Argumentation Logic as an Anhomomorphic Logic

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Abstract

Recent work proposed Anhomomorphic Logic, characterized in algebraic terms via answering maps on a Boolean algebra of propositions, as a logical framework appropriate for physics, including both classical and quantum theory. In this paper we study how another type of logic, called Argumentation Logic, generalizing classical logic to accommodate various forms of commonsense reasoning, exhibits properties of Anhomomorphic Logic.

1 Introduction

Argumentation Logic (AL) captures entailment in classical logic as well as forms of defeasible commonsense reasoning [4, 5, 6]. AL-entailment is defined through argumentation, in terms of the least fixed points of 'acceptability' and 'non-acceptability' operators, for a given (possibly empty) theory in a propositional language equipped with an underlying notion of 'direct derivation'. This is some fragment of derivability in classical logic, but excluding reductio ad absurdum.

Anhomomorphic Logic (AnhomL) was developed to deal with the modified rules of inference that may be needed for quantum physics [7, 10, 1]. It is framed in terms of a Boolean algebra of propositions about a system and a set of allowed answering maps. A *Scheme* for AnhomL is a set of conditions the allowed answering maps must satisfy which define the type of inferences allowed in the logic. It remains an open question which Scheme – if any – will successfully account for the physics we know.

The central feature of both AnhomL and AL is that they "tolerate contradiction" without collapsing into triviality, so it is interesting to explore the possible connections between them. We make a start in this direction by defining an answering map, χ^T, corresponding to AL-entailment, and show that χ^T has algebraic properties that can be compared to those of answering maps in AnhomL.

The paper is organised as follows: we first briefly recall essentials of AnhomL (in section 2) and AL (in section 3); we then define and study χ^T (in section 4), illustrating it with the 3-slit example from quantum physics (in section 5), and conclude in section 6.

2 Anhomomorphic Logic

Anhomomorphic Logic [7, 10, 1] pertains to a collection, $\mathscr{U}$, of propositions about a (physical) system. $\mathscr{U}$ is a Boolean algebra and closed under the propositional logical connectives of $\wedge, \vee, \neg$. Elements of $\mathscr{U}$ are also referred to as *events*. This structure naturally arises in physics where there is an underlying set, Ω, of spacetime histories of the system such that elements of $\mathscr{U}$ are subsets of Ω. Then $\wedge, \vee, \neg$ are defined in the canonical way through the intersection, union and complement set operations. A possible world is an answering map, $\phi : \mathscr{U} \to \mathscr{Z}_2 = \{0, 1\}$ (called a *co-event* in the literature, since ϕ maps events to a set of scalars), and if $\phi(A) = 1$ (respectively $\phi(A) = 0$) we say A is *affirmed* (respectively *denied*) by the world ϕ. A Scheme is a set of conditions that an *allowed* answering map must satisfy. We define here some conditions on ϕ that may or may not hold within any given Scheme. First it is useful to define the logical connective 'exclusive or', $\oplus$, on events in $\mathscr{U}$: $A \oplus B := (A \vee B) \wedge \neg(A \wedge B)$. Then ϕ is *additive* if $\phi(A \oplus B) = \phi(A) + \phi(B)$ for all A and B in $\mathscr{U}$. ϕ is *multiplicative* if $\phi(A \wedge B) = \phi(A)\phi(B)$ for all A and B in $\mathscr{U}$. ϕ is a *homomorphism* if $\phi \neq 0$ (namely there is some A such that $\phi(A) \neq 0$) and is both additive and multiplicative.

The rules of logical inference are determined by the Scheme conditions. For example, if the Scheme condition is that ϕ is a homomorphism then the rules of inference are those of classical logic [7]. Hence the term *Anhomomorphic Logic* to denote the framework: in a Scheme with less restrictive conditions, allowed worlds will not be homomorphisms and the rules of inference about the world will not be classical. The Multiplicative Scheme (MS) for AnhomL [7] is defined by the conditions that ϕ is multiplicative and not constant, together with a minimality condition and a dynamical condition known as preclusion (see the triple slit example below in section 5). In the MS scheme the rule of inference $(\phi(A) = 1 \Rightarrow \phi(\neg A) = 0) \forall A \in \mathscr{U}$ holds but the rule of inference $(\phi(A) = 0 \Rightarrow \phi(\neg A) = 1) \forall A \in \mathscr{U}$ does not.

