Counterpart Semantics at work: An Incompleteness Result in Quantified Modal Logic

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ABSTRACT. In this paper we make use of counterpart semantics to prove an original incompleteness result in quantified modal logic (QML), that is, the system $Q^E.K{+}BF$ based on free logic and containing the Barcan formula is incomplete with respect to Kripke semantics. This incompleteness result extends to the system $Q^E.K{+}CBF{+}BF$ obtained by adding the converse of the Barcan formula to $Q^E.K{+}BF$.

Keywords: Quantified Modal Logic, Kripke and Counterpart Semantics, Incompleteness.

1 Kripke Semantics

In this paper we consider a first-order modal alphabet $\mathcal{A}$ containing a denumerable infinite set Var of individual variables $x_1, x_2, \ldots$; a denumerable infinite set of n-ary predicative constants $P_1^n, P_2^n, \ldots$, for $n \in \mathbb{N}$; the connectives $\neg$ and $\rightarrow$; the quantifier $\forall$; the operator $\Box$; the existence predicative constant E. The terms $t_1, t_2, \ldots$ are only individual variables.

DEFINITION 1. The formulas in the first-order modal language $\mathcal{L}$ are defined in the Backus-Naur form as follows:

$$\phi \quad ::= \quad P^n(t_1, \ldots, t_n) \mid E(t) \mid \neg\phi \mid \phi \rightarrow \psi \mid \forall x\phi \mid \Box\phi$$

The symbols $\wedge$, $\vee$, $\leftrightarrow$, $\exists$, $\diamond$ are standardly defined; $\phi[\vec{y}/\vec{t}]$ denotes the simultaneous substitution of some, possibly all, free occurrences of $\vec{y} = y_1, \ldots, y_n$ in ϕ with $\vec{t} = t_1, \ldots, t_n$, renaming bounded variables if necessary.

DEFINITION 2. A Kripke frame, or K-frame, is a tuple $\mathcal{F} = \langle W, R, D, d\rangle$ such that W is a non-empty set; $R \subseteq W^2$; for $w, w' \in W$, $D(w)$ is a non-empty set and wRw' implies $D(w) \subseteq D(w')$; for $w \in W$, $d(w) \subseteq D(w)$.

A K-frame $\mathcal{F}$ has *constant* (resp. *increasing*, *decreasing*) inner domains iff wRw' implies $d(w) = d(w')$ (resp. $d(w) \subseteq d(w')$, $d(w) \supseteq d(w')$).

DEFINITION 3. A Kripke model of language $\mathcal{L}$ based on a K-frame $\mathcal{F}$, or K-model, is a pair $\mathcal{M} = \langle \mathcal{F}, I\rangle$ where I is an interpretation of $\mathcal{L}$ such that (i) if P^n is an n-ary predicative constant and $w \in W$, then $I(P^n, w)$ is an n-ary relation on $D(w)$; (ii) $I(E, w) = d(w)$.

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A w-assignments is any function $\sigma : Var \to D(w)$. The variant $\sigma\binom{x}{a}$ does not coincide with σ at most on x, and assigns $a \in D(w)$ to x.

DEFINITION 4. The satisfaction relation $\models$ for a world $w \in \mathcal{M}$, a formula $\phi \in \mathcal{L}$, and a w-assignment σ is defined as follows:

$$\begin{array}{ll}
(\mathcal{M}^\sigma, w) \models P^n(t_1, \ldots, t_n) & \text{iff } \langle \sigma(t_1), \ldots, \sigma(t_n) \rangle \in I(P^n, w) \\
(\mathcal{M}^\sigma, w) \models \neg\psi & \text{iff } (\mathcal{M}^\sigma, w) \not\models \psi \\
(\mathcal{M}^\sigma, w) \models \psi \to \psi' & \text{iff } (\mathcal{M}^\sigma, w) \not\models \psi \text{ or } (\mathcal{M}^\sigma, w) \models \psi' \\
(\mathcal{M}^\sigma, w) \models \Box\psi & \text{iff for every } w' \in W,\ wRw' \text{ implies } (\mathcal{M}^\sigma, w') \models \psi \\
(\mathcal{M}^\sigma, w) \models \forall x\psi & \text{iff for every } a \in d(w),\ (\mathcal{M}^{\sigma\binom{x}{a}}, w) \models \psi
\end{array}$$

