beta$, then $\langle \alpha_1, \ldots, \alpha_m \rangle \preceq \beta$ doesn't hold in the *free* Gentzen structure $\mathbf{Q}^+$ for $\mathbf{FL_{ew}}$.

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- We can show that the singleton $\{(\langle \alpha_1, \ldots, \alpha_m \rangle, \beta)\}$ is finitely based.

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- By the lemma above, $\{(\langle \alpha_1, \ldots, \alpha_m \rangle, \beta)\}$ is embedded into a Gentzen structure $(\mathbf{Q}^+)^\star$ for $\mathbf{FL_{ew}}$ with a relation $\preceq^\star$ such that the closed base is finite. Moreover, $\langle \alpha_1, \ldots, \alpha_m \rangle \preceq^\star \beta$ doesn't hold in $(\mathbf{Q}^+)^\star$ by definition.

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- By a previous lemma, the quasi-completion $\mathbf{R}$ of $(\mathbf{Q}^+)^\star$ is finite. Since $\langle \alpha_1, \ldots, \alpha_m \rangle \preceq^\star \beta$ doesn't hold in $(\mathbf{Q}^+)^\star$, $(\alpha_1 \cdot \ldots \cdot \alpha_m) \leq \beta$ doesn't hold either in $\mathbf{R}$, which is a $\mathbf{FL_{ew}}$-algebra.

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- By the lemma above, $\{(\langle \alpha_1, \ldots, \alpha_m \rangle, \beta)\}$ is embedded into a Gentzen structure $(\mathbf{Q}^+)^\star$ for $\mathbf{FL_{ew}}$ with a relation $\preceq^\star$ such that the closed base is finite. Moreover, $\langle \alpha_1, \ldots, \alpha_m \rangle \preceq^\star \beta$ doesn't hold in $(\mathbf{Q}^+)^\star$ by definition.
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- This proof of the finite model property can be extended to the first-order substructural logic $\mathbf{QFL_{ew}}$.

Thank you!

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