3 Argumentation Logic

Argumentation Logic (AL) [4, 5, 6] defines entailment in terms of notions of acceptability and non-acceptability of arguments, seen as sets of propositional formulae in a given language $\mathscr{L}$, equipped with a notion of *direct derivation* based on a subset of standard inference rules in Natural Deduction (see appendix A). When a formula A in $\mathscr{L}$ is directly derived (using the chosen inference rules) from a theory T in $\mathscr{L}$, we write $T \vdash_{DD} A$. Whereas in [4, 5, 6] the chosen inference rules, and therefore $\vdash_{DD}$, are fixed, and amounting to all inference rules in appendix A except RA (Reduction ad Absurdum), in this paper we do not commit to a specific $\vdash_{DD}$, and leave it instead as a parameter for AL, while imposing that it does not include RA. We say that a theory T in $\mathscr{L}$ is *directly inconsistent* if $T \vdash_{DD} \bot$, where $\bot$ stands for inconsistency and amounts to $A \wedge \neg A$ for any A in $\mathscr{L}$. We say that a theory is *directly consistent* if it is not directly inconsistent. Throughout this paper we will assume as given a directly consistent theory T.

The notions of (non-)acceptability are defined in terms of notions of *attack* and *defence* amongst arguments (i.e. T extended by sets of formulae), as follows, for Δ, Γ sets of formulae in $\mathscr{L}$:

- argument $a = T \cup \Delta$ *attacks* argument $b = T \cup \Gamma$, with $\Gamma \neq \emptyset$, iff $a \cup b \vdash_{DD} \bot$;
- argument d *defends against* argument $a = T \cup \Delta$, iff
  1. $d = T \cup \{\neg A\}$ ($d = T \cup \{A\}$) for some $A \in \Delta$ (respectively $\neg A \in \Delta$), or
  2. $d = T \cup \emptyset$ and $a \vdash_{DD} \bot$.

(Non-)Acceptability is defined as the least fixed point of an operator: given the set of binary relations $\mathscr{R}$ over all sets of arguments in $\mathscr{L}$,

- the *acceptability operator* $\mathscr{A}_T : \mathscr{R} \to \mathscr{R}$ is defined as follows: for any $acc \in \mathscr{R}$ and arguments a, a_0: $(a, a_0) \in \mathscr{A}_T(acc)$ iff
  - $a \subseteq a_0$, or
  - for any argument b such that b attacks a,
    - $b \not\subseteq a_0 \cup a$, and
    - there is argument d that defends against b such that $(d, a_0 \cup a) \in acc$.
- the *non-acceptability operator* $\mathscr{N}_T : \mathscr{R} \to \mathscr{R}$ is defined as follows: for any $nacc \in \mathscr{R}$ and arguments a, a_0: $(a, a_0) \in \mathscr{N}_T(nacc)$ iff
  - $a \not\subseteq a_0$, and

– there is argument b such that b attacks a and
  - $b \subseteq a_0 \cup a$, or
  - for any argument d that defends against b, $(d, a_0 \cup a) \in nacc$.

ACC^T and $NACC^T$ denote the least fixed points of $\mathscr{A}_T$ and $\mathscr{N}_T$ respectively. We say that *a is acceptable wrt* a_0 *in* T iff $ACC^T(a, a_0)$, and *a is not acceptable wrt* a_0 *in* T iff $NACC^T(a, a_0)$.

The definition of entailment in AL is given as follows:

- a formula A in $\mathscr{L}$ is *AL-entailed (from T, given* $\vdash_{DD}$*)*, written $\models_{AL}^{DD(T)} A$, iff $ACC^T(\{A\}, \emptyset)$ and $NACC^T(\{\neg A\}, \emptyset)$.

In the remainder of the paper we often omit T from arguments and say that a formula A is acceptable (non-acceptable, respectively) when $ACC^T(\{A\}, \{\})$ holds ($NACC^T(\{A\}, \{\})$ holds, respectively).

4 An algebraic view of AL-entailment

Here and in the remainder of this section we assume $\vdash_{DD}$ and (a directly consistent theory) T in $\mathscr{L}$ as given, and that A, B are formulae in $\mathscr{L}$. (Note that T may be classically inconsistent).