A formula ϕ is *true at a world* w iff it is satisfied by every w-assignment σ; ϕ is *valid on a* K*-model* $\mathcal{M}$ iff it is true at every world in $\mathcal{M}$; ϕ is *valid on a* K*-frame* $\mathcal{F}$ iff it is valid on every K-model based on $\mathcal{F}$.

2 The Systems $Q^E.K{+}BF$ and $Q^E.K{+}CBF{+}BF$

We now introduce the systems $Q^E.K{+}BF$ and $Q^E.K{+}CBF{+}BF$ based on free logic. We will consider the following principles in what follows.

$Taut$	tautologies of classical propositional calculus	
K	$\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$	distribution axiom
MP	$\phi \to \psi, \phi \Rightarrow \psi$	modus ponens
Nec	$\phi \Rightarrow \Box\phi$	necessitation
$E\text{-}Ex$	$\forall x\phi \to (E(y) \to \phi[x/y])$	E-exemplification
$E\text{-}Gen$	$\phi \to (E(x) \to \psi) \Rightarrow \phi \to \forall x\psi$, x not free in ϕ	E-generalization
BF	$\forall x\Box\phi \to \Box\forall x\phi$	Barcan formula
CBF	$\Box\forall x\phi \to \forall x\Box\phi$	converse of BF
$N\neg E$	$\neg E(x) \to \Box\neg E(x)$	necessity of fictionality
NE	$E(x) \to \Box E(x)$	necessity of existence

DEFINITION 5. The system $Q^E.K{+}BF$ includes the schemes of axioms $Taut$, K, $E\text{-}Ex$, BF, and the inference rules MP, Nec, $E\text{-}Gen$. The system $Q^E.K{+}CBF{+}BF$ extends $Q^E.K{+}BF$ by adding CBF.

We consider the standard definitions of *proof* and *theorem*: $S \vdash \phi$ means that ϕ is a theorem in the system S. A K-frame $\mathcal{F}$ is a *K-frame for S* iff all the theorems of S are valid on $\mathcal{F}$, i.e., $S \vdash \phi$ implies $\mathcal{F} \models \phi$.

LEMMA 6. *For any system S in the first column, $\mathcal{F}$ is a K-frame for S iff it satisfies the constraint on inner domains in the second column:*

calculi	inner domain
$Q^E.K{+}BF$	*decreasing*
$Q^E.K{+}CBF{+}BF$	*constant*

LEMMA 7. *Every K-frame for $Q^E.K{+}BF$ validates the necessity of fictionality, i.e., $Q^E.K{+}BF \models N\neg E$.*

We leave to the interested reader the proof of this standard result, which is due to decreasing inner domains. In the incompleteness result in section 4 we will show that $Q^E.K{+}BF$ does not prove $N\neg E$. Lemma 7 applies also to the system $Q^E.K{+}CBF{+}BF$.

3 Counterpart Semantics

For introducing the counterpart semantics for QML we make use of typed languages. First, every variable x_i in the alphabet $\mathcal{A}$ is a term of type n, or n-term, for $n \geq i$. If x_j is an n-term and $t_1, \ldots, t_n$ are m-terms, the substituted m-term $x_j[t_1, \ldots, t_n]$ is the m-term t_j, or $t_j : m$ in short.