We define $\chi^T : \mathscr{L} \to \mathscr{Z}_2$ corresponding to AL-entailment, as follows:

$$\chi^T(A) = 1 \text{ iff } \models_{AL}^{DD(T)} A$$

The following basic property of χ^T follows directly from the property of AL-entailment, that $T \vdash_{DD} A$ implies $ACC^T(\{A\}, \emptyset)$ and $NACC^T(\{\neg A\}, \emptyset)$:

Lemma 1. *If* $T \vdash_{DD} A$ *then* $\chi^T(A) = 1$.

The map χ^T satisfies axiomatic properties of ϕ depending on the inference rules in the underlying $\vdash_{DD}$. This correspondence follows from the following *closure property* of AL (see appendix B for a sketch of the proof):

Lemma 2. *if* $\models_{AL}^{DD(T)} A$ *and* $T \cup \{A\} \vdash_{DD} B$*, then* $\models_{AL}^{DD(T)} B$.

This result essentially indicates that AL respects its underlying basic inference rules ($\vdash_{DD}$). As a consequence, depending on the choice of $\vdash_{DD}$, χ^T satisfies various corresponding algebraic properties as considered in AnhomL. For example:

- if $\vdash_{DD}$ includes the $\wedge E$ rule then
  if $\chi^T(A \wedge B)$=1 then $\chi^T(A)$=1 and $\chi^T(B) = 1$.
- if $\vdash_{DD}$ includes the $\wedge I$ rule then
  if $\chi^T(A)$=1 and $\chi^T(B) = 1$ then $\chi^T(A \wedge B)$=1.

Hence when $\vdash_{DD}$ includes both of the standard inference rules for conjunction then AL satisfies the multiplicative property:

Proposition 1. $\chi^T(A \wedge B) = \chi^T(A)\chi^T(B)$.

With respect to negation the properties of χ^T are:

- if $\chi^T(A)$=1 then $\chi^T(\neg A)$=0.

This follows since non-acceptability of $\neg A$ makes it impossible for $\neg A$ to be AL-entailed. Note though that the reverse property, i.e. if $\chi^T(A)$=0 then $\chi^T(\neg A)$=1, does not hold. This is similar to the properties of ϕ in the MS Scheme for AnhomL.

Similarly, if we include the $\vee I$ rule in $\vdash_{DD}$ then the algebraic semantics of AL has the property:

- if $\chi^T(A)$=1 then $\chi^T(A \vee B)$=1.

In AnhomL the similar condition on ϕ holds in the MS Scheme due to the Boolean algebra structure of $\mathscr{U}$: we have $A \wedge (A \vee B) = A$ and so $\phi(A)\phi(A \vee B) = \phi(A)$, and thus if $\phi(A) = 1$ then $\phi(A \vee B) = 1$.

4.1 χ^T is anhomomorphic

Note that when $\neg(A \wedge B)$ holds in AL, i.e. $\chi^T(\neg(A \wedge B))$=1, then $\chi^T(A)$=0 or $\chi^T(B)$=0 (as they cannot both take the value 1). Does this impose that only one of the disjuncts will take the value 0 and hence the other will take the value 1 (i.e. that one of A or B will hold)? In other words, as in the AnhomL approach, is the map χ^T homomorphic in the full algebra of $\mathscr{Z}_2$, i.e. satisfying the additive property: $\chi^T(A \oplus B) = \chi^T(A) + \chi(B)$?

For disjoint formulae A, B, i.e. such that $\chi^T(\neg(A \wedge B))$=1, given, additionally, that $\chi^T(A \vee B)$=1 this homomorphic property (wrt + in $\mathscr{Z}_2$) forces one of A and B to take the value 1: the disjunction can hold and yet neither disjunct holds.

When we take $\vdash_{DD}$ to contain also the $\vee E$ inference rule (i.e. reasoning by cases) and the theory T is classically consistent then this property is satisfied (as AL and classical logic coincide in this case [4, 5, 6]) and hence χ^T is a homomorphism. However, in general, the answer to the questions above is no, as this property does not always hold in AL: the disjunction can hold and yet neither disjunct holds.

For the case of directly consistent theories that we are considering, if we exclude the $\vee E$ inference rule from $\vdash_{DD}$ then a simple example shows how this property is violated:

Example 1. *Let $T = \{\alpha \vee \beta, \alpha \rightarrow \perp, \beta \rightarrow \perp\}$. Then, $\chi^T(\alpha)$=0 and $\chi^T(\beta)$=0 although $\chi^T(\alpha \vee \beta)$=1.*

Note that the well known “Barber of Seville” paradox is a concrete variant of this example:
$T_{BS} = \{SBarber \vee SHimself, \neg(SBarber \wedge SHimself), SBarber \leftrightarrow SHimself\}$.

The last formula in T_{BS} makes both $SBarber$ and $SHimself$ non-acceptable and hence both take the value 0 under χ^T_{BS} and yet $SBarber \vee SHimself$ and $SBarber \oplus SHimself$ take the value 1 under χ^T_{BS} as they are directly derivable from the theory.

The following example shows that even when $\vee E$ is present in $\vdash_{DD}$, χ^T is not homomorphic:

Example 2. $T = \{\alpha \vee \beta, \neg(\alpha \wedge \gamma), \neg(\alpha \wedge \neg\gamma), \neg(\beta \wedge \delta), \neg(\beta \wedge \neg\delta)\}$.
Again, both $\chi^T(\alpha)$=0 and $\chi^T(\beta)$=0 although $\chi^T(\alpha \vee \beta)$=1.

We see that in general, AL gives rise to an answering map on $\mathscr{L}$ that shares properties comparable to the properties of the co-events in the MS Scheme for AnhomL. Through this, AL avoids classical logic paradoxes, that arguably do not exist in common sense reasoning. Since AnhomL was proposed as a logical framework for physics motivated by the need to encompass results in quantum mechanics that seem to many paradoxical by the lights of classical reasoning, it is interesting to investigate and compare how both AnhomL and AL treat an example from Quantum Physics.

5 An illustrative example from Quantum Physics

In the 3-slit experiment [8], particles (photons say) are incident on a barrier in which there are three equally spaced, parallel slits, labelled A, B and C. Beyond the barrier is a screen on which the particles are detected if they make it through the slits. The experiment is run and a particle is detected at a particular position P on the screen. The distance from the barrier to the screen and the slit spacing are such that the amplitude for the particle to pass through slit B and arrive at P is equal to the amplitude for the particle to pass through slit C and arrive at P, and minus the amplitude for the particle to pass through slit A and arrive at P. There are three atomic propositions which are, "the particle passed through slit A", "the particle passed through slit B" and "the particle passed through slit C". Treating this experiment in AnhomL, the event algebra is $\mathscr{U} = \{\emptyset, A, B, C, A \vee B, B \vee C, C \vee A, A \vee B \vee C\}$ where we use A to denote "the particle passed through slit A" *etc.*

Quantum destructive interference – cancellation of equal and opposite amplitudes – means that the *quantum measure* [8] of each event $A \vee B$ and $A \vee C$ is zero and the *preclusion* condition [3, 8] implies that those events are denied by every allowed co-event: $\phi(A \vee B) = \phi(A \vee C) = 0$. Multiplicativity in AnhomL then implies also that $\phi(A) = \phi(B) = \phi(C) = 0$. The *minimality* condition [7] in the MS Scheme means that there is exactly one allowed co-event, ϕ, in this example: $\phi(B \vee C) = \phi(A \vee B \vee C) = 1$ are the only affirmations, all other events are denied by ϕ.

In AL, we represent this experiment and the quantum dynamics by $T = \{A \vee B \vee C, \neg(A \vee B), \neg(A \vee C)\}$ and $\vdash_{DD}$ given by $\wedge I$, $\wedge E$ and $\vee I$. Here, T is a directly consistent theory (since $\vee E$ is not included in $\vdash_{DD}$).

Then A, B and C on their own are non-acceptable and hence $\chi^T(A) = \chi^T(B) = \chi^T(C) = 0$. Yet $\chi^T(A \vee B \vee C) = 1$.

We can also see that $\neg A$ is acceptable as there are no attacks against this apart from $T \cup \{A\}$, defended by $T \cup \{\neg A\}$ trivially, and apart from attacks that contain the negation of some direct consequence, D, of $\neg A$. But then the attack can be defended by $T \cup \{D\}$ which is acceptable wrt $\{\neg A\}$. Hence $\chi^T(\neg A) = 1$. Similarly, $\chi^T(\neg B) = 1$ and $\chi^T(\neg C) = 1$ hold.