DEFINITION 8. The typed first-order modal language $\mathcal{L}_T$ contains all and only the formulas ϕ of type n, or $\phi : n$, for $n \in \mathbb{N}$, defined as follows:

- if P^m is an m-ary predicative constant and $(t_1, \ldots, t_m)$ is an m-tuple of n-terms, then $P^m(t_1, \ldots, t_m)$ is a (atomic) formula of type n;
- if ψ, ψ' are n-formulas, then $\neg\psi$ and $\psi \to \psi'$ are formulas of type n;
- if ψ is an m-formula and $(t_1, \ldots, t_m)$ is an m-tuple of n-terms, then $(\Box\psi)(t_1, \ldots, t_m)$ is a formula of type n;
- if ψ is an $n{+}1$-formula, then $\forall x_{n+1}\psi$ is a formula of type n.

The formula $\Box\phi : n$ is a shorthand for $(\Box\phi)(x_1, \ldots, x_n) : n$. Let ϕ be an n-formula and $\vec{s}$ an n-tuple of k-terms, the substituted k-formula $\phi[\vec{s}]$ is inductively defined as follows:

- ϕ is the atomic formula $P^m(t_1, \ldots, t_m)$, then $\phi[\vec{s}]$ is $P^m(t_1[\vec{s}], \ldots, t_m[\vec{s}])$;
- $\phi = \neg\psi$, then $(\neg\psi)[\vec{s}] = \neg(\psi[\vec{s}])$;
- $\phi = \psi \to \psi'$, then $(\psi \to \psi')[\vec{s}] = \psi[\vec{s}] \to \psi'[\vec{s}]$;
- $\phi = (\Box\psi)(t_1, \ldots, t_m)$, then $(\Box\psi)(t_1, \ldots, t_m)[\vec{s}] = (\Box\psi)(t_1[\vec{s}], \ldots, t_m[\vec{s}])$;
- $\phi = \forall x_{n+1}\psi$, then $(\forall x_{n+1}\psi)[\vec{s}] = \forall x_{k+1}(\psi[\vec{s}, x_{k+1}])$.

Note that substitution does not commute with the modal operator, therefore it is not the case that $(\Box\phi)[t_1, \ldots, t_m]$ is equivalent to $\Box(\phi[t_1, \ldots, t_m])$.

DEFINITION 9. A counterpart frame, or c-frame, is a tuple $\mathcal{F} = \langle W, R, D, d, C\rangle$ such that W is a non-empty set; $R \subseteq W^2$; for $w \in W$, $D(w)$ is a non-empty set and $d(w) \subseteq D(w)$; for wRw', $C_{w,w'} \subseteq D(w) \times D(w')$.

In this paper we focus on the following classes of c-frames:

existentially faithful iff wRw', $a \in d(w)$ and $C_{w,w'}(a, b)$, imply $b \in d(w')$
fictionally faithful iff wRw', $a \in D(w) \setminus d(w)$ and $C_{w,w'}(a, b)$, imply $b \in D(w') \setminus d(w')$
everywhere-defined iff wRw' and $a \in D(w)$, imply there is $b \in D(w')$ s.t. $C_{w,w'}(a, b)$
surjective iff wRw' and $b \in d(w')$, imply there is $a \in d(w)$ s.t. $C_{w,w'}(a, b)$
functional iff wRw', $C_{w,w'}(a, b)$ and $C_{w,w'}(a, b')$, imply $b = b'$

DEFINITION 10. A counterpart model for the language $\mathcal{L}_T$ based on a c-frame $\mathcal{F}$, or c-model in short, is a couple $\mathcal{M} = \langle \mathcal{F}, I\rangle$ where I is an interpretation of $\mathcal{L}_T$ such that (i) if P^n is an n-ary predicative constant and $w \in W$, then $I(P^n, w)$ is an n-ary relation on $D(w)$; (ii) $I(E, w) = d(w)$.

A finitary assignment of type n, or n-assignment, in a world w is an n-tuple $\vec{a}$ of elements in $D(w)$. Let t be the n-term x_j, the valuation $\vec{a}(t)$ for the n-assignment $\vec{a}$ is equal to a_j.