Regarding $B \vee C$ this can be shown to be acceptable for the same reason as above, i.e. that there are no attacks against this except those that contain the negation of a direct consequence of $B \vee C$. The same holds for $\neg(B \vee C)$. Therefore χ^T gives 0 for both. Here we see a difference with the MS Scheme in AnhomL in which $\phi(B \vee C) = 1$.

6 Conclusions

We have initiated a comparison of two attempts to address the limitations of classical logic, one in the realm of commonsense reasoning and logical paradoxes and the other in quantum physics. In Argumentation Logic the central notion of acceptability of a formula gives a logical framework where the so called "tetralemma" [9] is naturally accommodated. In the future we will seek to understand better the relationship between AL and AnhomL by studying further quantum examples such as the Kochen-Specker theorem in AnhomL [2]. Quantum mechanics is sometimes described as "counter-intuitive" and "paradoxical": it would be striking if understanding it requires an approach to logic that is actually closer to human, commonsense reasoning than the rigid rules of classical logic.

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A Natural Deduction

We use (some of) the following inference rules, for any propositional formulae A,B,C in $\mathscr{L}$:

$$\wedge I : \frac{A,B}{A\wedge B} \quad \wedge E : \frac{A\wedge B}{A} \quad \wedge E : \frac{A\wedge B}{B} \quad \vee I : \frac{A}{A\vee B} \quad \vee I : \frac{B}{A\vee B} \quad \rightarrow I : \frac{\lceil A\ldots B\rfloor}{A\rightarrow B} \quad \bot I : \frac{A,\neg A}{\bot}$$

$$\neg E : \frac{\neg\neg A}{A} \quad \neg I/RA : \frac{\lceil A\ldots\bot\rfloor}{\neg A} \quad \vee E : \frac{A\vee B, \lceil A\ldots C\rfloor, \lceil B\ldots C\rfloor}{C} \quad \rightarrow E : \frac{A, A\rightarrow B}{B}$$

where $\lceil\zeta,\ldots\rfloor$ is a (sub-)derivation with ζ referred to as the *hypothesis*. $\neg I$ is also called Reduction ad Absurdum (RA). $\bot$ stands for inconsistency.

B Sketch of proof of lemma 2

We need to show that (i) $ACC^T(\{B\},\emptyset)$ and (ii) $NACC^T(\{\neg B\},\emptyset)$.

(i) Any attack, $T\cup\{C\}$, against $\{B\}$ is also an attack against $\{A\}$ since $T\cup\{A\}\vdash_{DD} B$. Then the defence against C given from $ACC^T(\{A\},\emptyset)$ will also form a defence for the acceptability of $\{B\}$. Otherwise, there will be an attack A' against some defence D' in the acceptability tree of A such that $A'\subseteq Branch(D')\cup\{B\}$ where $Branch(D')$ is the union of defences up to and including D' in the acceptability tree of A. But then $A'' = (A'-\{B\})\cup\{A\}$ will also be an attack against D' (since $T\cup\{A\}\vdash_{DD} B$) such that $A''\subseteq Branch(D')$, thus contradicting the acceptability of A.
(ii)The set $\{A\}$ is an attack against $\{\neg B\}$ (since $T,A\vdash_{DD} B$). Then because $ACC^T(\{A\},\emptyset)$ holds it follows that $T\cup A\not\vdash_{DD}\bot$ and hence the only possible defence against A is $\{\neg A\}$. From $NACC^T(\{\neg A\},\emptyset)$ then (when $B\neq A$ as is our case here) $NACC^T(\{\neg A\},\{\neg B\})$ would also hold. Otherwise, if it does not hold then in the non-acceptability proof of $\{\neg event\}$ there will be some defence equal to $\{\neg B\}$ and so $NACC^T(\{\neg B\},\{\neg A\})$ will hold. But then from this we have that $NACC^T(\{\neg B\},\emptyset)$ also holds, as required, since if $NACC^T(\{\neg B\},\{\neg A\})$ comes from an attack containing $\neg A$ then (when this is not part of the branch) its possible defence $\{A\}$ is attacked by $\{\neg B\}$ (since $T\cup\{A\}\vdash_{DD} B$) and therefore this defence will also be non-acceptable in the non-acceptability proof of $\{\neg B\}$ wrt the empty set. $\square$