DEFINITION 11. The satisfaction relation $\models$ for a world $w \in \mathcal{M}$, a typed formula $\phi : n$, and an n-assignment $\vec{a}$ is defined as follows:

$$
\begin{array}{ll}
(\mathcal{M}^{\vec{a}}, w) \models P^m(t_1, \ldots, t_m) & \text{iff } \langle \vec{a}(t_1), \ldots, \vec{a}(t_m) \rangle \in I(P^m, w) \\
(\mathcal{M}^{\vec{a}}, w) \models \neg\psi & \text{iff } (\mathcal{M}^{\vec{a}}, w) \not\models \psi \\
(\mathcal{M}^{\vec{a}}, w) \models \psi \to \psi' & \text{iff } (\mathcal{M}^{\vec{a}}, w) \not\models \psi \text{ or } (\mathcal{M}^{\vec{a}}, w) \models \psi' \\
(\mathcal{M}^{\vec{a}}, w) \models (\Box\psi)(t_1, \ldots, t_m) & \text{iff for } w' \in W, \text{ for } b_1, \ldots, b_m \in D(w'), \\
 & \quad wRw' \text{ and } C_{w,w'}(\vec{a}(t_i), b_i) \text{ imply } (\mathcal{M}^{\vec{b}}, w') \models \psi \\
(\mathcal{M}^{\vec{a}}, w) \models \forall x_{n+1}\psi & \text{iff for every } a^* \in d(w), (\mathcal{M}^{\vec{a}\cdot a^*}, w) \models \psi
\end{array}
$$

where $\vec{a} \cdot a^*$ is the $n+1$-assignment $(a_1, \ldots, a_n, a^*)$.

A typed formula $\phi : n$ is said to be *true at a world* w iff it is satisfied by every n-assignment; ϕ is *valid on a c-model* $\mathcal{M}$ iff it is true at every world in $\mathcal{M}$; ϕ is *valid on a c-frame* $\mathcal{F}$ iff it is valid on every c-model based on $\mathcal{F}$.

4 Incompleteness of QML Systems

This section is devoted to the incompleteness proofs for systems $Q^E.K{+}BF$ and $Q^E.K{+}CBF{+}BF$, which are inspired to a similar result in [3]. We first show that $Q^E.K{+}BF$ is Kripke-incomplete, that is, there is no class of Kripke frames which validates all and only the theorems of $Q^E.K{+}BF$.

THEOREM 12. *The system $Q^E.K{+}BF$ is Kripke-incomplete, i.e., every K-frame for $Q^E.K{+}BF$ validates $N\neg E$, but $Q^E.K{+}BF \nvdash N\neg E$.*

In section 2 we remarked that $Q^E.K{+}BF \models N\neg E$. In order to show that $Q^E.K{+}BF$ does not prove $N\neg E$ we need two lemmas. By the first one if a formula $\phi \in \mathcal{L}$ is a theorem in $Q^E.K{+}BF$, then its translation $\tau_n(\phi) \in \mathcal{L}_T$ as defined below holds in a suitable c-frame. By the second lemma this suitable c-frame does not validate $\neg E(x_n) \to \Box\neg E(x_n)$, i.e., the translation of $N\neg E$ according to τ_n. By contraposition we obtain that $Q^E.K{+}BF$ does not prove $N\neg E$.

Following [3, 6] we define a translation function from untyped to typed first-order modal languages.

DEFINITION 13. Let $\phi \in \mathcal{L}$ be an untyped formula and define $g(\phi)$ as the maximum k such that x_k occurs in ϕ. For $n \geq g(\phi)$, the formula $\tau_n(\phi) \in \mathcal{L}_T$ of type n is inductively defined as follows:

$$
\begin{array}{ll}
\tau_n(P^m(t_1, \ldots, t_m)) & := P^m(t_1, \ldots, t_m) \\
\tau_n(\neg\psi) & := \neg\tau_n(\psi) \\
\tau_n(\Box\psi) & := \Box\tau_n(\psi) \\
\tau_n(\psi \to \psi') & := \tau_n(\psi) \to \tau_n(\psi') \\
\tau_n(\forall x_i\psi) & := \forall x_{n+1}(\tau_n(\psi)[x_1, \ldots, x_{i-1}, x_{n+1}, x_{i+1}, \ldots, x_n])
\end{array}
$$

By the first lemma theoremhood in $Q^E.K{+}BF$ implies validity in everywhere-defined, surjective, functional c-frames, *modulo* the translation function τ_n.

LEMMA 14. *Let $\phi \in \mathcal{L}$, $n \geq g(\phi)$ and let $\mathcal{F}$ be an everywhere-defined, surjective, and functional c-frame, then*

$$Q^E.K + BF \vdash \phi \quad \textit{implies} \quad \mathcal{F} \models \tau_n(\phi)$$

The proof of this lemma requires the following auxiliary result, in which the assumptions of everywhere-definiteness and functionality are essential.

LEMMA 15. *If ϕ is a formula in $\mathcal{L}$, $\mathcal{F}$ is an everywhere-defined and functional c-frame, and $x_{i_1}, \ldots, x_{i_m}$ are free for $x_1, \ldots, x_m$ in ϕ, then*

$$\mathcal{F} \models \tau_m(\phi)[x_{i_1}, \ldots, x_{i_m}] \leftrightarrow \tau_n(\phi[x_{i_1}, \ldots, x_{i_m}])$$

If $Q^E.K{+}BF$ proves $N\neg E$, then any everywhere-defined, surjective, and functional c-frame models $\tau_n(N\neg E)$. But the latter fact is negated by the next lemma.

LEMMA 16. *There exists an everywhere-defined, surjective, and functional c-frame $\mathcal{F}$ such that $\mathcal{F} \not\models \neg E(x_n) \rightarrow \Box\neg E(x_n) : n$.*

Proof. Consider the c-frame $\mathcal{F}$, where $W = \{w, w'\}$; $R = \{(w, w')\}$; $D(w) = \{a, a'\}$, $D(w') = \{b\}$; $d(w) = \{a\}$, $d(w') = \{b\}$; $C_{w,w'} = \{(a, b), (a', b)\}$. By definition $\mathcal{F}$ is everywhere-defined, surjective, and functional, but $N\neg E$ fails in $\mathcal{F}$ as it is not fictionally faithful. Consider a c-model $\mathcal{M}$ based on $\mathcal{F}$ and an n-assignment $\vec{a}$ such that $a_n = a'$ and $(\mathcal{M}^{\vec{a}}, w) \models \neg E(x_n)$. We have that $C_{w,w'}(a', b)$ and $b \in d(w')$, so $(\mathcal{M}^{\vec{a}}, w) \models \Diamond E(x_n)$. Thus, $(\mathcal{M}^{\vec{a}}, w) \models \neg E(x_n) \wedge \Diamond E(x_n)$ and $\mathcal{F} \not\models N\neg E$. ■

By lemmas 14 and 16 the system $Q^E.K{+}BF$ does not prove $N\neg E$, which is nonetheless valid on every K-frame for $Q^E.K{+}BF$. As a result, theorem 12 holds.

Note that also the system $Q^E.K{+}CBF{+}BF$ is Kripke-incomplete, as lemma 14 holds also for $Q^E.K{+}CBF{+}BF$ with respect to existentially faithful, everywhere-defined, surjective, and functional c-frames. Further, the c-frame in lemma 16 is also existentially faithful.

THEOREM 17. *The system $Q^E.K{+}CBF{+}BF$ is Kripke-incomplete, i.e., $Q^E.K{+}CBF{+}BF \models N\neg E$, but $Q^E.K{+}CBF{+}BF \nvdash N\neg E$.*

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